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# Feedback Based On Multi-Qubit Measurements, Multiple-Particle Interference And Quantum Error Correction And Other Models

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**[1]Dr. K.N.P.Kumar**

Independent Researcher and Post Doctoral Scholar, Department of Mathematics, Kuvempu University, Janna Sahyadri, Shankaraghatta, Karnataka, India

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**Abstract:** Feedback Based On Multi-Qubit Measurements, Multiple-Particle Interference And Quantum Error Correction, Theory Of Quantum Error-Correcting Codes, Theory Of Quantum Error-Correcting Codes-2, Theory Of Quantum Error-Correcting Codes-3, New Class Of Quantum Error-Correcting Codes For A Bosonic Mode, New Class Of Quantum Error-Correcting Codes For A Bosonic Mode, Bosonic Quantum Memories, Photonic Quantum Communication, And Optical-To-Microwave Up- And Down-Conversion, Mixed State Entanglement And Quantum Error Correction, Mixed State Entanglement And Quantum Error Correction-2, Mixed State Entanglement And Quantum Error Correction-3, Universal Operator Theoretic Framework For Quantum Fault Tolerance, Efficiently Controllable Graphs. Foregoing system and its connected properties are studied in detail.

**Key Words**: Quantum error correcting codes, Quantum Fault Tolerance, Bosonic Mode

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**Introduction:**

We study the following system and its attendant properties: Feedback Based On Multi-Qubit Measurements, Multiple-Particle Interference And Quantum Error Correction, Theory Of Quantum Error-Correcting Codes, Theory Of Quantum Error-Correcting Codes-2, Theory Of Quantum Error-Correcting Codes-3, New Class Of Quantum Error-Correcting Codes For A Bosonic Mode, New Class Of Quantum Error-Correcting Codes For A Bosonic Mode, Bosonic Quantum Memories, Photonic Quantum Communication, And Optical-To-Microwave Up- And Down-Conversion, Mixed State Entanglement And Quantum Error Correction, Mixed State Entanglement And Quantum Error Correction-2, Mixed State Entanglement And Quantum Error Correction-3, Universal Operator Theoretic Framework For Quantum Fault Tolerance, Efficiently Controllable Graphs

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| **SECTION ONE**<br>**Feedback Based On Multi-Qubit Measurements** | |
| **INTRODUCTION—VARIABLES USED**<br>**Repeated quantum error correction on a continuously encoded qubit by real-time feedback J.** | |

**Cramer, N. Kalb,M. A. Rol, B. Hensen, M. S. Blok, M. Markham, D. J. Twitchen, R. Hanson & T. H. Taminiau Nature Communications 7, Article number: 11526 doi:10.1038/ncomms11526**

(1) A promising way to correct errors in encoded quantum states is (=) to perform feedback based on (e) multi-qubit measurements known as stabilizer measurements1, 2, 3 (see Fig. 1a for details).
(2) These measurements are performed non-destructively using (e) extra qubits (ancillas) and are (=) frequently repeated to detect errors before (e) they accumulate.
**(3)** The measurement outcomes are then processed in (eb) classical logic that identifies the error syndrome, and, in order to enable (eb) universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed.
**(4)** The key experimental challenge is to perform such complete error-correction cycles including (e) non-destructive stabilizer measurements and real-time feedback well within the coherence time

**NOTATION**

**Module One**

A promising way to correct errors in encoded quantum states is (=) to perform feedback based on (e) multi-qubit measurements known as stabilizer measurements1, 2, 3 (see Fig. 1a for details)

$G_{13}$ : Category one of promising way to correct errors in encoded quantum states

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of perform feedback based on (e) multi-qubit measurements known as stabilizer measurements1, 2, 3 (see Fig. 1a for details)

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

A promising way to correct errors in encoded quantum states is to perform feedback based on (e) multi-qubit measurements known as stabilizer measurements1, 2, 3 (see Fig. 1a for details)

$G_{16}$ : Category one of multi-qubit measurements known as stabilizer measurements1, 2, 3 (see Fig. 1a for details)

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of promising way to correct errors in encoded quantum states is to perform feedback

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

**Module three**

These measurements are performed non-destructively using (e) extra qubits (ancillas) and are (=) frequently

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| repeated to detect errors before (e) they accumulate | |
| $G_{20}$ : Category one of extra qubits (ancillas) and are (=) frequently repeated to detect errors before (e) they accumulate<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of measurements are performed non-destructively<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>These measurements are performed non-destructively using extra qubits (ancillas) and are (=) frequently repeated to detect errors before (e) they accumulate | |
| $G_{24}$ : Category one of measurements are performed non-destructively using extra qubits (ancillas)<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of repeated to detect errors before (e) they accumulate<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>These measurements are performed non-destructively using extra qubits (ancillas) and are frequently repeated to detect errors before (e) they accumulate | |
| $G_{28}$ : Category one of they accumulate; measurements are performed non-destructively using extra qubits (ancillas) and are frequently repeated to detect errors<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of measurements are performed non-destructively using extra qubits (ancillas) and are frequently repeated to detect errors they accumulate<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The measurement outcomes are then processed in (eb) classical logic that identifies the error syndrome, and, | |

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| in order to enable (eb) universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed | |
| $G_{32}$ : Category one of **measurement outcomes;** classical logic that identifies the error syndrome, and, in order to enable (eb) universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of classical logic that identifies the error syndrome, and, in order to enable (eb) universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed **;measurement outcomes**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable (eb) universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed | |
| $G_{36}$ : Category one of measurement outcomes are then processed in classical logic that identifies the error syndrome<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback is applied to (e&eb) the encoded system to correct (e&eb) errors where needed | |
| $G_{40}$ : Category one of **measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback;** encoded system to correct (e&eb) errors where needed<br><br>$G_{41}$ : Category two of SAS | |

$G_{42}$ : Category three of SAS

T$_{40}$ : Category one of encoded system to correct (e&eb) errors where needed; **measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback**

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

The measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback is applied to the encoded system to correct (e&eb) errors where needed

$G_{44}$ : Category one of **measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback is applied to the encoded system;** errors where needed

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

T$_{44}$ : Category one of errors where needed **;measurement outcomes are then processed in classical logic that identifies the error syndrome, and, in order to enable universal computations1, active feedback is applied to the encoded system**

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$

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| $(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)},$ $(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)},$ $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$ are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |

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| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |

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| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |

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| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3 | |

$+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3

$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)} & -(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt}=(a_{16})^{(2)}G_{17}-\left[\begin{array}{ccc}(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt}=(a_{17})^{(2)}G_{16}-\left[\begin{array}{ccc}(a_{17}^{'})^{(2)}+(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt}=(a_{18})^{(2)}G_{17}-\left[\begin{array}{ccc}(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3

$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt}=(b_{16})^{(2)}T_{17}-\left[\begin{array}{ccc}(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t)\\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t)\end{array}\right]T_{16} \quad (64)$$

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| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18}$ | 66 |
| where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3<br>$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22}$ | 69 |
| $\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3 | |

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| $+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3* | |

$\boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)}\boxed{-(b''_{25})^{(4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

*Where* $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$
*are fourth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$
*are fifth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$
*are sixth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$
*are seventh detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$
*are eighth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

*Where* $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* $1, 2$ *and* $3$

*And* $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ *are second augmentation*

| | |
|---|---|
| $coefficient\ for\ category\ 1,2\ and\ 3$ $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ $are\ third\ \ augmentation$ $coefficient\ for\ category\ 1,2\ and\ 3$ $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | 84 |
| $where$ $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ $are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ $are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ $are\ \ third\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3 $-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3 $-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3 $-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition | |

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| coefficients for category 1,2, and 3 $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)}\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{28}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{32}$ | *85* |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)}\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{29}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{33}$ | *86* |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)}\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{30}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{34}$ | *87* |
| $\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{28}^{''})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5)}(T_{29},t)}$ *are second augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)}$ *are third augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$ - are fourth augmentation coefficients $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$ - fifth augmentation coefficients $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$ sixth augmentation coefficients $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$ seventh augmentation coefficients $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$ Eighth augmentation coefficients $\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}^{'})^{(6)}\boxed{-(b_{32}^{''})^{(6)}(G_{35},t)} & \boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)} & \boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{32}$ | *88* |

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

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Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

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| Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[\begin{matrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{matrix} (a_{41}^{'})^{(8)}\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{matrix} (a_{42}^{'})^{(8)}\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{15}$ | |
| Where $\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second | |

augmentation coefficient for category 1, 2 and 3
$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3
$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3
$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3
$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}(G_{43},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}-(b_{41}^{''})^{(8)}(G_{43},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}(G_{43},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3
$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3
$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3
$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

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| $-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt}$<br>$= (a_{44})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt}$<br>$= (a_{45})^{(9)}G_{44}$<br>$-\begin{bmatrix} (a_{45}^{'})^{(9)}+(a_{45}^{''})^{(9)}(T_{45},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt}$<br>$= (a_{46})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3

$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b_{44}^{'})^{(9)}\left[-(b_{44}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\left[-(b_{45}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\left[-(b_{46}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{44}^{''})^{(9)}(G_{47},t)$, $-(b_{45}^{''})^{(9)}(G_{47},t)$, $-(b_{46}^{''})^{(9)}(G_{47},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

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| $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| Where we suppose | |
| $(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$<br><br>The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | *97* |
| $lim_{T_2 \to \infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ | 101 |

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| satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(2)}(T_{17}', t) - (a_i'')^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}' \rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i'')^{(2)}((G_{19})', t) - (b_i'')^{(2)}((G_{19}), t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})'\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}', t)$ and $(a_i'')^{(2)}(T_{17}, t)$ . $(T_{17}', t)$ and $(T_{17}, t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i'')^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants | |

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| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$ satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$<br>$\lim_{G \to \infty}(b_i^{''})^{(3)}(G_{23}, t) = (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$$

$$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$$ 116

Where we suppose

$$(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$$ 117

The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:

$$(a_i^{''})^{(4)}(T_{25}, t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$$

$$(b_i^{''})^{(4)}((G_{27}), t) \le (r_i)^{(4)} \le (b_i^{'})^{(4)} \le (\hat{B}_{24})^{(4)}$$

$$lim_{T_2 \to \infty}(a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$$ 118

$$\lim_{G \to \infty}(b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$$

**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :

Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$

They satisfy Lipschitz condition: 119

$$|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)| \le (\hat{k}_{24})^{(4)}|T_{25} - T_{25}^{'}|e^{-(\hat{M}_{24})^{(4)}t}$$

$$|(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}((G_{27}), t)| < (\hat{k}_{24})^{(4)}||(G_{27}) - (G_{27})^{'}||e^{-(\hat{M}_{24})^{(4)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ : 120

$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants

$$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$$

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| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}(G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants | 125 |

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| $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\| \leq (\hat{k}_{32})^{(6)} \|T_{33} - T_{33}^{'}\| e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\|(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\| < (\hat{k}_{32})^{(6)} \|(G_{35}) - (G_{35})'\| e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants | 129 |

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(A) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(B) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \le (p_i)^{(7)} \le (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \le (r_i)^{(7)} \le (b_i')^{(7)} \le (\hat{B}_{36})^{(7)}$$

*(C)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$ 132

*(D)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \le (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the

| | |
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| system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(E) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(F) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i'')^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i')^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i'')^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i'')^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}'\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |

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|---|---|
| $\|(b_i^{''})^{(8)}((G_{43})',t)-(b_i^{''})^{(8)}\big((G_{43}),t\big)\|<(\hat{k}_{40})^{(8)}\|(G_{43})-(G_{43})'\|e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}',t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}',t)$ and $(T_{41},t)$ are points belonging to the interval $\big[(\hat{k}_{40})^{(8)},(\hat{M}_{40})^{(8)}\big]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)}=1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\ ,\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}<1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)},(\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},(\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)},(a_i')^{(8)},(b_i)^{(8)},(b_i')^{(8)},(p_i)^{(8)},\ (r_i)^{(8)},i=40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)}+(a_i')^{(8)}+\ (\hat{A}_{40})^{(8)}+(\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}]<1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\ (b_i)^{(8)}+(b_i')^{(8)}+\ (\hat{B}_{40})^{(8)}+(\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}]<1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)},(a_i')^{(9)},(a_i^{''})^{(9)},(b_i)^{(9)},(b_i')^{(9)},(b_i^{''})^{(9)}>0,\quad i,j=44,45,46$<br><br>The functions $(a_i^{''})^{(9)},(b_i^{''})^{(9)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(9)},\ (r_i)^{(9)}$:<br><br>$(a_i^{''})^{(9)}(T_{45},t)\le(p_i)^{(9)}\le(\hat{A}_{44})^{(9)}$<br><br>$(b_i^{''})^{(9)}(G_{47},t)\le\ (r_i)^{(9)}\le(b_i')^{(9)}\le(\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i^{''})^{(9)}(T_{45},t)=(p_i)^{(9)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(9)}(G_{47},t)=\ (r_i)^{(9)}$<br><br>**Definition of** $(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)}$ :<br><br>Where $\boxed{(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}}$ are positive constants and $\boxed{i=44,45,46}$ | |
| They satisfy Lipschitz condition: | |

$$|(a_i^{''})^{(9)}(T_{45}^{'},t)-(a_i^{''})^{(9)}(T_{45},t)|\le(\hat{k}_{44})^{(9)}|T_{45}-T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'},t)-(b_i^{''})^{(9)}\big((G_{47}),t\big)|<(\hat{k}_{44})^{(9)}||(G_{47})-(G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'},t)$ and$(a_i^{''})^{(9)}(T_{45},t)$ . $(T_{45}^{'},t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)}=1$ then the function $(a_i^{''})^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)}and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i^{'})^{(9)},(b_i)^{(9)},(b_i^{'})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i^{'})^{(9)}+(\hat{A}_{44})^{(9)}+(\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)}+(b_i^{'})^{(9)}+(\hat{B}_{44})^{(9)}+(\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0),T_i(0)$ :

$$G_i(t)\le(\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le(\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0),T_i(0)$

$$G_i(t)\le(\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ G_i(0)=G_i^0>0$$

$$T_i(t)\le(\hat{Q}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}t}\ ,\ G_i(0)=G_i^0>0$$

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| $T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 B |

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| $G_i(t) \leq \left(\hat{P}_{44}\right)^{(9)} e^{(\bar{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)}t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)}t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)}t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{20})^{(3)}$ , $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |

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| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}')^{(7)} - (b_{37}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}')^{(7)} - (b_{38}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br><br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br><br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}')^{(8)} + a_{40}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}')^{(8)} + (a_{41}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}')^{(8)} + (a_{42}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}')^{(8)} - (b_{40}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}')^{(8)} - (b_{41}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}')^{(8)} - (b_{42}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$$

Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$

The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$$

$$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$$

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From which it follows that

$$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[((\hat{P}_{13})^{(1)} + G_{14}^0) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 1

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Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$$

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From which it follows that

$$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[((\hat{P}_{16})^{(2)} + G_{17}^0) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$$

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Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

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$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that 172

$$(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$$

From which it follows that 174

$$(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that 175

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

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(a) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{36}(t) \le G_{36}^0 + \int_0^t\left[(a_{36})^{(7)}\left(G_{37}^0+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$
$$\left(1+(a_{36})^{(7)}t\right)G_{37}^0+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

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From which it follows that

$$(G_{36}(t)-G_{36}^0)e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^0)e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^0}{G_{37}^0}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t\left[(a_{40})^{(8)}\left(G_{41}^0+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$
$$\left(1+(a_{40})^{(8)}t\right)G_{41}^0+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$

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From which it follows that

$$(G_{40}(t)-G_{40}^0)e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^0)e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^0}{G_{41}^0}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

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The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t\left[(a_{44})^{(9)}\left(G_{45}^0+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)} =$$
$$\left(1+(a_{44})^{(9)}t\right)G_{45}^0+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^0)e^{-(\hat{M}_{44})^{(9)}t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^0)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^0}{G_{45}^0}\right)}+(\hat{P}_{44})^{(9)}\right]$$

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| $(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose<br>$(\widehat{\mathrm{P}}_{13})^{(1)}$ and $(\widehat{\mathrm{Q}}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G}, \tilde{T}$ : $(\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{13})^{(1)}\left\|G_{14}^{(1)} - G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$<br>$\int_0^t\{(a'_{13})^{(1)}\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$G_{13}^{(2)}\left\|(a''_{13})^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a''_{13})^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right\| \; e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a'_{13})^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$ | 186 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results | |

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| $T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded. The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{-(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\hat{M}_{16})^{(2)}s_{(16)}}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\hat{M}_{16})^{(2)}t} \leq$ | |

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| $\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+\ (a'_{16})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)\mathrm{d}\left(((G_{19})^{(1)},(T_{19})^{(1)};\ (G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a''_{16})^{(2)}$ and $(b''_{16})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(2)}$ and $(b''_i)^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a'_i)^{(2)}-(a''_i)^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}\mathrm{d}s_{(16)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{(-(b'_i)^{(2)}t)}>0$ for $t>0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1,\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16}<(\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt}\leq\left((\widehat{M}_{16})^{(2)}\right)_1-(a'_{17})^{(2)}G_{17}$ and by integrating<br><br>$G_{17}\leq\left((\widehat{M}_{16})^{(2)}\right)_2=G_{17}^0+2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a'_{17})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18}\leq\left((\widehat{M}_{16})^{(2)}\right)_3=G_{18}^0+2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a'_{18})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(2)}((G_{19})(t),t))=(b'_{17})^{(2)}$ then $T_{17}\to\infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t>t_2$<br><br>$(b_{17})^{(2)}-(b''_i)^{(2)}((G_{19})(t),t)<\varepsilon_2, T_{16}(t)>(m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt}\geq(a_{17})^{(2)}(m)^{(2)}-\varepsilon_2 T_{17}$ which leads to | 204 |

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| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1-e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t), t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left(\widetilde{(G_{23})}, \widetilde{(T_{23})}\right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}'')^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |

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| $\left\|{G_{23}}^{(1)} - {G_{23}}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)}t} \leq$ $\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right) d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$ And analogous inequalities for $G_i$ $and$ $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a''_{20})^{(3)}$ and $(b''_{20})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ $and$ $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(3)}$ and $(b''_i)^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ $and$ $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(3)} - (a''_i)^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ $and$ $\left((\widehat{M}_{20})^{(3)}\right)_3$ : **Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ $and$ $G_{22}$ . indeed if $G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a'_{21})^{(3)} G_{21}$ and by integrating $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1 / (a'_{21})^{(3)}$ In the same way , one can obtain $G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2 / (a'_{22})^{(3)}$ If $G_{21}$ $or$ $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ $is$ bounded, from below, the same property holds for $G_{21}$ $and$ $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(3)}\left((G_{23})(t), t\right)) = (b'_{21})^{(3)}$ then $T_{21} \to \infty$. **Definition of** $(m)^{(3)}$ and $\varepsilon_3$ : Indeed let $t_3$ be so that for $t > t_3$ $(b_{21})^{(3)} - (b''_i)^{(3)}\left((G_{23})(t), t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to | 220 |

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1-e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.
The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose

$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have

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$$\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)} \qquad 222$$

$$\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)} \qquad 223$$

In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself 224

The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric 225

$$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{24})^{(4)}\left|G_{25}^{(1)} - G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

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| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\| e^{-(\widehat{M}_{24})^{(4)}t} \leq$ $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a'_{24})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$ And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a''_{24})^{(4)}$ and $(b''_{24})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(4)}$ and $(b''_i)^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(4)} - (a''_i)^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ : **Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating $G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a'_{25})^{(4)}$ In the same way , one can obtain $G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2 / (a'_{26})^{(4)}$ If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(4)}((G_{27})(t), t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$. **Definition of** $(m)^{(4)}$ and $\varepsilon_4$ : Indeed let $t_4$ be so that for $t > t_4$ $(b_{25})^{(4)} - (b''_i)^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)}$$

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$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)}$$

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In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric

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$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{31})}, \widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})}, \widetilde{(T_{31})}\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)}, s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

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| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$ | 242 |

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| $(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\,\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)}, s_{(32)}\right)\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $G_{32}^{(2)}\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br><br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}\ and\ (\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32{,}33{,}34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{(-(b_i^{'})^{(6)}t)}>0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1,\left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32}<(\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt}\leq\left((\widehat{M}_{32})^{(6)}\right)_1-(a_{33}^{'})^{(6)}G_{33}$ and by integrating<br><br>$G_{33}\leq\left((\widehat{M}_{32})^{(6)}\right)_2=G_{33}^0+2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34}\leq\left((\widehat{M}_{32})^{(6)}\right)_3=G_{34}^0+2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}\ is$ bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t))=(b_{33}^{'})^{(6)}$ then $T_{33}\to\infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ : | 253 |

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| Indeed let $t_6$ be so that for $t > t_6$<br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t), t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}^{'})^{(6)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$<br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left|G_{37}^{(1)} - G_{37}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$ | 258 |

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| $G_{36}^{(2)}\|(a_{36}^{''})^{(7)}(T_{37}^{(1)},s_{(36)})-(a_{36}^{''})^{(7)}(T_{37}^{(2)},s_{(36)})\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}\ and\ (\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)}>0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1,\left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36}<(\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt}\leq\left((\widehat{M}_{36})^{(7)}\right)_1-(a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37}\leq\left((\widehat{M}_{36})^{(7)}\right)_2=G_{37}^0+2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38}\leq\left((\widehat{M}_{36})^{(7)}\right)_3=G_{38}^0+2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |

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| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$. | 264 |
| **Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br>Indeed let $t_7$ be so that for $t > t_7$<br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1-e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}((G_{39})(t),t) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results | 271 |

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| $\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)}-G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$ $\int_0^t \{(a_{40}^{'})^{(8)} \left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$ $(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$ $G_{40}^{(2)} \left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$ | |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(\left((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)} t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$ | 276 |

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| If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t),t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1-e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t),t(t),t\big) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ and to choose $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[(\hat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\Big(((G_{47})^{(1)},(T_{47})^{(1)}),\big((G_{47})^{(2)},(T_{47})^{(2)}\big)\Big) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}$ : $\big(\widetilde{(G_{47})},\widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}),(T_{47})\big)$<br>It results | |

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{44})^{(9)} \left|G_{45}^{(1)}-G_{45}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\, ds_{(44)} +$$
$$\int_0^t \{(a_{44}^{'})^{(9)} \left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\} ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)} (\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{45}^{'})^{(9)}$ then $T_{45}\to\infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t>t_9$

$$(b_{45})^{(9)}-(b_i^{''})^{(9)}\left((G_{47})(t),t\right)<\varepsilon_9, T_{44}(t)>(m)^{(9)}$$

Then $\frac{dT_{45}}{dt}\geq(a_{45})^{(9)}(m)^{(9)}-\varepsilon_9 T_{45}$ which leads to

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1-e^{-\varepsilon_9 t})+T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t}=\frac{1}{2}$ it results

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t=log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)}\leq-(a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}-(a_{13}^{''})^{(1)}(T_{14},t)+(a_{14}^{''})^{(1)}(T_{14},t)\leq-(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)}\leq-(b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G,t)-(b_{14}^{''})^{(1)}(G,t)\leq-(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)},(\nu_2)^{(1)},(u_1)^{(1)},(u_2)^{(1)},\nu^{(1)},u^{(1)}$ : 281

By $(\nu_1)^{(1)}>0,(\nu_2)^{(1)}<0$ and respectively $(u_1)^{(1)}>0,(u_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_1)^{(1)}\nu^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_1)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(\bar{\nu}_1)^{(1)},,(\bar{\nu}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)}>0,(\bar{\nu}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0,(\bar{u}_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_2)^{(1)}\nu^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)}$ by
$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$

$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)},if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$

and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$

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| $(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | |
| and analogously<br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$ are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})}[e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t}\right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a'_{13})^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b'_{13})^{(1)}$<br>$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |

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| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(T_{17}, t) + (a_{17}^{''})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities | 310 |

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|---|---|
| $G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^{0}e^{(S_1)^{(2)}t}$ | |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}}G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}}G_{16}^{0}e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t}-e^{-(S_2)^{(2)}t}\right]+G_{18}^{0}e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and | 320 |

By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

**Definition of** $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ by

$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,

and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$

$(p_i)^{(3)}$ is defined by equation

$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ 323

$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le\right.$

$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a_{22}')^{(3)}t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)}t}\right)$ 324

$\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ 325

$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ 326

$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b_{22}')^{(3)}t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)}t} \le T_{22}(t) \le$ 327

$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(\nu_1)^{(4)}, (\nu_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, \nu^{(4)}, u^{(4)}$ : 329

By $(\nu_1)^{(4)} > 0$, $(\nu_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_1)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(4)}, , (\bar{\nu}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{\nu}_1)^{(4)} > 0$, $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$
and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$

$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$, ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,

and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$

$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$

and analogously 331

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ : | 339 |

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|---|---|
| By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$, $(v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b'_{30})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b'_{30})^{(5)}t}\right]+T_{30}^{0}e^{-(b'_{30})^{(5)}t}\le T_{30}(t)\le$$

$$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a'_{28})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b'_{28})^{(5)}$

$(R_2)^{(5)}=(b'_{30})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a'_{32})^{(6)}+(a'_{33})^{(6)}-(a''_{32})^{(6)}(T_{33},t)+(a''_{33})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b'_{32})^{(6)}+(b'_{33})^{(6)}-(b''_{32})^{(6)}((G_{35}),t)-(b''_{33})^{(6)}((G_{35}),t)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}(u^{(6)})^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}(u^{(6)})^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b'_{34})^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b'_{34})^{(6)} t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$<br>$(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}\big((G_{39}), t\big) - (b''_{37})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0$, $(\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0$, $(u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0$, $(\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0$, $(\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

Then the solution of global equations satisfies the inequalities 364

$$G^0_{36} e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G^0_{36} e^{(S_1)^{(7)} t}$$

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| where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$ $\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$ $\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- <br> Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ <br> $(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$ <br> $(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$ <br> $(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define <br> **Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ : <br> $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying <br> $-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$ <br> $-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br> By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(v^{(8)})^2 + (\sigma_1)^{(8)} v^{(8)} - (a_{40})^{(8)} = 0$ | 372 |

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| and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(8)},,(\bar{v}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the<br>roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br>**Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |

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| | |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br><br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br><br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br><br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br><br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br><br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :<br><br>By $(\nu_1)^{(9)} > 0$, $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$, $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :<br><br>By $(\bar{\nu}_1)^{(9)} > 0$, $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$, $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(\nu^{(9)})^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$<br>**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-<br><br>If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by<br><br>$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$ | |

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , $\boldsymbol{if}$ $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, $\boldsymbol{if}$ $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , $\boldsymbol{if}$ $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}, \boldsymbol{if}$ $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{44}(t) \leq G_{44}^0 e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{45}(t) \leq \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)} t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})} \left[ e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)} t} \right] + G_{46}^0 e^{-(S_2)^{(9)} t} \leq G_{46}(t) \leq \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a'_{46})^{(9)})} [e^{(S_1)^{(9)} t} - e^{-(a'_{46})^{(9)} t}] + G_{46}^0 e^{-(a'_{46})^{(9)} t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)} t} - e^{-(b'_{46})^{(9)} t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)} t} \leq T_{46}(t) \leq$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)} + (r_{44})^{(9)} + (R_2)^{(9)})} \left[ e^{((R_1)^{(9)} + (r_{44})^{(9)})t} - e^{-(R_2)^{(9)} t} \right] + T_{46}^0 e^{-(R_2)^{(9)} t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

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$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

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$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain

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$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq v^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{v^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $v^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

| | |
|---|---|
| In the same manner , we get<br>$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | 391 |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br>**Definition of** $u^{(2)}(t)$ :-<br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |

**Proof** : From global equations we obtain 398

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \leq \frac{dv^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \geq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \leq v^{(3)}(t) \leq (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \leq v^{(3)}(t) \leq (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \leq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \leq v^{(3)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \leq (\bar{v}_1)^{(3)} \leq (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \leq v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

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From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)\le\frac{dv^{(5)}}{dt}\le-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(5)},(v_0)^{(5)}$ :-

For $0<\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}<(v_1)^{(5)}<(\bar{v}_1)^{(5)}$

$$v^{(5)}(t)\ge\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}{5+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}\quad,\quad\boxed{(C)^{(5)}=\frac{(v_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(v_2)^{(5)}}}$$

it follows $(v_0)^{(5)}\le v^{(5)}(t)\le(v_1)^{(5)}$

In the same manner , we get

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$$v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{5+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(5)}=\frac{(\bar{v}_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(\bar{v}_2)^{(5)}}}$$

From which we deduce $(v_0)^{(5)}\le v^{(5)}(t)\le(\bar{v}_5)^{(5)}$

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If $0<(v_1)^{(5)}<(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}<(\bar{v}_1)^{(5)}$ we find like in the previous case,

$$(v_1)^{(5)}\le\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}\le v^{(5)}(t)\le$$

$$\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(\bar{v}_1)^{(5)}$$

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If $0<(v_1)^{(5)}\le(\bar{v}_1)^{(5)}\le\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(v_1)^{(5)}\le v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(v_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(5)}(t)$ :-

$$(m_2)^{(5)}\le v^{(5)}(t)\le(m_1)^{(5)},\quad\boxed{v^{(5)}(t)=\frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)}\le u^{(5)}(t)\le(\mu_1)^{(5)},\quad\boxed{u^{(5)}(t)=\frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case, 414

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain 415

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\quad \boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

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**Proof** : From global equations we obtain 421

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \leq \frac{dv^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$v^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq v^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}')^{(1)}(r_{14})^{(1)} - (b_{14}')^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i'')^{(2)} and\ (b_i'')^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}')^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}')^{(2)}(r_{17})^{(2)} - (b_{17}')^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i'')^{(3)} and\ (b_i'')^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}')^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}')^{(3)}(r_{21})^{(3)} - (b_{21}')^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i'')^{(4)} and\ (b_i'')^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}')^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}')^{(4)}(r_{25})^{(4)} - (b_{25}')^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i'')^{(5)} and\ (b_i'')^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}')^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}')^{(5)}(b_{29}')^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ , | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$ | 436 A |

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| $(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |

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| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |

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| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if | 486 |

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| $F(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) + (a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) + (a''_{28})^{(5)}(T_{29})(a''_{29})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a'_{32})^{(6)}(a'_{33})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a'_{32})^{(6)}(a''_{33})^{(6)}(T_{33}) + (a'_{33})^{(6)}(a''_{32})^{(6)}(T_{33}) + (a''_{32})^{(6)}(T_{33})(a''_{33})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a'_{36})^{(7)}(a'_{37})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a'_{36})^{(7)}(a''_{37})^{(7)}(T_{37}) + (a'_{37})^{(7)}(a''_{36})^{(7)}(T_{37}) + (a''_{36})^{(7)}(T_{37})(a''_{37})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a'_{40})^{(8)}(a'_{41})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a'_{40})^{(8)}(a''_{41})^{(8)}(T_{41}) + (a'_{41})^{(8)}(a''_{40})^{(8)}(T_{41}) + (a''_{40})^{(8)}(T_{41})(a''_{41})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a'_{44})^{(9)}(a''_{45})^{(9)}(T_{45}) + (a'_{45})^{(9)}(a''_{44})^{(9)}(T_{45}) +$ | 492 A |

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| $(a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first | 499 |

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| equations<br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |

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| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |

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| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$ | 515 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)} (T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} ((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)} ((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)} (T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)} (T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)} ((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)} ((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)} (T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)} (T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)} ((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)} ((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and<br>$G_{36}^* = \frac{(a_{36})^{(7)} G_{37}^*}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^*)]}$ , $G_{38}^* = \frac{(a_{38})^{(7)} G_{37}^*}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^*)]}$<br>$T_{36}^* = \frac{(b_{36})^{(7)} T_{37}^*}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^*)]}$ , $T_{38}^* = \frac{(b_{38})^{(7)} T_{37}^*}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^*)]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^*$ given by $\varphi((G_{43})^*) = 0$ , $T_{41}^*$ given by $f(T_{41}^*) = 0$ and | 523 |

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| $G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 531 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)}$ , $\frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |

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| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b'_{20})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T^*_{20}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b'_{21})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T^*_{21}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b'_{22})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T^*_{22}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T^*_{25}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a'_{24})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G^*_{24}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a'_{25})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G^*_{25}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a'_{26})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G^*_{26}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b'_{24})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T^*_{24}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b'_{25})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T^*_{25}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b'_{26})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T^*_{26}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T^*_{29}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |

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| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 571 |

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \quad , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain from

$$\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37} \quad (573)$$

$$\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37} \quad (574)$$

$$\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37} \quad (575)$$

$$\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right) \quad (576)$$

$$\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right) \quad (578)$$

$$\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right) \quad (579)$$

Obviously, these values represent an equilibrium solution

**ASYMPTOTIC STABILITY ANALYSIS**

**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- (580)

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \quad , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41} \quad (581)$$

$$\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41} \quad (582)$$

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| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$<br><br>$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ | |

$$+\left(\left((\lambda)^{(1)}+(a_{14}^{'})^{(1)}+(p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^{*}+(a_{13})^{(1)}(q_{14})^{(1)}G_{14}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(13)}T_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$+\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}^{'})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}^{'})^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}^{'})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left(\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}^{'})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}^{'})^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}^{'})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

| | |
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| $+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$ $\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$ $\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$ $\left(\left((\lambda)^{(3)}\right)^{2}+\left((b'_{20})^{(3)}+(b'_{21})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$ $+\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$ $+\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$ $\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\}=0$ $+$ | |
| $(\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\}\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$ $\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$ $+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$ $\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$ $\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$ $+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$ $+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$ $+$ | |
| $(\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$ $\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$ $\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$ | |

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b'_{34})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a'_{34})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(33)}T^{*}_{33}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^{*}_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^{*}_{33}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(32)}T^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(34)}T^{*}_{32}\right)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(37)}T^{*}_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^{*}_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^{*}_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(36)}T^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(38)}T^{*}_{36}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\Big(\left((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\Big(\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\Big(\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future**. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_{i,}$ $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i Xi(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \qquad (1)$$

Where $x_i$ is the abundance of population $i$, $x= (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 1. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K $\subset$ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta_0 > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B*.

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I}\boldsymbol{C}_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $\mathrm{A}_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A}_i(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A}_i(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N\,\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots,\tilde{A}_{k})$ by

$$\tilde{A}_i(x) = \{\frac{A(x)}{A(x)-\frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1,\ldots,k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x$,$W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\mathrm{diag}(f^i_2(E_1^j))^m_{j=1}$. Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E_1^j) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.1) $$P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.2) $$D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.3) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, \text{'} & (1.4) \\ \rho_1(0, t) = \int_0^{\infty} b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^{\infty} b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s), a.0, s\epsilon [-\tau, 0], \quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^{\infty} \varphi_i(a, s)\, da, s\in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1, \varphi_2, f_1, f_2, b_1, b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i, b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 1. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +}$ [$-\tau, T] \rightarrow R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +}$ [$\{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +}$ [$-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t,0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right)ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t)= B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$

$$+\int_0^\infty b_i\left(a+t,P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i\,(t)\right)da$$

$$i = 1,\ 2,\ t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*\,(t-a)] \ \le\ 1, if\ \ t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^*\,\beta_i(a)\exp[-d_i^*(t-a)]\,da + \beta_i^*\Phi_i(0)\exp[-d_i^* t]\,,\ \ i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^*\Phi_i(0)\exp[\delta_i t],\ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i\,(t) - 0_i\,(0)| \le \beta(0)\,exp[\delta_i\,a - d_i^*(t-a)]\,da$$

$$+\int_0^\infty \varphi_i\ (a,0)|1-exp(-d_i^* t)|da$$

$$\le \beta_i^*\Phi_i\ (0)\,exp(\beta_i^* t\,)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0)d_i^* t$$

$$\le [\beta_i^*\phi_i(0)\,exp(\beta_i^* t)\,\phi_i(0)d_i^*]t\ \ for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then ▯ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , ▯ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|▯(P_t^{(1)}) - ▯(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|▯(P^{(1)}(t)) - ▯(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta}\, 1^{a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\,P_2^*(a)}{da} + f_i(\mathrm{a}, P_1^*, P_2^*)\ P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty\ P_i^*(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty\ b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ \ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1 = \int_o^\infty\ b_1(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_1(s, P_1^*, P_2^*)ds\right]da,$$

$$(3.5)\ 1 = \int_o^\infty\ b_2(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_2(s, P_1^*, P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

$$(3.6)\ F_{i.}\ (P_1, P_2) = \int_o^\infty\ b_i(a, P_1, P_2)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1, P_2)ds\right]da,$$

$$i = 1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

$$(3.7)\ A_{4.}\quad F_{i.}\ (0,\ 0) > 1,\ \ i = 1, 2:$$

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

$$(3.8)\qquad F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

$$(3,9)\ F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

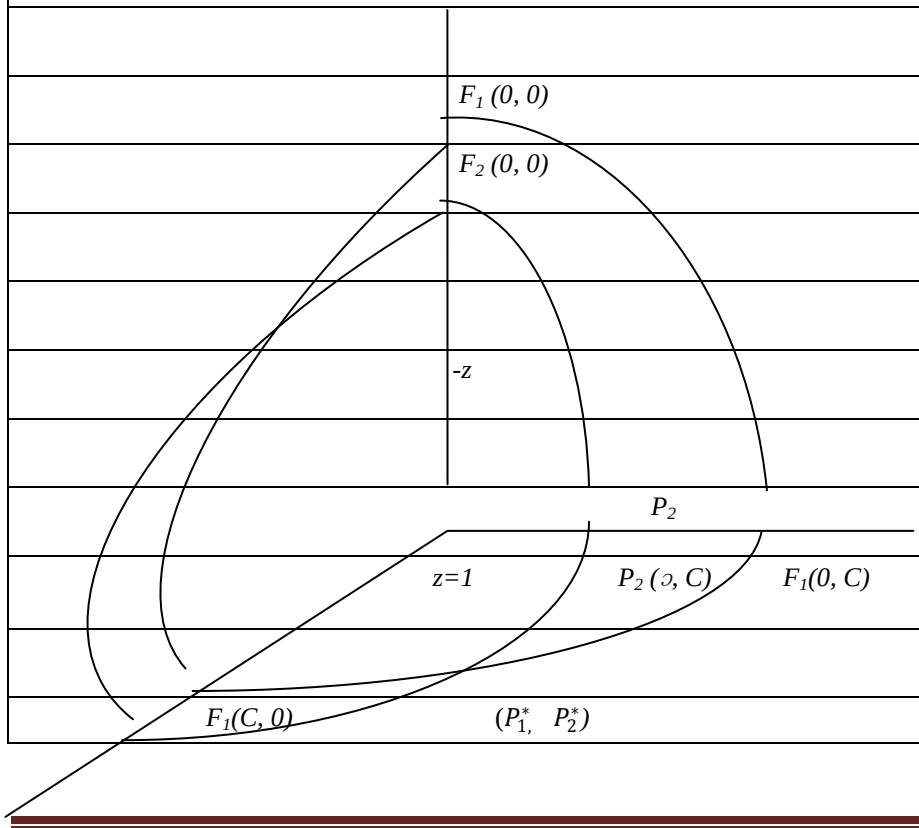
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11)\quad \begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

$$(3.12)\quad \begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)

$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

$$(3.14)\quad \begin{cases} u_i(a,t) = u_i(a)\exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{\varphi}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\varphi(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\varphi(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

From the first of (3.15) we get

(3.18) $$\varphi(a) = \left\{\varphi(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\varphi(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 1. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \left\{ \text{an individual's contribution to population growth} \right\} N(t) \qquad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right)$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right)\right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\left\{\frac{dN(t)}{dt} = N(t) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds\right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(i) $b_1(0) = 0,\quad m_1(0, x_2) \equiv 0, b_2(0) = 0,\quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(ii) There exist positive constants $\delta_1$, $\delta_2$ such that

$$b_1(\delta_1) - m_1(\delta_1, x_2) < 0,$$

$b_2\ (\delta_2) - m_2(x_1, \delta_2) < 0,$ for $x_1$ 0, $x_2$ 0, (2.5)

(iii) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i, \; j = 1, 2; i \neq j \;)$.

## 2. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij} \; (i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij} \; , i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1 \; (i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\,(t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\,(t_2)\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\,(t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\,(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\,(t - \tau_{11}) - m_{12}Y_2\,(t - \tau_{12}),$$

$$\frac{dY_2\,(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\,(t - \tau_{21}) - m_{22}Y_2\,(t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \qquad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(i)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \qquad g_2(0) = 0, \tag{4.3}$$

(ii) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(iii) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(iv) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{vmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{vmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 5. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t $\geq$ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jj} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{bmatrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{bmatrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{a\tau} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}), \end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{11}\left|Y_1(t)\right| + b_{11}\left|Y_1(t-\tau_{11})\right| + m_{12}\left|Y_2(t-\tau_{12})\right|, \\ \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{22}\left|Y_2(t)\right| + m_{21}\left|Y_1(t-\tau_{21})\right| + b_{22}\left|Y_2(t-\tau_{21})\right|, \end{aligned}\right\} \quad (A.2)$$

Where

$$\frac{d}{dt}\left|Y_1(t)\right| = \left\{\begin{aligned} &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0, \\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0, \end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(s) > \left|Y_1(s)\right|, \quad s \in [-\tau, 0], \; i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(t) \text{ for } t \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(t) = Z_i(t) - \left|Y_1(s)\right|$, $i = 1,2$. $t \ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\ge -m_{11}U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\ge -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(s) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \ge U_1(0) \exp[-m_{11}t]$

$$+\int_0^t \left[b_{11}U_1(s-\tau_{11}) + m_{12}U_2(s-\tau_{12})\right] \exp[-m_{11}(t-s)ds > 0,] \qquad (A.9)$$

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22},)$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds+\int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds+\int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0\leq \mid Y_{1,}(t) \mid \leq Z_1(t)\to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}+\frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\to 0, Z_2(t)\to 0$ as $t\to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty sk_{ij}(s)ds< \infty;\ \int_0^\infty k_{ij}(s)ds=1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon)$, $t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 1. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ $and$ $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 2. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0, \qquad \textbf{(2.1)}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| < b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G} =$ closure of $G_2$ in $B$ and

$\frac{\bar{G}_2}{G_1} =$ complement of $G_1$ in $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ Ω and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ Ω then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ ∈ $-\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega + t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(i) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(ii) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(iii) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(a)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(b) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 3. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(a) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(b) N is monotonic nondecreasing on $K^*$.

(c) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cup K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \|x_1\| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \|x_2\| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique ω - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ –}\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ –}\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ –}\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x^*_1(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21})\, x^*_1(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} -\ _{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha,\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter T:

$$\frac{dx_1(t)}{dt} x_1(t)[1 - x_1(t) + 0.4x_2(t - \mathcal{T})/k_1(t)]$$

$$\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$$

Where $k_1(t) = 20 + \sin\ (\pi t)$ and $k_2(t)$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 1. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \ldots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots., n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t\in R} b_l(t) = b_i^l; \qquad \sup_{t\in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t\in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t\in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l\{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.1) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \left( b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left( b_j^u / a_{jj}^l \right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \geq x_i^l \quad for\ t \geq t_0,\ i = 1, 2, \ldots, n. \qquad (1.7)$$

2. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \leq x_l \leq x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1 \leq i \leq n} x_l^l; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(1) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(2) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(3) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(4) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(5) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(6) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(7) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(8) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(9) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(10) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(11) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(12) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(13) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (14) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (15) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (16) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (17) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (18) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (19) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (20) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (21) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (22) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception:

"what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left|\left|f\ (t+\ \tau,x) - \ f(t,x)\right|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{ij}\ (\sigma+\tau_k) - a_{ij}\,(\sigma+\tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k), x(t+\tau_m)\big) \le v\big(x(t_0), x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$$\le [(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] \ + (\varepsilon/2x^*)$$

$$< (\varepsilon/x^*) \text{ On using (2.8), (2.10)} \qquad (2.14)$$

For $m \ge k \ge n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \le x_i(t) \le x^*, i = 1,2,\dots,n; t \ge t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \ge k \ge n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*, x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*, x^*], i = 1,2,\dots,n;$ for $t \ge t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ & $k$ > $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ & $k$ > $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\,(t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad \beta_1 \leq y_i(t) \leq \beta_2, \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \ \rho_1(t, a_1) \ \rho_1^*(t, a_1),$$

$$0 \ Q_1(t) \ Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0)$$

$$= \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1\,(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1\,(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})\tau_1}(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1 \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ħ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** **d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu} [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \quad P_3 \text{ 1}, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \text{ ?}\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \text{ ?}\infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_3 a_3-(f_{30})_3 a_3-\int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3,a_3)da_3\right)u_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0, \qquad (24)$$

$$\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^\infty Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^a(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^l)_3\int_0^\infty Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3))\int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\quad da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{,3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2[a_3^2]_3<4[a_2]_3<4a_2e$,

$\mathrm{Re}\,\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2]_3 < 0.$

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then$|n_3(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2<4a_2e$ and $[a_3] > 2\Delta$,or $[a_3^2]$ ⊠4$[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3}da = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3(\theta(t)),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

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Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. 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Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell^{-1}\log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 1. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \; 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \; \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 2. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^\infty Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^\infty Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a (\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^\tau \mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a \mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4[a_2]_4<1\, a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])^3_4\ [\mathrm{K}]_4[P_0]^3_4} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}^4_{2*}] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}^4_*] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}^4_*] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae_4^{mu} \; [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 3. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \; 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \; \infty. \tag{1}$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \; \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt}e^{-ra}\,da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\,da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6}\rho_6(t,a_6)\,da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 4. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0) = \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(1) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(2) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(3) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(4) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(5) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical**

**language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(6) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(7) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(8) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(9) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(10) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

**Physical applications**

**Some phenomena where quantum gravity plays a major role are the following.**

**The microscopic structure of spacetime**

(11) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(12) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(13) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(14) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

**Early cosmology**

(15) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(16) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(17) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(18) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(19) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

**Black holes**

(20) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(21) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(22) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(23) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(24) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

**Astrophysical effects**

(25) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(26) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(27) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

**Main approaches**

(28) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out**

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| **certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (29) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (30) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

**String theory**

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| (31) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (32) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (33) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (34) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (35) **However, when summed the entire series appears to be (=) divergent.** |
| (36) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (37) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (38) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (39) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (40) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (41) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (42) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (43) **The granular properties of space can be (=) explicitly computed.** |

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| (44) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (45) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (46) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (47) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (48) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (49) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (50) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (51) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (52) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (53) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (54) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (55) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (56) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (57) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (58) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (59) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection** |

**among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.**

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(60) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.**

(61) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.**

(62) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.**

(63) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.**

(64) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI**

(65) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].**

(66) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.**

(67) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.**

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(68) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.**

(69) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.**

(70) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.**

(71) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.**

(72) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(73) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

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| (74) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (75) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (76) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (77) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (78) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (79) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (80) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (81) **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (82) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (83) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (84) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (85) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (86) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (87) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (88) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system** |

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| **at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (89) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (90) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (91) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (92) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (93) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (94) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (95) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (96) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (97) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (98) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (99) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (100) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (101)**It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (102) **This is a realistic scenario for (e) all known systems.** |
| (103)**On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (104)**On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (105)**No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |

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| (106) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (107)**Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (108)**Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (109)**Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (110)**So in those systems, the quantum phase transition does indeed exist.** |
| (111)**But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (112)**Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (113)**The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (114)**Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (115)**They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (116)**Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (117)**These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (118)**Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (119)**Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (120)**They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (121) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (122)**Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |

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| (123)**They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (124)**These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (125) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (126)**A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (127)**At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (128)**The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (129)**The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (130) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (131)**The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (132)**The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (133)**"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (134) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (135) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (136)**There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (137)**Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (138)**Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (139) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (140)**Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (141)**They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (142) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (143)**Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (144)**These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (145)**K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (146)**The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (147)**(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (148)**According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (149)**Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (150)**These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (151)**Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (152) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (153)**That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |

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| (154)**Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (155)**The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (156)**The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (157)**"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (158)**Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (159)**The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (160)**However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (161)**The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (162)**"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (163)**If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (164)**Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (165) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (166)**In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (167)**. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (168)**More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (169)**Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (170)**Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (171) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (172)**Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |

(173)**The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.**

(174)**Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.**

(175) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.**

(176)**They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios.**
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**A new look at the galaxy-shaping power of black holes July 6, 2016**

(177)**It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.**

(178)**Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.**

(179)**Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.**

(180)**But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.**

(181)**New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.**

(182)**"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."**

**Champagne bubbles of plasma**

(183)**During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.**

(184) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.**

(185) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.**

(186)**Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun**

**New look at the galaxy-shaping power of black holes**

(187)**Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.**

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| (188)**"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (189) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (190)**This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (191)**The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (192)**The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (193)**The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (194)**Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (195)**"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (196)**"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (197)**That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (198)**Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (199)**In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (200)**Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (201)**"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (202)**"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (203) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

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| **A new look at the galaxy-shaping power of black holes** |

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| (204)**An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (205)**McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (206)**Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

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| **Doomed satellite missions** |

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| (207)**Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (208)**At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (209)**Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (210)**The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (211)**These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (212)**"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (213)**. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (214)**The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |
| (215) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website** |

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

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| (216)**Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.** |
| (217)**They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.** |
| (218)**However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.** |
| (219) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.** |
| (220)**X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.** |

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| (221)**The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.** |
| (222)**These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.** |
| (223)**Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.** |
| (224)**A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.** |
| (225)**Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.** |

**Dark matter guides growth of supermassive black holes February 18, 2015**

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| (226)**This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.** |
| (227)**Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.** |
| (228) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.** |
| (229)**A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.** |
| (230) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (231)**"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (232)**This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (233)**However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (234)**In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (235) **We know dark matter exists only from (e) its gravitational effects.** |
| (236) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (237) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (238)**To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (239) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (240) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |

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| (241)**They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (242)**This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (243) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (244)**Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (245)**"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (246)**The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (247)**Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (248)**Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (249)**"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (250)**Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (251)**"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (252)**Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb)** |

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| **consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (253)**The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (254) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (255)**Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (256)**These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (257)**Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (258)**The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (259)**Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (260) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (261)**"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (262)**"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (263)**The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (264)**Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (265)**"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (266) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (267)**Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (268) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (269)**Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (270)**"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (271)**The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (272)**However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |
| (273)**The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (274) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (275) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (276)**Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (277)**The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (278)**Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (279)**"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (280)**The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |

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| (281)**The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (282)**(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (283) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (284)**This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (285)**According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (286)**This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (287) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (288)**Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (289)**In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (290) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (291)**Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (292)**They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (293) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (294)**By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (295) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |

(296)**The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.**

(297)**As a result, it provides (eb) an important clue for finding the global minimum.**

(298)**It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.**

(299)**This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.**

**Thompson's problem 2**

(300)**By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society**

(301)**The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.**

(302) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.**

**Implications for Thomson's and Smale's problems**

(303)**Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.**

(304)**"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."**

(305)**As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.**

(306)**"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."**

(307)**In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.**

(308) **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.**

(309)**"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.**

(310)**"For larger numbers, there will likely be (=) multiple funnels.**

(311)**Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."**

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| (312) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (313) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (314) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(1)** | **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(2)** | **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(3)** | **Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(4)** | **Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(5)** | **Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(6)** | **Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(7)** | **No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(8)** | **This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(9)** | **Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(10)** | **In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |

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| (11) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors. |

**Classical vs. relativistic black holes**

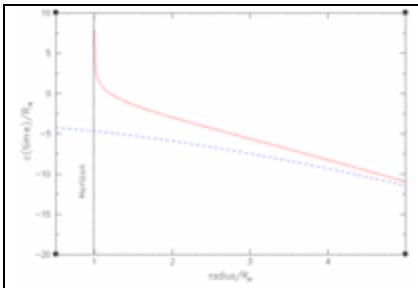
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| (12) Something like a black hole exists within (eb) Newton's classical theory of gravity. |
| (13) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R |
| (14) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity. |
| (15) Thus the condition for such an "unseeable" object is R<2GM/c2 |
| (16) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape. |
| (17) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity. |
| (18) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it. |
| (19) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c . |
| (20) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed. |
| (21) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity. |
| (22) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable. |

**Stationary black holes**

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| (23) In the Newtonian theory, gravity is described by (e) the potential Φ. |
| (24) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass. |
| (25) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object. |
| (26) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward. |
| (27) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory. |
| (28) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution. |
| (29) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H. |
| (30) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution. |
| (31) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity. |
| (32) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own. |

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| (33) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole. |
| (34) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density. |
| (35) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed. |
| (36) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole. |
| (37) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter. |
| (38) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum. |
| (39) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior. |
| (40) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object. |
| (41) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum. |
| (42) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.] |
| (43) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time. |
| (44) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?" |
| (45) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point. |
| (46) One of these coordinates is (=) called the "coordinate time." |
| (47) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate. |



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| (48) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole |
| (49) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole. |
| (50) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations. |
| (51) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree |
| (52) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events. |
| (53) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving. |
| (54) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle. |
| (55) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return. |
| (56) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon. |
| (57) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future. |
| (58) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?" |

**Black hole parameters**

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| (59) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin). |
| (60) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun. |
| (61) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits. |
| (62) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form. |
| (63) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds. |
| (64) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2. |
| (65) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed. |
| (66) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer. |
| (67) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2) |

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| **(68) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(69) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(70) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(71) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(72) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(73) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(74) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(75) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(76) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(77) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(78) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(79) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√ )2+a2].** |
| **(80) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(81) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(82) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(83) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(84) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(85) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(86) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |

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| **(87) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(88) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(89) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(90) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(91) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(92) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(93) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(94) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(95) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(96) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\,(\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1-\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(a_6)da_6-\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^\infty E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^\infty \beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^a \alpha_6\,(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^\infty E_6(a_6)da_6.$$

Since $f_6^{'}(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[\dot{f_{6*}}\right]_6$ denoting $\left[\dot{f}_6(P^*)\right]_6\left[\dot{f_{2*}}\right]_6$ denoting $[f_2]_6$ $(Q_6^*$, $h_6^*$ denoting $\dot{h}_*$ denoting $\dot{h_6}$ $(Q_6^*)$, and $\left[\dot{g}_*\right]_6$

Denoting $\left[\dot{g}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[\dot{f}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[\dot{f_{2*}}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[\dot{f}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[\dot{f_{6*}}\right]_6 P_6^*\right)M_6 - \left[\dot{f_{2*}}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6$<4 $[a_2]_6$<4 $a_2$e,

Re $\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0$.

Let ⊿=max$_{2<u<\infty}\left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then$|n_6(t)|$ ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_6^2$ <4 $a_2$ e and $[a_6] > 2\Delta$,or $[a_6^2]$ [$a_2$]e and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\, P^*\, [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\, [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\, g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^2} + \frac{2\omega_6}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}] \, P_6^*$$

$$[a_2]_6 = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [g_*^6]_6 \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_6 \beta_6 \left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right] M_6 - [a_2]_6 W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 5. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^{\infty} \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \; P_7 \; 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^{\infty} n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(315)(Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(316)In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(317)As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(318) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(319)But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(320)For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(321) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(322) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(323)It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(324)Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(325)The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(326)The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(327)Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(328)This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(329)Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(330)Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(331)The electronic and (e&eb) structural variability among them is extraordinary.

(332)The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(333)The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(334) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(335)Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(336)Nudge your bath tub just right and (eb) you might end up reversing time.

(337)Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(338)To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(339)Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(340)But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(341)The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(342)Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(343)Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(344)To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(345)The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(346)Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(347)"The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(348)Wave control is usually performed by (e) spatially engineering the properties of a medium.

(349) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(350) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(351)Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(352)They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(353)Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(354)The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(355)Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(356)Quantum mechanics rules the dynamics of (e) light and matter.

(357) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(358) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(359)Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(360)The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit.

(361)It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations.

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| (362)Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (363)The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (364)All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (365)When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (366) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (367)To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (368). Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (369)As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (370)Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (371) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (372)Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (373)However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (374) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (375)So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (376)Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (377)The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (378) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (379)The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (380) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (381) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (382)Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |

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| (383)This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (384)First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (385) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (386)Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (387)A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (388)Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (389) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (390)The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (391) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (392) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (393) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (394)Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (395)Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (396)This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (397) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (398)In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (399)Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (400) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (401) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (402) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (403)This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |

(404) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ 0 $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt}$$

$$= -b_7Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \quad \rho_7(t, a_7) \quad \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7\, Q_7^* + g_7(P_7^*) Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \quad (10)$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \qquad (H1)$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7 Q_7 - c_7 P_7 Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 6. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7(P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau}\, e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7(P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7 N_7 + o_7(N_7), \quad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \quad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7(0) \quad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7 Q_7$$





The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{\mathrm{u}}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_7)$ gives,

$$\tilde{\mathrm{u}}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{\mathrm{u}}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a(\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds+y\tau}\alpha_7(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^a(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty} Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty} Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)e^{\gamma_7\tau_7+(f_{70})_7\tau+\int_0^{\tau}\mu_7(s_7)ds_7}d\tau$$

$$=\frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7a_7}-1)$$

Let

$$E_7(a_7)=e^{-(f_{70})_7a_7-\int_0^{a}\mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)da_7 = 1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)\ \int_0^a \alpha_7\ (\tau_7)Y^{-7}(\gamma_7, \tau_7)d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7(\rho_0)_7 \int_0^{\infty} E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^{\infty} \beta_7(a_7)p_7^*(a_J)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^{\infty} p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7']_7(z_7))\int_0^\infty a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}da_{7}$$

$-h'(z_7)\int_0^{\infty}\beta_7(a_7)\,e^{\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7-[f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^-(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\,e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^\infty m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - [p^*]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M,_7$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}']_7$ denoting $[f'_7(P^*)]_7 [f_{7*}']_7$ denoting $[f_7]_7$ $(Q_7^*$, $h_{*7}$ denoting $h_*'$ denoting $h_7'$ $(Q_7^*)$, and $[g_*']_7$

Denoting $[g']_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7 [p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7) m_7(t, a_7) da_7 + \frac{[h_*^7]_7}{[h^*]_7} [p^*]_7(0)\, W,_7 \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7 [Q_7^*] M_7$$

Integrating $[m_7]_7(t, a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t, a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t, a_7)da_7 - \int_0^\infty \mu(a_7)m(t, a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$= m_7(t, 0) - \int_0^\infty \mu_7(a_7)\, m_7(t, a_7)da_7$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [\dot{f}_{7*}]_7 P_7^*\right)M_7 - [\dot{f}_{7*}]_7 P_7^* W_{\cdot 7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f'_{7*}]_7 P_7^*\right)M_7 + \left(\frac{[h'_*]_7}{[h_*]^7}\, P_7^*(0) - f'_{7*}P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}[g'_*]_7\, Q_7^*\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{\cdot 7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $\frac{7}{7}$ [a $\frac{7}{7}$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_{\,7}\left[\lambda_{\,7}\right]_{\,7} = \mathrm{Re}\lambda_{\,7}\left[\lambda_{\,7}\right]_{\,7} = -\frac{7}{7}a_7$[a2 ] $_7 < 0$.

Let ⊿=max$_{7<u<\infty}\left|\mathrm{h}_{\,*}\,[\mathrm{h}_{\,*}]_{\,7}\beta_{\,7}(a_7) - \mu_7(a_7)\right|$. Then|$n_7$(t)| ⊿∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $\frac{7}{7}$ <4 a $_7$ e and [a $_7$] $> 1⊿$,or [a $\frac{7}{7}$] ⑭[a $_7$]e and (a $_7 - \sqrt{(\mathrm{a}\,\frac{7}{7})_{\,7} - 4(\mathrm{a}_{\,7})_{\,7}\,\mathrm{e}} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\mathrm{a}}{\mathrm{a}+7}\right)$ and $\beta_{\,7}(a_7) = \mathrm{ae}^{\text{-ma}}$ Then ⊿=[$\mu_0$] + 2 Simple calculation leads to

$$\int_0^{\infty} \beta_{\,7}(a_7)\mathrm{e}^{-\int_0^{\mathrm{a}}\mu_7(\tau_{\,7}) + d\tau_7}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^{\,7}} + \frac{1}{(\mathrm{m}+\mu_0+1)^{\,3}}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7$ + [$\mu_0$] ⓪.5. Assume[$f_7$]($x_7$)= $K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)$= [0$x_7$]/(2 + $x_7$); the functional responses [[$f_7$]($x$) = [0] $x_7$/(1 + $x_7$)and $h_7(x_7)$=(1+$\omega_7x_7$)/(1 + $x_7$)(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])\,\frac{3}{7}\;[\mathrm{K}]_{\,7}[P_0]\,\frac{3}{7}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_7$ i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^*\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\ 7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7} [\ \mathrm{g}_*^{\ 7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\ 7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\ 7}} + \frac{1\omega_7}{([m_7]_{\ 7} + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\ 3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\ 7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_{\ 7} + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\ \mathrm{Q}_7^*)^{\ 3}$

$= m_7 + [\mu_0]_{\ 7} + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + \mathrm{Q}^*)(1 + \omega_7 \mathrm{Q}_7^*$

Now

$$\left[\mathrm{a}\begin{matrix}1\\1\end{matrix}\right]-4\left[\mathrm{a}_{7}\right]\mathrm{e}=\left(\frac{1K_7b_7}{\delta_7-b_7}\right)^{7}+\frac{4K_7b_7\sigma_7\,\mathrm{Q}_{7}^{*}}{(\delta_7-b_7)(7+\mathrm{Q}_{7}^{*})}+\frac{\sigma_{7}^{7}\mathrm{Q}_{7}^{*7}}{\left(1+\mathrm{Q}_{7}^{*}\right)^{7}}$$

$$\frac{4\,b_7\sigma_7\,(\delta_{7}-\mathrm{b})\mathrm{Q}^{*}}{\delta_{7}(1+\mathrm{Q}_{7}^{*})^{7}}-\frac{4\,(\delta_{7}-b_7)^{7}(1-\omega_7)\mathrm{Q}_{7}^{*}}{\delta_{7}\left(1+\omega_7\mathrm{Q}_{7}^{*}\right)(1+\mathrm{Q}_{7}^{*})}\mathrm{P}^{*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{7}\,ae_{7}^{mu}\;\;[\mu_0]_{7}-\frac{a_7}{a_7+7}$$

Hence if $[\mathrm{a}_{7}^{7}>4(a_7)_{7}\mathrm{e}$ and

$$[a_7]=\frac{1K_7b_7}{\delta_7-b_7}+\frac{\sigma_7\,\mathrm{Q}_{7}^{*}}{1+\mathrm{Q}_{7}^{*}}>1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(111)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(112) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(113)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(114) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(115)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(116) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

(117)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States.

(118)Humans are not very good at understanding interacting many-body systems.

(119)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them.

(120)But what happens when we couple (e&eb) the particles strongly?

(121)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments.

(122)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water.

(123)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h.

(124) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1]

(125) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic

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| origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (126)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (127) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (128)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (129) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (130) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (131)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (132)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (133) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (134) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (135)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (136)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (137)When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (138)The energy levels of these quantized orbitals take on (e) discrete values |
| (139):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (140) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (141) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (142)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |

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| (143) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (144)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (145) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (146)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (147)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (148)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (149) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (150)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (151)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (152)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (153) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an "elementary particle." |
| (154)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (155)The remarkable thing is that solitons can become (=) "elementary," in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (156)An example of a duality, which we will use below, arises in (eb) Maxwell's theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (157)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (158)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (159)Another classic example of duality involves (e&eb) vortices. |
| (160) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (161)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (162) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (163)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (164)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |

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| (165) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (166)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (167)TIs are distinguished only by (e) their edge states. |
| (168) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (169)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (170) But the Dirac cone is not (e) the only possible signature. |
| (171) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (172)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (173)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (174)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (175) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (176). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (177)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (178)The bulk duality implies (eb) a duality between edge theories. |
| (179)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (180)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (181) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (182) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (183)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |

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| (184)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (185)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (186)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (187)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (188)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (189) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (190)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (191)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (192) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (193)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (194)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

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| IF we define |
| $$[a_7]_{\,7} = [f_{7*}]_{\,7} + [f_{7*}]_{\,7} + [f_{7*}]\,P^{*}_{7}$$ |
| $$[a_7]_{\,7} = \left[f^{\,7}_{7\,*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$ |
| $$e=[\,g^{7}_{*}]_{\,7}\,Q^{*}_{7}$$ |
| $$\eta_7(t) = \left[h_{*}\right]_{\,7}\beta_{\,7}\left(\theta_{\,7}(t)\right) - \mu_{\,7}\left(\theta(t)\right),$$ |

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{7}\right]M_7 - [a_7]_{7}W_7 - [\mathrm{n(t)}]M_7$$

$$w_7(\mathrm{t}) = \mathrm{e}M_7$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_7]_{7} & -[a_7]_{7} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{7} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{7}(0) + \int_0 \mathrm{e}^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality

function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 7. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(405)Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(406)Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(407) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(408)"Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(409)Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

(410) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems.

(411) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe.

(412)At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short.

(413) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another).

(414)And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited.

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| (415)Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (416)One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (417) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (418)"The underlying physics is (=) chaos, essentially," he said. |
| (419)The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (420)But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (421) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (422)To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (423) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (424) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (425)By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (426)The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (427)The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (428)"There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |
| (429) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (430)It turns out that they are actually the driving forces behind (eb) thermalization. |
| (431)"What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (432)The study's findings have fundamental implications for (e) quantum computing. |
| (433) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (434)"It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , |

http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

(435)A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box.

(436) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction.

(437)Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed.

(438)The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'-the basic component of a quantum computer.

(439)Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box.

(440)By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance.

(441)The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction.

(442)Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device.

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(443)Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(444)As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(445) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(446) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(447)Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(448)A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(449)For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(450) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(451)The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(452)Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(453)H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(454)This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(455)Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(456)At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(457)Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(458)The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(459) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(460)However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(461) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(462)In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(463)Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(464) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(465) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(466)When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(467) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(468)Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(469)Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(470)The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(471)Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(472)Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

**Physical applications**

Some phenomena where quantum gravity plays a major role are the following.

**The microscopic structure of spacetime**

(473)If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(474) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(475) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(476) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

**Early cosmology**

(477)According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(478) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(479) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(480) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(481)One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(482)Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(483) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(484) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(485)What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(486) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(487)Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(488) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(489)The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(490)The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(491) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(492)The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(493)String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(494) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(495)String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(496)Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(497)However, when summed the entire series appears to be (=) divergent.

(498)In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(499)Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(500)Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(501) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(502)Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(503)The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(504)Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(505)The granular properties of space can be (=) explicitly computed.

(506) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(507)The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(508) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(509) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(510)Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(511)This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(512)Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(513) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(514)It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(515) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(516)Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(517)This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(518) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(519). Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(520) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(521) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \infty,$$

$$\rho_8(t,0) = \int_0^\infty n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \qquad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \qquad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8 P_8) + (f_8)_8(Q_8))(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8) \int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t, 0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t, a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t, a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \tag{10}$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \tag{$H1$}$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8}\rho_8(t,a_8)\,da_8$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + c_8P_8Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8P_8 - \lambda_8P_8Q_8$$

$$\frac{dQ_8}{dt} = b_8Q_8 - c_8P_8Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 8. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t, 0) = \int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t, a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8\,(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8\,(0) \qquad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0'Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8\,(a_8)u_8(t,a_8)da_8 + (h'_{a_8})(\rho_0)_8 Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8 u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8 Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8 a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t,a_8)da_8$$

$$u_8(t,0) = \int_0^\infty \beta_8\,(a_8)u_8(t,a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t,a_8) = \tilde{\mathrm{u}}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^{\infty} \tilde{u}(a) e^{-\gamma a_8}\, da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^{\tau}\mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^{\tau}\mu_8(s_8)ds}d\tau_8\, da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\, da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1 - \int_0^{\infty} e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8 + \int_0^{\tau}\mu_8(s_8)ds + y\tau}\alpha_8(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1 - \int_0^{\infty}\beta_8\,(a_8)Y_8(\gamma_8, a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^{a}(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8=0,$$

(24)

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^{a}(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^{a}(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$= \frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8 a_8} - 1)$$

Let

$$E_8(a_8) = e^{-(f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8) d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8, a_8) = E_8(a_8)e^{-\gamma_8 a_8}$. by a simple calculation,

$$\int_0^\infty \beta_8(a_8)Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma}\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}\ da\right);$$

$$\int_0^\infty Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\ (\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1 + \frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(\ a_8)da_8 - \int_0^\infty E_8(\ a_8)e^{-\gamma_8 a_8}da_8\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1 + \frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(\ a_8)da_8) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^{a}\alpha_8\,(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0.$ Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^{\infty}E_8(\,a_8)da_8.$$

Since $f_8^{'}(P_0)_8 > 0,$ we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8),0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(\,p^*(a),Q^*\,)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(\,P_8^*) + [f_8]_8(Q_8^*)\,)P_8^* = 0, \qquad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^{\infty}\ \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \qquad (27)$$

$$g_8(P_8^*) - b_8 = 0, \qquad (28)$$

Where $P_8^*=\int_0^\infty p_8^*(a_8)\,da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}da_{,8}$$

And assume that

$$[k_3]_8\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\,da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)\left( [f_8^{'}]_8(z_8) \right) \int_0^{\infty} a_8 \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_8$$

$$-h^{'}(z_8) \int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8} \quad da_8$$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y\frac{a}{8}}} da_8$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\left(f_8(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t,a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - [p^*]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}^{'}]_8$ denoting $[f^{'}{}_8(P^*)]_8[f_{8*}^{'}]_8$ denoting $[f_8]_8$ $(Q_8^*,\quad h_{8}^*$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $[g_*^{'}]_8$

Denoting $[g^{'}]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - [f^{'}{}_{8*}]_8[p_8^*](a_8)M_8 - [f_{8*}^{'}]_8 p_8^*(a_8)W_8$

$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\,W_{,8}$ (30)

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^{\infty} [m_a]_8\,(t, a_8)da_8 - \int_0^{\infty}\ \mu(a_8)m(t, a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [f'_{8*}]_8 \int_0^{\infty} p_8^*(a_8)da_8\right) M_8 - [f_{8*}^8]_8 \int_0^{\infty} p_8^*(a_8)da_8 W_8$$

$= m_8(t, 0) - \int_0^{\infty} \mu_8(a_8)\, m_8(t, a_8)da_8$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [f_{8*}^{'}]_8 P_8^*\right) M_8 - [f_{8*}^{'}]_8 P_8^* W_{.8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^{\infty} ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t, a_8)da_8$$

$$-\left([f_{8*}] + [f_{8*}]_8 + [f_{8*}^{'}]_8 P_8^*\right) M_8 + \left(\frac{[h_*^{'}]_8}{[h_*]^8}\ P_8^*(0) - f_{8*}^{'} P_8^*\right) W_8,$$

(31)

$$\frac{dW_8(t)}{dt} [g'_*]_8\, Q_8^*\, M_{.8}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_8\beta_8(a_8) - \mu_8(a_8))m_8(t, a_8)da_8 = ([h_*]_8\beta_8(\theta_8) - \mu_8(\theta_8))M_{,8}$$

Where $0< \theta_8 = \theta_8(t) < \infty$.

And if $a_8^8\,[a_8^8]2<4\,[a_{\ 8}]2<4\,a_{\ 8}e$,

Re $\lambda_8[\lambda_8]_8 = \text{Re}\lambda_8[\lambda_8]_8 = -\frac{8}{8}a_8[\text{a2}\ ]_8 < 0.$

Let $\Delta=\max_{8<u<\infty}\left|h_*[h_*]_8\beta_8(a_8) - \mu_8(a_8)\right|$. Then$|n_8(t)|$ 2Δ∀t 0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_8^8 < 4\,a_8\,e$ and $[a_8] > 1\Delta$,or $[a_8^8]$ 4$[a_8]$e and $(a_8 - \sqrt{(a_8^8)_8 - 4(a_8)_8\,e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+8}\right)$ and $\beta_8(a_8) = \text{ae-}^{\text{ma}}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_8(a_8)\text{e}^{-\int_0^a \mu_8(\tau_8) + d\tau_8}\text{da} = \frac{1}{(\text{m} + \mu_0 + 1)^8} + \frac{1}{(\text{m} + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\,x_8/(1 + x_8)$and $h_8(x_8)$=(1+$\omega_8x_8$)/$(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\text{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\text{K}][P_0])_8^3\ [\text{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\text{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^{*}\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^{*}}{8+\mathrm{Q}_8^{*}} \quad [h_{*}] = \frac{8+\omega_8 \mathrm{Q}_8^{*}}{8+\mathrm{Q}_8^{*}}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{\ 8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^{*})^8}[\ \mathrm{g}_{*}^{\ 8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^{*})^8} \quad [\mathrm{h}_{*}^{\ 8}] = \frac{\omega_8 - 8}{(8+\mathrm{Q}_8^{*})^8}$

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^{\ 8}} + \frac{1\omega_8}{([m_8]_{\ 8} + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^{\ 3}} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 8}} + \frac{1\omega_8}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the following equation:

$(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_{\ 8} + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\ \mathrm{Q}_8^{*})^{\ 3}$

$= m_8 + [\mu_0]_{\ 8} + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + \mathrm{Q}^{*})(1 + \omega_8 \mathrm{Q}_8^{*}$

Now

$$\left[\mathrm{a}_{1}^{\ 1}\right] - 4\left[\mathrm{a}_{\ 8}\right]\mathrm{e} = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^{8} + \frac{4K_8 b_8 \sigma_8\ \mathrm{Q}_8^{*}}{(\delta_8 - b_8)(8 + \mathrm{Q}_8^{*})} + \frac{\sigma_8^{\ 8}\ \mathrm{Q}_8^{*8}}{(1 + \mathrm{Q}_8^{*})^{\ 8}}$$

$$\frac{4\,b_8\sigma_8\,(\delta_{\,8}-\mathrm{b})\mathrm{Q}^{\,*}}{\delta_{\,8}\,(1+\mathrm{Q}_8^{\,*})^{\,8}}-\frac{4\,(\delta_{\,8}-b_8)^{\,8}\,(1-\omega_8)\mathrm{Q}_{\,8}^{\,*}}{\delta_{\,8}\,(1+\omega_8\mathrm{Q}_8^{\,*})(1+\mathrm{Q}_8^{\,*})}\mathrm{P}^{\,*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{\,8}\,ae_{\,8}^{\,mu}\;\;[\mu_0]_{\,8}-\frac{a_8}{a_8+8}$$

Hence if $[\mathrm{a}_8^{\,8}>4(a_8)_{\,8}\mathrm{e}$ and

$$[a_8]=\frac{1K_8b_8}{\delta_8-b_8}+\frac{\sigma_8\,\mathrm{Q}_8^{\,*}}{1+\mathrm{Q}_8^{\,*}}>1\Delta$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8}=[f_{8*}]_{\,8}+[f_{8*}]_{\,8}+[f_{8*}]\,\mathrm{P}_8^{\,*}$$

$$[a_8]_{\,8}=\left[\mathrm{f}_{8\,*}^{\,8}\right]\mathrm{P}_8^{\,*}-\frac{\mathrm{h}_{\,*}^{\,8}}{\mathrm{h}_{\,*}}\mathrm{P}^{\,*}(0)$$

$$\mathrm{e}=[\,\mathrm{g}_*^{\,8}]_{\,8}\,\mathrm{Q}_8^{\,*}$$

$$\eta_8(\mathrm{t})=\left[\mathrm{h}_{\,*}\right]_{\,8}\beta_{\,8}\left(\theta_{\,8}(\mathrm{t})\right)-\mu_{\,8}\left(\theta(\mathrm{t})\right),$$

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{8}\right]M_8 - [a_8]_{\,8}W_8 - [\mathrm{n(t)}]M_8$$

$$w_8(\mathrm{t}) = \mathrm{e}M_8$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_8]_{\,8} & -[a_8]_{\,8} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{\,8} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$$

The solution of (32)is

$$X_8(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{\,8}(0) + \int_0 \mathrm{e}^{At}$$

This equation notwithstanding its importance in all the models stand out for its importance for the following models

to which it is applied:

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(111)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.

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| (112)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (113) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (114) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (115)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (116) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (117)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (118)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (119)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems. |
| (120)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions. |
| (121) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field. |
| (122)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface. |
| (123)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator. |
| (124)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states. |
| (125) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions. |
| (126)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(127)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(128)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(129)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(130)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(131)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(132)Bringing them to thermal equilibrium requires (e) an external bath.

(133) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(134)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(135)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(136)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(137)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(138)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(139) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(140) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(141) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(142)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(143)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

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| (144) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL. |
| (145)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked. |
| (146)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice. |
| (147) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site. |
| (148)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself). |
| (149)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition. |
| (150)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (151) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (152)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (153) This is a realistic scenario for (e) all known systems. |
| (154)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (155)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (156)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (157) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (158)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (159)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (160)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (161)So in those systems, the quantum phase transition does indeed exist. |
| (162)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (163)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus |

find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity**

**avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2} \in [-0.99, 0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2} \in [2,48] M_\odot$, and (e&eb) total mass $M_{total} \leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced**

**LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 9. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \ P_9 \ 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \ \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth

rate with $\beta(a) \geq 0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h'(Q) \leq 0, \forall Q \geq 0$, and $\lim_{Q\to\infty} h(Q) = k_3 \geq 0$.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9 P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9) \int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \leq \rho_9(t, a_9) \leq \rho_9^*(t, a_9),$$

$$0 \leq Q_9(t) \leq Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9)da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9)da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \tag{10}$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \tag{$H1$}$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9Q_9 + c_9P_9Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9P_9 - \lambda_9P_9Q_9$$

$$\frac{dQ_9}{dt} = b_9Q_9 - c_9P_9Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t, 0)$$

$$= \int_0^\infty \beta_9\,(a_9)\rho_9(t, a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t, a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9\,(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9\,(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9 u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t, a_9) = \tilde{u}_9(a_9)^{e^{\gamma(t-a_9)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_9}\, da_9$. Then,

A solution of the first differential equation for $\tilde{u}(a_9)$ gives,

$$\tilde{u}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{u}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})\tau_9}\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a (\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds}\,d\tau_9\, da_9$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\,da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds + y\tau}\,\alpha_9(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty}Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90_9}^{l}\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^{a}\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9, a_9) = \int_0^a (\rho_0)_9(\tau_9) e^{\gamma_9\tau_9 + (f_{90})_9\tau + \int_0^\tau \mu_9(s_9)ds_9} d\tau$$

$$= \frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9 a_9} - 1)$$

Let

$$E_9(a_9) = e^{-(f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ \left(\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(a_9)da_9 - \int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(a_9)da_9) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) da_9 = 1 - \int_0^\infty \beta_9(a_9) E_9(a_9) e^{-\gamma_9 a_9} da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) \int_0^a \alpha_9(\tau_9) Y^{-9}(\gamma_9, \tau_9) d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}')_9 (\rho_0)_9 \int_0^\infty E_9(a_9) da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$p_9^*(0) = h_9(Q_9^*) \int_0^\infty \beta_9(a_9) p_9^*(a_J) p_9^*(a_9) da_9$ (27)

$g_9(P_9^*) - b_9 = 0,$ (28)

Where $P_9^* = \int_0^\infty p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b),$

From (26),

$p_9^*(a) = p_9^*(0) e^{\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9}$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*) \int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9} da_9\right) p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9) \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(z_9) a_9} da_{,9}$$

And assume that

$$[k_3]_9 \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9 a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left([f_9^{'}]_9(z_9)\right)\int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9)\int_0^{\infty}\beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}\, da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty}\beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty}\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y_9^a} da_9}$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r$, $R\left(f_9(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$M_9(t) \equiv \int_0^\infty m_9(t, a_9)da_9 = P_9(t) - P_9^*,$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - \left[p^*\right]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9)m_9(t,a_9)\,da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0)W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $\left[f_{9*}^{'}\right]_9$ denoting $\left[f^{'}{}_9(P^*)\right]_9\left[f_{9*}^{'}\right]_9$ denoting $[f_9]_9$ $(Q_9^*,$ $h_{9}^{*}$ denoting $h_*^{'}$ denoting $h_9^{'}$ $(Q_9^*)$, and $\left[g_*^{'}\right]_9$

Denoting $\left[g^{'}\right]_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9[p_9^*](a_9)M_9 - [f'_{9*}]_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9)m_9(t,a_9)da_9 + \frac{[h_*^9]_9}{[h^*]_9}[p^*]_9(0)\,W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9[Q_9^*]M_9$$

Integrating $[m_9]_9(t,a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t,a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t,a_9)da_9 - \int_0^\infty \mu(a_9)m(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f'_{9*}]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$$= m_9(t,0) - \int_0^\infty \mu_9(a_9)\,m_9(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}']_9 P_9^*\right)M_9 - [f_{9*}']_9 P_9^* W_{,9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + [f_{9*}']_9 P_9^*\right)M_9 + \left(\frac{[h_*']_9}{[h_*]^9}\,P_9^*(0) - f_{9*}' P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_{*}\right]_9 Q_9^{*} M_{.9}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9 = ([h_{*}]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9[a_9^9]2<4\ [a_9]2<4\ a_9e$,

$\text{Re}\,\lambda_9[\lambda_9]_9 = \text{Re}\lambda_9[\lambda_9]_9 = -\frac{9}{9}a_9[a2\ ]_9 < 0.$

Let $\Delta=\max_{9<u<\infty}\left|h_{*}[h_{*}]_9\beta_9(a_9) - \mu_9(a_9)\right|$. Then$|n_9(t)|$ ?4∀t ?0. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^{*}(a), \boldsymbol{Q}^{*}$) is asymptotically stable if either $a_9^9 <4\ a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ?4$[a_9]e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae\text{-}^{ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9) + d\tau_9}da = \frac{1}{(m+\mu_0+1)^9} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ 0.5. Assume$[f_9](x_9)= K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)= [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9)=(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])\,{}^{3}_{9}\ [\mathrm{K}]_{\,9}[P_0]\,{}^{3}_{9}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)\ [P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}^{*}_{9}}{9+\mathrm{Q}^{*}_{9}}$ $[h_*] = \frac{9+\omega_9 \mathrm{Q}^{*}_{9}}{9+\mathrm{Q}^{*}_{9}}$

$[f_{9*}] = K_9\ [\mathrm{f}\,{}^{9}_{9*}] = \frac{\sigma_9}{(9+\mathrm{Q}^{*}_{9})^{9}}[\ \mathrm{g}\,{}^{9}_{*}] = \frac{\sigma_9}{(9+\mathrm{P}^{*}_{9})^{9}}$ $[\mathrm{h}\,{}^{9}_{*}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}^{*}_{9})^{9}}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_9 + [\mu_0]_9 + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_9 + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\, Q_9^*)^3$$

$$= m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1 + Q^*)(1 + \omega_9 Q_9^*$$

Now

$$\left[a_{1}^{1}\right] - 4\left[a_{9}\right] e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9\, Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9\, Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9\, (\delta_9 - b) Q^*}{\delta_9 (1 + Q_9^*)^9} - \frac{4\, (\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{mu}_9\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9\, Q_9^*}{1 + Q_9^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| |
| IF we define |
| $$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$ |
| $$[a_9]_9 = \left[f^9_{9*}\right]P^*_9 - \frac{h^9_*}{h_*}P^*(0)$$ |
| $$e=[g^9_*]_9\,Q^*_9$$ |
| $$\eta_9(t) = \left[h_*\right]_9\beta_9\left(\theta_9(t)\right) - \mu_9(\theta(t)),$$ |
| The system(32)can be written as the following system: |
| $$[M(t)]\left[a_9\right]M_9 - [a_9]_9W_9 - [n(t)]M_9$$ |
| $$w_9(t) = eM_9$$ |
| Write |
| $$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \quad 0 \end{pmatrix}$$ |
| $$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$ |

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(t) = e^{A,t}[X]_9(0) + \int_0 e^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual**

**oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of**

**classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed**

**huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy**

**holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION ONE A

## Multiple-Particle Interference And Quantum Error Correction

### INTRODUCTION—VARIABLES USED

**Multiple-Particle Interference and Quantum Error Correction Andrew Steane Published 8 November 1996.DOI: 10.1098/rspa.1996.0136**

(1) The concept of multiple-particle interference is discussed, using insights provided by (e) the classical theory of error correcting codes.
(2) This leads to a discussion of error correction in a quantum communication channel or a quantum computer. Methods of error correction in the quantum regime are presented, and their limitations assessed. A quantum channel can recover from (e) arbitrary decoherence of x qubits if K bits of quantum information are encoded using (e) n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n).
(3) This implies (eb) exponential reduction of decoherence with (e&eb) only a polynomial increase in the computing resources required.
**(4)** Therefore quantum computation can be made free of errors in the presence of (e) physically realistic levels of decoherence.
**(5)** The methods also allow (eb) isolation of quantum communication from (e) noise and eavesdropping (quantum privacy amplification).

### NOTATION

### Module One

The key experimental challenge is to perform such complete error-correction cycles including (e) non-destructive stabilizer measurements and real-time feedback well within the coherence time

$G_{13}$ : Category one of key experimental challenge is to perform such complete error-correction cycles including (e) non-destructive stabilizer measurements and real-time feedback well within

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| $G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of coherence time<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>The concept of multiple-particle interference is discussed, using insights provided by (e) the classical theory of error correcting codes | |
| $G_{16}$ : Category one of classical theory of error correcting codes<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of concept of multiple-particle interference<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>This leads to a discussion of error correction in a quantum communication channel or a quantum computer.<br><br>Methods of error correction in the quantum regime are presented, and their limitations assessed.<br><br>A quantum channel can recover from (e) arbitrary decoherence of x qubits if K bits of quantum information are encoded using (e) n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n). | |
| $G_{20}$ : Category one of arbitrary decoherence of x qubits if K bits of quantum information are encoded using (e) n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n).<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of quantum channel can recover<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>A quantum channel can recover from arbitrary decoherence of x qubits if K bits of quantum information are encoded using (e) n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - | |

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| 2H (x/n). | |
| $G_{24}$ : Category one of n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n).<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of quantum channel can recover from arbitrary decoherence of x qubits if K bits of quantum information are encoded<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>A quantum channel can recover from arbitrary decoherence of x qubits if K bits of quantum information are encoded using n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n). | |
| $G_{28}$ : Category one of K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n).<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of quantum channel can recover from arbitrary decoherence of x qubits if K bits of quantum information are encoded using n quantum bits<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This implies (eb) exponential reduction of decoherence with (e&eb) only a polynomial increase in the computing resources required | |
| $G_{32}$ : Category one of quantum channel can recover from arbitrary decoherence of x qubits if K bits of quantum information are encoded using n quantum bits, where (e) K/n can be greater than 1 - 2H (2x/n), but must be less than 1 - 2H (x/n).<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of exponential reduction of decoherence with (e&eb) only a polynomial increase in the computing resources required | |

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| $T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>This implies (eb) exponential reduction of decoherence with (e&eb) only a polynomial increase in the computing resources required | |
| $G_{36}$ : Category one of **exponential reduction of decoherence;** only a polynomial increase in the computing resources required<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of only a polynomial increase in the computing resources required **;exponential reduction of decoherence**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Therefore quantum computation can be made free of errors in the presence of (e) physically realistic levels of decoherence | |
| $G_{40}$ : Category one of quantum computation can be made free of errors<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of physically realistic levels of decoherence<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The methods also allow (eb) isolation of quantum communication from (e) noise and eavesdropping (quantum privacy amplification). | |
| $G_{44}$ : Category one of noise and eavesdropping (quantum privacy amplification).<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of isolation of quantum communication<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |

| | |
|---|---|
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |

| | |
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| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |

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|---|---|
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}')^{(1)}\boxed{-(b_{15}'')^{(1)}(G,t)} & \boxed{-(b_{18}'')^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}'')^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}'')^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}'')^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15} \quad (60)$$

Where $\boxed{-(b_{13}'')^{(1)}(G,t)}$, $\boxed{-(b_{14}'')^{(1)}(G,t)}$, $\boxed{-(b_{15}'')^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}'')^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}'')^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}'')^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}'')^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}'')^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}'')^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}'')^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}'')^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}'')^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}'')^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}'')^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}'')^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}')^{(2)}\boxed{+(a_{16}'')^{(2)}(T_{17},t)} & \boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}'')^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}'')^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}'')^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}')^{(2)}\boxed{+(a_{17}'')^{(2)}(T_{17},t)} & \boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}'')^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}'')^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}')^{(2)}\boxed{+(a_{18}'')^{(2)}(T_{17},t)} & \boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}'')^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}'')^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a_{16}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}'')^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

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| $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}\,-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}\,-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}\,-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3 | |

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| $-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)} & +(a_{22}^{''})^{(3)}(T_{21},t) & +(a_{18}^{''})^{(2,2,2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | 70 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | 71 |

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* $1, 2\ 3$

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$
*are fourth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$
*are fifth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$
*are sixth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$
*are seventh augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$
*are eighth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$
$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$
$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$
$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$
*are fourth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$
*are fifth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$
*are sixth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$
*are seventh detrition coefficients for category* $1,2$ *and* $3$
$-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$
*are eighth detrition coefficients for category* $1,2$ *and* $3$

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|---|---|
| $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | 80 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | 81 |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{30} \quad (84)$$

where $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[\begin{array}{ccc} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{34})^{(6)}\boxed{+(a''_{34})^{(6)}(T_{33},t)} & \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{34} \quad (87)$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6)}(T_{33},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)}$ are third augmentation

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| *coefficients for category* 1, 2 *and* 3<br>$+(a_{13}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}')^{(6)} & -(b_{32}'')^{(6)}(G_{35},t) & -(b_{28}'')^{(5,5,5)}(G_{31},t) & -(b_{24}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{13}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}')^{(6)} & -(b_{33}'')^{(6)}(G_{35},t) & -(b_{29}'')^{(5,5,5)}(G_{31},t) & -(b_{25}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{14}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}')^{(6)} & -(b_{34}'')^{(6)}(G_{35},t) & -(b_{30}'')^{(5,5,5)}(G_{31},t) & -(b_{26}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{15}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}'')^{(6)}(G_{35},t)$, $-(b_{33}'')^{(6)}(G_{35},t)$, $-(b_{34}'')^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{28}'')^{(5,5,5)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}'')^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3<br>$-(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3<br>$-(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3<br>$-(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3 | |

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$
are eighth detrition coefficients for category 1, 2, and 3
$-(b_{46}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}')^{(7)}+(a_{36}'')^{(7)}(T_{37},t) & +(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}+(a_{37}'')^{(7)}(T_{37},t) & +(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}+(a_{38}'')^{(7)}(T_{37},t) & +(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}'')^{(7)}(T_{37},t)$, $(a_{37}'')^{(7)}(T_{37},t)$, $(a_{38}'')^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3
$+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3
$+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3
$+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3
$+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$
are eighth augmentation coefficient for 1,2,3
$+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = \quad (94)$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\boxed{-(b_{41}^{''})^{(8)}(G_{43},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\boxed{-(b_{42}^{''})^{(8)}(G_{43},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}^{'})^{(9)}\boxed{+(a_{45}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

| | |
|---|---|
| $\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br><br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},(r_i)^{(2)}}$ are positive constants and $\boxed{i=16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |

| | |
|---|---|
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t)-(a_i^{''})^{(2)}(T_{17},t)\rvert \le (\hat{k}_{16})^{(2)}\lvert T_{17}-T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t)-(b_i^{''})^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)}\lvert\lvert(G_{19})-(G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and$(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},$ are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i^{'})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i^{'})^{(2)}+(\hat{A}_{16})^{(2)}+(\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)}+(b_i^{'})^{(2)}+(\hat{B}_{16})^{(2)}+(\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i^{'})^{(3)},(a_i^{''})^{(3)},(b_i)^{(3)},(b_i^{'})^{(3)},(b_i^{''})^{(3)}>0,\quad i,j=20,21,22$<br>The functions $(a_i^{''})^{(3)},(b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21},t)\le(p_i)^{(3)}\le(\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23},t)\le\ (r_i)^{(3)}\le(b_i^{'})^{(3)}\le(\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t)=(p_i)^{(3)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t)=\ (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)},(p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i=20,21,22}$ | 113 |

| | |
|---|---|
| They satisfy Lipschitz condition:<br><br>$\|(a_i'')^{(3)}(T_{21}', t) - (a_i'')^{(3)}(T_{21}, t)\| \leq (\hat{k}_{20})^{(3)}\|T_{21} - T_{21}'\|e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\|(b_i'')^{(3)}(G_{23}', t) - (b_i'')^{(3)}(G_{23}, t)\| < (\hat{k}_{20})^{(3)}\|\|G_{23} - G_{23}'\|\|e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}', t)$ and$(a_i'')^{(3)}(T_{21}, t)$ . $(T_{21}', t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i'')^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i'')^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition: | 119 |

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| $\|(a_i^{''})^{(4)}(T_{25}^{'},t)-(a_i^{''})^{(4)}(T_{25},t)\| \leq (\hat{k}_{24})^{(4)}\|T_{25}-T_{25}^{'}\|e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})^{'},t)-(b_i^{''})^{(4)}\big((G_{27}),t\big)\| < (\hat{k}_{24})^{(4)}\|\|(G_{27})-(G_{27})^{'}\|\|e^{-(\hat{M}_{24})^{(4)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)},(\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)}=1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}<1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)},(\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)},(\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)},(a_i^{'})^{(4)},(b_i)^{(4)},(b_i^{'})^{(4)},(p_i)^{(4)},\ (r_i)^{(4)}, i=24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)}+(a_i^{'})^{(4)}+\ (\hat{A}_{24})^{(4)}+\ (\hat{P}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}]<1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)}+(b_i^{'})^{(4)}+\ (\hat{B}_{24})^{(4)}+\ (\hat{Q}_{24})^{(4)}\ \ (\hat{k}_{24})^{(4)}]<1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)},(a_i^{'})^{(5)},(a_i^{''})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(b_i^{''})^{(5)}>0, \quad i,j=28,29,30$<br><br>The functions $(a_i^{''})^{(5)},(b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)},\ (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t)\leq(p_i)^{(5)}\leq(\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big)\leq\ (r_i)^{(5)}\leq(b_i^{'})^{(5)}\leq(\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \leq (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i')^{(6)} \leq (\hat{B}_{32})^{(6)}$$

$$lim_{T_2 \to \infty}(a_i'')^{(6)}(T_{33}, t) = (p_i)^{(6)}$$ 128

$$\lim_{G \to \infty}(b_i'')^{(6)}((G_{35}), t) = (r_i)^{(6)}$$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

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| Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32{,}33{,}34}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}\big((G_{35}), t\big)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32{,}33{,}34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + \ (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i')^{(6)} + \ (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} \ (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (G) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36{,}37{,}38$<br><br>(H) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(I)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(J)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = \ (r_i)^{(7)}$ | 132 |

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| **Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}'\rvert e^{-(\hat{M}_{36})^{(7)} t}$<br><br>$\lvert(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})'\rvert\rvert e^{-(\hat{M}_{36})^{(7)} t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(K) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(L) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i, j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |

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| $(b_i^{''})^{(8)}((G_{43}),t) \leq \ (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'},t) - (a_i^{''})^{(8)}(T_{41},t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'},t) - (b_i^{''})^{(8)}\left((G_{43}),t\right)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i^{'})^{(8)} + \ (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(b_i)^{(8)} + (b_i^{'})^{(8)} + \ (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i^{'})^{(9)}, (a_i^{''})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (b_i^{''})^{(9)} > 0, \quad i,j = 44,45,46$ | 146 A |

The functions $(a_i^{''})^{(9)}, (b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i^{'})^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i^{''})^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i^{''})^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)} |T_{45} - T_{45}^{'}| e^{-(\hat{M}_{44})^{(9)} t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}((G_{47}), t)| < (\hat{k}_{44})^{(9)} ||(G_{47}) - (G_{47})^{'}|| e^{-(\hat{M}_{44})^{(9)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

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| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ : | 153 |

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| $G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| Where $s_{(13)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |

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| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{\mathrm{T}}_{22}(\mathrm{t}) = \mathrm{T}_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{\mathrm{T}}_{26}(\mathrm{t}) = \mathrm{T}_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
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| $\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{32})^{(6)}$, $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a_{36}')^{(7)} + a_{36}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a_{37}')^{(7)} + (a_{37}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a_{38}')^{(7)} + (a_{38}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b_{36}')^{(7)} - (b_{36}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b_{37}')^{(7)} - (b_{37}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b_{38}')^{(7)} - (b_{38}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a_{40}')^{(8)} + a_{40}'')^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a_{41}')^{(8)} + (a_{41}'')^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |

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| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that | 168 |

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|---|---|
| $(G_{13}(t)-G_{13}^0)e^{-(\hat{M}_{13})^{(1)}t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{P}_{13})^{(1)}+G_{14}^0)e^{\left(-\frac{(\hat{P}_{13})^{(1)}+G_{14}^0}{G_{14}^0}\right)}+(\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0+(\hat{P}_{16})^{(6)}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1+(a_{16})^{(2)}t\right)G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)}t}-1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t)-G_{16}^0)e^{-(\hat{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | |

that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 (175)

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (176)

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6 (177)

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(b) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (178)

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$ (180)

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| $\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$ | |
| From which it follows that<br>$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$<br>$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left((G^{(1)},T^{(1)}),(G^{(2)},T^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right|\leq\int_{0}^{t}(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_{0}^{t}\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

(186)

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_{+}$.

If instead of proving the existence of the solution on $\mathbb{R}_{+}$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)},i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

From 19 to 24 it results

$$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\geq 0$$

$$T_i\,(t)\geq T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$:

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13}<(\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt}\leq\left((\widehat{M}_{13})^{(1)}\right)_1-(a_{14}^{'})^{(1)}G_{14}$ and by integrating

$$G_{14}\leq\left((\widehat{M}_{13})^{(1)}\right)_2=G_{14}^0+2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$$

(187)

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| In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\widehat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br><br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br><br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i$ (t) $= 0$ and $T_i$ (t) $= 0$<br><br>it results<br><br>$G_i$ (t) $\geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i$ (t) $\geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for t $> 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

In the same way , one can obtain

$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$

If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively.

**Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. 202

**Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$. 203

**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :

Indeed let $t_2$ be so that for $t > t_2$

$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$

Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to 204

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. 205
The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose 207

$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have

$$\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)} \quad (208)$$

$$\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)} \quad (209)$$

In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself 210

The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric 211

$$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t}\}$$

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} \} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t} \; and \; (\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23}) (and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0 \; and \; T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0 \;$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2 \; and \; \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21} \; and \; G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left((\widehat{M}_{20})^{(3)}\right)_1 / (a_{21}^{'})^{(3)}$ | 217 |

| | |
|---|---|
| In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}} < 1$ and to choose<br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}\}$ | 225 |

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})}, \widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\, ds_{(24)} +$$

$$\int_0^t \{(a_{24}^{'})^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} \left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)}, s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

(226)

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

(227)

**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)} t\right)} > 0 \quad \text{for t} > 0$$

(228)

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}^{'})^{(4)}$$

(229)

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|---|---|
| In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$ | 236 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br><br>It results<br><br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br><br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br><br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(5)}t\right)}>0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1,\left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if | 240 |

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| $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a'_{30})^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b''_{30})^{(5)}\left((G_{31})(t),t\right) = (b'_{30})^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |

The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric 247

$$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{32})^{(6)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$

It results

$$\left|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{32})^{(6)}\left|G_{33}^{(1)}-G_{33}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)}+$$

$$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)}-G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$$

$$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left|G_{32}^{(1)}-G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$$

$$G_{32}^{(2)}\left|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$$

Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|(G_{35})^{(1)}-(G_{35})^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}t}\le$$
$$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$$ 248

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$. 249

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32,33,34$ depend only on $\mathrm{T}_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 27:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$ 250

it results

$$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$$

$$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(6)}t\right)}>0\quad \text{for t}>0$$

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a'_{33})^{(6)} G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)} \quad (257)$$

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| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\le$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\ge 0$ | 261 |

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| $T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(7)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}')^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}')^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}')^{(7)}$<br><br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}\left((G_{39})(t),t\right) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\widehat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
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| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$ $\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}\ and\ (\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$ | 275 |

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| it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}')^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(8)}((G_{43})(t), t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty} (b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)}+\left((\hat{P}_{44})^{(9)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{44})^{(9)}\right]\leq(\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),\left((G_{47})^{(2)},(T_{47})^{(2)}\right)\right)=$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\;e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\;(G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}')^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}')^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}')^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(9)} ((G_{47})(t), t)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}((G_{47})(t), t) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty} (b_{46}'')^{(9)} ((G_{47})(t), t) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14}, t) + (a_{14}'')^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

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| $-(\tau_2)^{(1)} \le -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \le -(\tau_1)^{(1)}$ | |
| **Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ :<br><br>By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 281 |
| **Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)},\ if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}^{'})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}^{'})^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}^{'})^{(1)}t}\right)$ | 286 |

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| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(\mathrm{T}_{17}, t) + (a_{17}'')^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}\big((G_{19}), t\big) - (b_{17}'')^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |

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| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a_{18}')^{(2)} t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b_{18}')^{(2)} t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ | 317 |

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| $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$<br>$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)} \le -(a'_{20})^{(3)} + (a'_{21})^{(3)} - (a''_{20})^{(3)}(T_{21}, t) + (a''_{21})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)} \le -(b'_{20})^{(3)} + (b'_{21})^{(3)} - (b''_{20})^{(3)}(G_{23}, t) - (b''_{21})^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ :<br>By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and<br>By $(\bar{v}_1)^{(3)} > 0$, $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | 320 |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$, ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T^0_{20}}{T^0_{21}}}$ | 322 |

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| $(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},$ ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br>Then the solution of global equations satisfies the inequalities<br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$<br>$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b_{24}')^{(4)} + (b_{25}')^{(4)} - (b_{24}'')^{(4)}((G_{27}), t) - (b_{25}'')^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0, (v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations | 329 |

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| $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$<br>$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$ | 330 |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}^{'})^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}^{'})^{(4)}t} \le T_{26}(t) \le$ | 336 |

$\frac{(a_{26})^{(4)}T_{24}^{0}}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^{0}e^{-(R_2)^{(4)}t}$

**Definition of** $(S_1)^{(4)},(S_2)^{(4)},(R_1)^{(4)},(R_2)^{(4)}$:- 337

Where $(S_1)^{(4)}=(a_{24})^{(4)}(m_2)^{(4)}-(a_{24}^{'})^{(4)}$

$(S_2)^{(4)}=(a_{26})^{(4)}-(p_{26})^{(4)}$

$(R_1)^{(4)}=(b_{24})^{(4)}(\mu_2)^{(4)}-(b_{24}^{'})^{(4)}$

$(R_2)^{(4)}=(b_{26}^{'})^{(4)}-(r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)}\le-(a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}-(a_{28}^{''})^{(5)}(T_{29},t)+(a_{29}^{''})^{(5)}(T_{29},t)\le-(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)}\le-(b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31}),t)-(b_{29}^{''})^{(5)}((G_{31}),t)\le-(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)},(v_2)^{(5)},(u_1)^{(5)},(u_2)^{(5)},v^{(5)},u^{(5)}$ : 339

By $(v_1)^{(5)}>0$ , $(v_2)^{(5)}<0$ and respectively $(u_1)^{(5)}>0$ , $(u_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and

**Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$

**Definition of** $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-

If we define $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)}$ by

$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)}=\frac{G_{28}^{0}}{G_{29}^{0}}}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$

and analogously 341

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| $(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, $\boldsymbol{if}\ (u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$, $\boldsymbol{if}\ (u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying | 349 |

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| $-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br><br>$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39}),t)-(b_{37}^{''})^{(7)}((G_{39}),t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

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| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{36}(t) \leq G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{37}(t) \leq \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \leq G_{38}(t) \leq$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \leq T_{38}(t) \leq$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ | 370 |

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,
and $\boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

| | |
|---|---|
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}, \ \boldsymbol{if}\ (\nu_0)^{(9)} < (\nu_1)^{(9)}$$

$$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}, \boldsymbol{if}\ (\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)},$$

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}, \ \boldsymbol{if}\ (\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$$

and analogously

$$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}, \ \boldsymbol{if}\ (u_0)^{(9)} < (u_1)^{(9)}$$

$$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}, \boldsymbol{if}\ (u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)},$$

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}, \boldsymbol{if}\ (\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G^0_{44} e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G^0_{44} e^{(S_1)^{(9)} t}$$

(

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t}\leq G_{46}(t)\leq$$

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t})$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t}\leq T_{44}(t)\leq T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t}\leq T_{44}(t)\leq\frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t}\leq T_{46}(t)\leq$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:-

Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$

$(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$

$(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$

$(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

383 384 385 386

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

$$\text{For } 0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

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| $\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t)=\frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain | 398 |

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \le \frac{dv^{(3)}}{dt} \le -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \ge \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \le v^{(3)}(t) \le (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \le v^{(3)}(t) \le (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \le \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \le v^{(3)}(t) \le$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \le (\bar{v}_1)^{(3)} \le (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \le v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$, we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} =$

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$(\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

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$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq v^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $v^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)v^{(7)} - (a_{37})^{(7)} v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)} v^{(7)} - (a_{36})^{(7)}\right) \leq \frac{dv^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$v^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq v^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

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**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

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In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

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If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

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If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

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| $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |

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| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ | 436 A |

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| $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |

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| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}))\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}))\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}))\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |

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| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(b) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) +$ | 486 |

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| $(a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |

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| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{13}=\frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}=\frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | 494 |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations | 499 |

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| $G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 504 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if | 510 |

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| $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |

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| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 523 |

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| $T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br>**Proof:**_Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions | 531 |

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| $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |

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| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}\ and\ (b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}\ ,\ \frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}\ ) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}\ and\ (b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}\ ,\ \frac{\partial(b_i^{''})^{(5)}}{\partial G_j}(\ (G_{31})^{*}\ ) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |

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| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |

| | |
|---|---|
| | |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \;,\; \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\Big)\right]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\Big)\right]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\Big)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\Big)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\Big)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al**

**Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal**

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(2) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and

review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 2. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For $K \subset C$ and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \|x - y\| < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta_0 > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B*.

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I}\boldsymbol{C}_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$ . If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $\mathrm{A_i}(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A_i}(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A_i}(x)\ and\ \mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i\ \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N\,\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots,\tilde{A}_{k})$ by

$$\tilde{A}_i(x) = \{\frac{A(x)}{A(x)-\frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1,\ldots,k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x$,$W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i\in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu)>0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.4) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.5) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.6) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a,t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a,t) = 0, & \text{'} (1.4) \\ \rho_1(0,t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a,t)da, & t>0, \\ \rho_2(0,t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a,t)da, \end{cases}$$

(1.5) $\rho_i(a,s) = \varphi_i(a,s), a.0, s\epsilon[-\tau, 0], \quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a,s)\,da, s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 2. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{\ i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +}$ [$-\tau, T] \rightarrow R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +}$ [$\{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +}$ [$-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t, 0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right) ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t)= B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

$$(2.5)\ L_i\left(a,t,\ p_i(t)\right) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a, t, p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a, t, P_i(t)\right)da$

$$+\int_0^{\infty} b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, \; t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \; \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^{\infty} \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \; t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2\ ),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in \mathrm{C}([0,T], R^+), \quad B_i \in \mathrm{C}([0,T], R^+)\ , i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in \mathrm{C}([-\tau, T], R\ ) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $\mathrm{C}^+[-\tau, T\ ]$. Then for any

$(x, y) \in \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$\mathrm{C}_T = \mathrm{C}^+[-\tau, T\ ], \quad \mathrm{C}_T^2 = \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ].$$

Members of $\mathrm{C}_T$ will be denoted by $Z_t$ with the meaning that

$Z\ (t+s) = z_t(s)$ *for* $s \in [-\tau, T\ ]$ *and* $t \in [0\ , T\ ]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in \mathrm{C}_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2\ )$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$\mathrm{B}\ (\mathrm{t}) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$ , $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^*\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{\sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^*\right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then ▯ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , ▯ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|▯(P_t^{(1)}) - ▯(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|▯(P^{(1)}(t)) - ▯(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\ \ 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

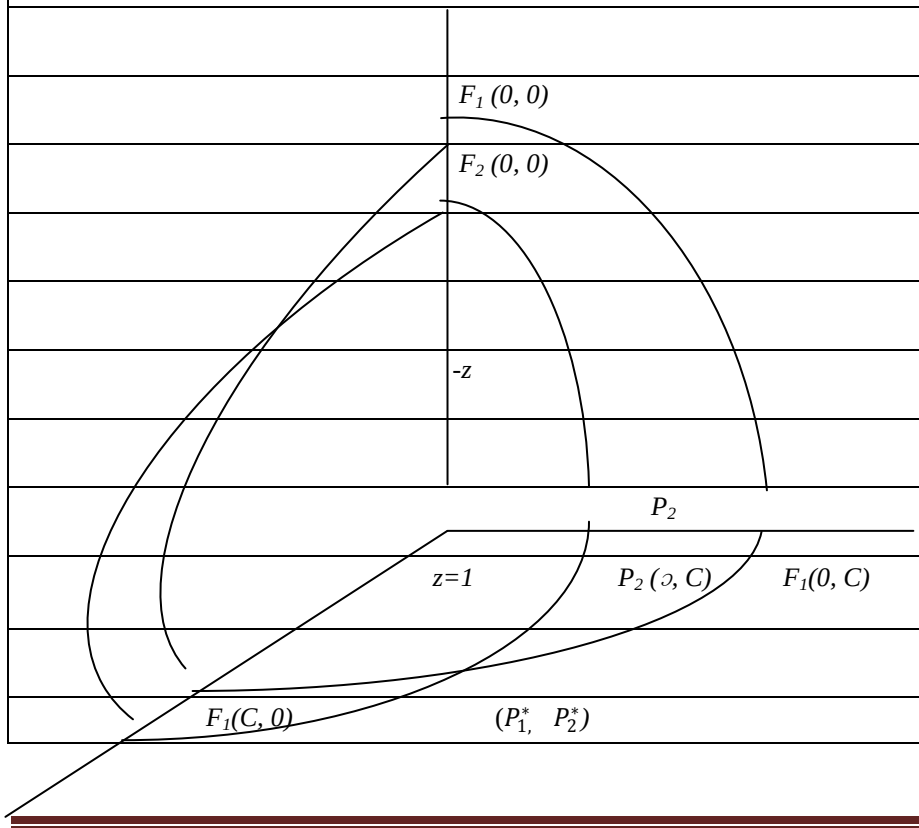
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{\varphi}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\varphi(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\varphi(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

From the first of (3.15) we get

(3.18) $$\varphi(a) = \left\{\varphi(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\varphi(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 3. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate}.$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(iv) $b_1(0) = 0,\quad m_1(0, x_2) \equiv 0, b_2(0) = 0,\quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x^*_{1,} > 0, x^*_{2,} > 0$ we have

$$b_1(x^*_1, -m_1(x^*_1, 0) = 0,$$

$$b_2(x^*_2, -m_1(0, x^*_2) = 0,$$

(2.4)

(v) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2\ (\delta_2) - m_2(x_1, \delta_2) < 0$, for $x_1$ 0, $x_2$ 0,

(2.5)

(vi) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha,\beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha,\beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

$$\text{Det} \begin{pmatrix} \lambda - \beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda - \beta_{22} \end{pmatrix} = 0$$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i,\; j = 1, 2; i \neq j)$.

## 4. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\ (t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\ (t_2)\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\ (t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\ (t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\ (t - \tau_{11}) - m_{12}Y_2\ (t - \tau_{12}),$$

$$\frac{dY_2\ (t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\ (t - \tau_{21}) - m_{22}Y_2\ (t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ [illegible]}; \quad \frac{\partial g_1}{\partial y_2} \text{ [illegible]},$$

$$\frac{\partial f_2}{\partial y_1} \text{ [illegible]}; \quad \frac{\partial g_2}{\partial y_2} \text{ [illegible]}; \quad \frac{\partial g_2}{\partial y_2} \text{ [illegible]}, \quad \text{Where } y_1 \text{ [illegible]}, y_1 \text{ [illegible]},$$

(4.2)

(v)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \quad g_2(0) = 0, \qquad (4.3)$$

(vi) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \qquad (4.4)$$

(vii) There exists a point $(\alpha^* \beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \qquad (4.5)$$

(viii) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ [illegible]},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \qquad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ [illegible] for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 6. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt} = -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}),$$

$t > 0$, (6.1)

Where $m_{1J}, b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt} = -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}),$$

$$\frac{dZ_2(t)}{dt} = -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}),$$

$t > 0$, (6.2)

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad ||Z|| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad ||B|| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{matrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{matrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{11}\left|Y_1(t)\right| + b_{11}\left|Y_1(t-\tau_{11})\right| + m_{12}\left|Y_2(t-\tau_{12})\right|, \\ \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{22}\left|Y_2(t)\right| + m_{21}\left|Y_1(t-\tau_{21})\right| + b_{22}\left|Y_2(t-\tau_{21})\right|, \end{aligned}\right\} \quad (A.2)$$

Where

$$\frac{d}{dt}\left|Y_1(t)\right| = \left\{\begin{aligned} &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0, \\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0, \end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(s) > \left|Y_1(s)\right|, \quad s \in [-\tau, 0], \; i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(t) \text{ for } t \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(t) = Z_i(t) - \left|Y_1(s)\right|$, $i = 1,2$. $t \ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\ge -m_{11}U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\ge -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(s) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[\tau, 0]$ : It will follow from (A.7) that $U_1(t) \ge U_1(0) \exp[-m_{11}t]$

$$+\int_0^t \left[b_{11}U_1(s-\tau_{11}) + m_{12}U_2(s-\tau_{12})\right] \exp[-m_{11}(t-s)ds > 0,] \qquad (A.9)$$

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds + \int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds + \int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0 \leq |\, Y_{1,}(t)\,| \leq Z_1(t) \to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22} m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t) \to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t) \to 0, Z_2(t) \to 0$ as $t \to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s \in [0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty s k_{ij}(s)ds < \infty;\ \int_0^\infty k_{ij}(s)ds = 1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 4. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 5. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big), \qquad t > 0, \tag{2.1}$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} {}^{|h(t)|, h \in B_\omega^\tau}\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| \geq b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| \geq b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ *in* $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ $\in -\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega + t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of $N$ *and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(iv) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(v) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(vi) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq 2\ max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(c)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(d) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 6. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(d) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(e) N is monotonic nondecreasing on $K^*$.

(f) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cup K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \| x_1 \| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \| x_2 \| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \leq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \leq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \leq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ –} \tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ –} \tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ –} \tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_1, X_2)$ where

$$v(t) = V(t, X_1, X_2) = |X_1(t)| + |X_2(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_2(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_1(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21}) x^*_1(t)|X_1(t)| + (|b_{22}| - b_{22}) x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_1, X_2) \geq |X_1(t)| + X_2(t)|$ and $dv/dt \leq 0$, which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_1(t)| + |X_2(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_1(s), X_2(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21}) x^*_1(t_m)|X_1(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} - {}_{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha$, $\beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha$,$\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that “In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$\frac{dx_1(t)}{dt} x_1(t)[1 - x_1(t) + 0.4x_2(t - \mathcal{T})/k_1(t)]$

$\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$

Where $k_1(t) = 20 + \sin(\pi t)$ and $k_2(t)$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 3. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \dots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.2) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \; b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \; (b_j^u / a_{jj}^l)) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \; x_i^l \; for \; t \; t_0, \; i = 1, 2, \ldots, n. \qquad (1.7)$$

4. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \; x_l \; x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1} x_l^l ; \qquad x_*$$

$$= \min_{1} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(23) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(24) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(25) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(26) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(27) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(28) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(29) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(30) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(31) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(32) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(33) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(34) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(35) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (36) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (37) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (38) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (39) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (40) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (41) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (42) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (43) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (44) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception:

"what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left|\left|f\ (t+\ \tau,x) - \ f(t,x)\right|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} \left( \sup_{t \in \boldsymbol{R}} a_{Jl}(t) \right) + \mu, \quad i = 1, 2, \dots, n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^{*}\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{ij}\;(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^{*})/(2x^{*}). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$\upsilon\big(x(t+\tau_k),x(t+\tau_m)\big)\ \upsilon\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_{*}(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_{*})/(2x^{*})]/\mu x_{*})$$

$$[(2x_n^{*})/x_{*}]\exp[-\mu x_{*}(t+t_k-t_0)]\ +(\varepsilon/2x^{*})$$

$$<(\varepsilon/x^{*})\ \text{On using (2.8), (2.10)} \qquad (2.14)$$

For $m\ k\ n_0(\varepsilon,\beta)$ and for all $t\in\cup\subset[\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n}|log x_i\,(t+\tau_k)-log x_l(t+\tau_m)|<(\varepsilon/x^{*}) \qquad (2.15)$$

And hence using $x_{*}\ x_i(t)\ x^{*}, i=1,2,\dots,n; t\ t_0$,

$$\sum_{i=1}^{n}|x_i\,(t+\tau_k)-x_i(t+\tau_m)|<\varepsilon \text{ for all } t\in\cup\subset[\beta,\infty], m\ k\ n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t)=\{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t)\in[x_{*},x^{*}], i=1,2,\dots,n; t\in[\beta,\infty]$ follows from (2.16) and the fact that $x_i(t)\in[x_{*},x^{*}], i=1,2,\dots,n;$ for $t\ t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k)\to w_l(t)$ as $k\to\infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i=1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ [illegible] $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ [illegible] $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} |\frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m)| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};\quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l};\quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \ge t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \ge t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \le y_1(t) \le \beta_2, t \ge t_0,\ \ i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad \beta_1 \leq y_i(t) \leq \beta_2, \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ 0 $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \ \rho_1(t, a_1) \ \rho_1^*(t, a_1),$$

$$0 \ Q_1(t) \ Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** **d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \; P_3 \; 0, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \; \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_3 a_3-(f_{30})_3 a_3-\int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3,a_3)da_3\right)u_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0, \qquad (24)$$

$$\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^\infty Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^a(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^l)_3\int_0^\infty Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3), 0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3))\int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\quad da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right) M_3 - [\dot{f}_{2*}]_3 P_3^* W_{\cdot 3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2\,[a_3^2]_3 < 4\,[a_2]_3 < 4\,a_2 e$,

$\mathrm{Re}\,\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4\,a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. 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In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. 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**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

**correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell^{-1}\log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 10. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \text{ 0}, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \text{ ∞},$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 11. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^\infty Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^\infty Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a (\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^\tau \mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a \mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(a_4)da_4-\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(a_4)da_4)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4 = 1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4 \neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\int_0^a \alpha_4(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4 > 0$. Therefore, the characteristic solution is

$$\gamma_4 = -(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(a_4)da_4.$$

Since $f_4^{'}(P_0)_4 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th]

(or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4[a_2]_4<1\, a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)\right|$. Then$|n_4(t)|$ ⊠Δ∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠4$[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])\,{}_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae_4^{mu} \; [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\left[a_{4}\right]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α~|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 12. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ } 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ } \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 13. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0)$$

$$= \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(522)**Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(523) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(524) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(525)**When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(526) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical**

**language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(527)**Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(528)**Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(529)**The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(530)**Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(531)**Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

**Physical applications**

**Some phenomena where quantum gravity plays a major role are the following.**

**The microscopic structure of spacetime**

(532)**If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(533) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(534) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(535) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

**Early cosmology**

(536)**According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(537) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(538) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(539) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(540)**One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

**Black holes**

(541)**Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(542) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(543) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(544)**What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(545) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

**Astrophysical effects**

(546)**Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(547) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(548)**The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

**Main approaches**

(549)**The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out**

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| **certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (550) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (551)**The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

| **String theory** |
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| (552)**String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (553) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (554)**String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (555)**Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (556)**However, when summed the entire series appears to be (=) divergent.** |
| (557)**In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (558)**Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (559)**Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (560) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

| **Loop quantum gravity and spin foams** |
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| (561)**Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (562)**The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (563)**Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (564)**The granular properties of space can be (=) explicitly computed.** |

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| (565) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (566)**The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (567) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (568) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (569)**Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (570)**This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (571)**Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (572) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (573)**It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (574) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (575)**Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (576)**This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (577) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (578)**. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (579) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (580) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection** |

**among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.**

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(581)**These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.**

(582)**The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.**

(583) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.**

(584) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.**

(585)**The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI**

(586) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].**

(587)**This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.**

(588)**More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.**

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(589)**Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.**

(590)**Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.**

(591) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.**

(592)**This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.**

(593)**The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(594)**This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

(595) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.**

(596)**The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society**

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(597)**Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.**

(598)**We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.**

(599)**When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.**

(600)**However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.**

(601)**Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).**

(602)**Bringing them to thermal equilibrium requires (e) an external bath.**

(603) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.**

(604)**The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).**

(605)**The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.**

(606)**But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.**

(607)**Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.**

(608)**For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.**

(609) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system**

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| **at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (610) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (611) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (612)**However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (613)**Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (614) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (615)**As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (616)**His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (617) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (618)**This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (619)**This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (620)**However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (621) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (622)**It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (623) **This is a realistic scenario for (e) all known systems.** |
| (624)**On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (625)**On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (626)**No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |

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| (627) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (628)**Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (629)**Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (630)**Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (631)**So in those systems, the quantum phase transition does indeed exist.** |
| (632)**But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (633)**Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (634)**The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (635)**Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (636)**They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (637)**Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (638)**These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (639)**Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (640)**Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (641)**They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (642) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (643)**Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |

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| (644)**They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (645)**These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (646) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (647)**A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (648)**At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (649)**The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (650)**The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (651) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (652)**The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (653)**The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (654)**"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (655) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (656) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (657)**There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (658)**Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (659)**Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (660) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (661)**Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (662)**They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (663) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (664)**Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (665)**These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (666)**K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (667)**The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (668)**(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (669)**According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (670)**Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (671)**These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (672)**Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (673) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (674)**That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |

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| (675)**Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (676)**The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (677)**The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (678)**"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (679)**Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (680)**The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (681)**However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (682)**The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (683)**"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (684)**If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (685)**Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (686) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (687)**In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (688)**. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (689)**More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (690)**Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (691)**Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (692) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (693)**Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |

(694)**The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.**

(695)**Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.**

(696) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.**

(697)**They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios.**
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**A new look at the galaxy-shaping power of black holes July 6, 2016**

(698)**It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.**

(699)**Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.**

(700)**Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.**

(701)**But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.**

(702)**New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.**

(703)**"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."**

**Champagne bubbles of plasma**

(704)**During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.**

(705) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.**

(706) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.**

(707)**Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun**

**New look at the galaxy-shaping power of black holes**

(708)**Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.**

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| (709)**"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (710) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (711)**This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (712)**The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (713)**The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (714)**The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (715)**Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (716)**"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (717)**"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (718)**That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (719)**Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (720)**In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (721)**Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (722)**"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (723)**"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (724) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (725)**An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (726)**McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (727)**Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (728)**Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (729)**At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (730)**Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (731)**The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (732)**These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (733)**"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (734)**. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (735)**The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |
| (736) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website** |

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

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| (737)**Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.** |
| (738)**They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.** |
| (739)**However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.** |
| (740) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.** |
| (741)**X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.** |

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| (742)**The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.** |
| (743)**These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.** |
| (744)**Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.** |
| (745)**A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.** |
| (746)**Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.** |

**Dark matter guides growth of supermassive black holes February 18, 2015**

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| (747)**This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.** |
| (748)**Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.** |
| (749) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.** |
| (750)**A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.** |
| (751) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (752)**"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (753)**This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (754)**However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (755)**In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (756) **We know dark matter exists only from (e) its gravitational effects.** |
| (757) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (758) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (759)**To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (760) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (761) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |

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| (762)**They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (763)**This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (764) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (765)**Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (766)**"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (767)**The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (768)**Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (769)**Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (770)**"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (771)**Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (772)**"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (773)**Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb)** |

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| **consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (774)**The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (775) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (776)**Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (777)**These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (778)**Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (779)**The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (780)**Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (781) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (782)**"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (783)**"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (784)**The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (785)**Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (786)**"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (787) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

(788)**Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.**

(789) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.**

(790)**Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.**

(791)**"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.**

(792)**The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.**

(793)**However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.**

(794)**The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.**

(795) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.**

(796) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.**

(797)**Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).**

(798)**The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.**

(799)**Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.**

(800)**"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp**

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

(801)**The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.**

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| (802)**The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (803)**(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (804) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (805)**This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (806)**According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (807)**This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (808) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (809)**Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (810)**In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (811) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (812)**Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (813)**They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (814) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (815)**By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (816) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |

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| (817)**The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (818)**As a result, it provides (eb) an important clue for finding the global minimum.** |
| (819)**It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (820)**This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (821)**By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |
| (822)**The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (823) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (824)**Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (825)**"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (826)**As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (827)**"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (828)**In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (829) **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (830)**"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (831)**"For larger numbers, there will likely be (=) multiple funnels.** |
| (832)**Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |

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| (833)**Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (834)**"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (835)**"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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| **@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@** |
| **Introduction:** |
| **Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.** |

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| **(97) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(98) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(99) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(100) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(101) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(102) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(103) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(104) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(105) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(106) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |

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| **(107) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(108) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(109) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(110) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(111) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(112) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(113) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |
| **(114) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(115) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(116) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(117) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(118) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

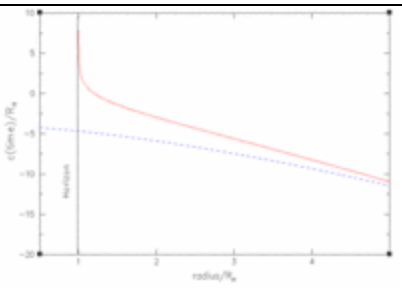
**Stationary black holes**

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| **(119) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(120) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(121) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(122) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(123) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(124) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(125) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(126) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |

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| (127) | In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity. |
| (128) | The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own. |
| (129) | For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole. |
| (130) | This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density. |
| (131) | Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed. |
| (132) | A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole. |
| (133) | In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter. |
| (134) | Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum. |
| (135) | In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior. |
| (136) | The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object. |
| (137) | Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum. |
| (138) | These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.] |
| (139) | All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time. |
| (140) | But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?" |
| (141) | The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point. |
| (142) | One of these coordinates is (=) called the "coordinate time." |
| (143) | Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate. |



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| **(144) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(145) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(146) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(147) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |
| **(148) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(149) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(150) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(151) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(152) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(153) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(154) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(155) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(156) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(157) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |

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| **(158) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(159) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(160) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(161) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(162) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(163) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(164) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(165) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(166) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(167) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(168) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(169) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(170) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(171) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(172) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(173) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(174) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(175) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−√)2+a2].** |
| **(176) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(177) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(178) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(179) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(180) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(181) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(182) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(183) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(184) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(185) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(186) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(187) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(188) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(189) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(190) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(191) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(192) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1 + \frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(\,a_6)da_6 - \int_0^\infty E_6(\,a_6)e^{-\gamma_6 a_6}da_6\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1 + \frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^\infty E_6(\,a_6)da_6) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_6(a_6)Y_6(\gamma_6 a_6)da_6 = 1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6 \neq 0,$$

Since $\int_0^\infty \beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^a \alpha_6\,(\tau_6)Y^{-6}(\gamma_6, \tau_6)d\tau_6 da_6 > 0$. Therefore, the characteristic solution is

$$\gamma_6 = -(f_{60}^{'})_6(\rho_0)_6 \int_0^{\infty} E_6(a_6)da_6.$$

Since $f_6'(P_0)_6 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^{\infty} \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^{\infty} p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}da_{,6}$$

And assume that

$$[k_3]_6\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)([f_2']_6(z_6))\int_0^\infty a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}da_{\square}$$

$-h'(z_6)\int_0^\infty \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\ da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$$H_6(0) = 1 - \int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-y_6^a}} da_6$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r, R\left(f_6(P^*)\right) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^\infty m_6(t,a_6)da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - [p^*]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^\infty \beta_6(a_6)m_6(t,a_6)\,da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $[\dot{f}_{6*}]_6$ denoting $[\dot{f}_6(P^*)]_6[\dot{f}_{2*}]_6$ denoting $[f_2]_6$ $(Q_6^*,\quad h_{6}^*$ denoting $\dot{h}_*$ denoting $\dot{h}_6$ $(Q_6^*)$, and $[\dot{g}_*]_6$

Denoting $[\dot{g}]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - [\dot{f}_{6*}]_6[p_6^*](a_6)M_6 - [\dot{f}_{2*}]_6 p_6^*(a_6)W_6$

$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6}$ (30)

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^{\infty} [m_a]_6\,(t, a_6)da_6 - \int_0^{\infty}\ \mu(a_6)m(t, a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6 \int_0^{\infty} p_6^*(a_6)da_6\right) M_6 - [f_{2*}^6]_6 \int_0^{\infty} p_6^*(a_6)da_6 W_6$$

$= m_6(t, 0) - \int_0^{\infty} \mu_6(a_6)\, m_6(t, a_6)da_6$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right) M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t, a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right) M_6 + \left(\frac{[h_*^{'}]_6}{[h_*]^6}\ P_6^*(0) - f_{2*}^{'} P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g^{'}{}_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t, a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{,6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6$<4 $[a_2]_6$<4 $a_2 e$,

Re $\lambda_6[\lambda_6]_6 = \mathrm{Re}\lambda_2[\lambda_2]_6 = -\frac{6}{2}a_6[\mathrm{a2}\ ]_6 < 0.$

Let $\Delta=\max_{2<u<\infty}\left|\mathrm{h}_*[\mathrm{h}_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then$|n_6(\mathrm{t})|$ 2Δ∀t 0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(\mathrm{a}), \boldsymbol{Q}^*$) is asymptotically stable if either $\mathrm{a}_6^2 < 4\,\mathrm{a}_2\,\mathrm{e}$ and $[\mathrm{a}_6] > 2\Delta$,or $[\mathrm{a}_6^2]$ 4[$\mathrm{a}_2$]e and $(\mathrm{a}_6 - \sqrt{(\mathrm{a}_6^2)_6 - 4(\mathrm{a}_2)_6\,\mathrm{e}} > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\mathrm{a}}{\mathrm{a}+6}\right)$ and $\beta_6(a_6) = \mathrm{ae}^{\text{-ma}}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_6(a_6)\mathrm{e}^{-\int_0^{\mathrm{a}}\mu_6(\tau_6)+d\tau_6}\mathrm{da} = \frac{6}{(\mathrm{m}+\mu_0+1)^2} + \frac{2}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)$=(1+$\omega_6x_6$)/(1 + $x_6$)(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [\mathrm{K}][P_0])}{[m_6]_6 + (\mu_0)_6 + [\mathrm{K}][P_0])_6^3\ [\mathrm{K}]_6[P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ \mathrm{P}^{*}\ [f_{2*}] = \frac{\sigma \mathrm{Q}_6^{*}}{6+\mathrm{Q}_6^{*}} \quad [h_{*}] = \frac{6+\omega_6 \mathrm{Q}_6^{*}}{6+\mathrm{Q}_6^{*}}$

$[f_{6*}] = K_6\ \left[\mathrm{f}_{2*}^{\ 6}\right] = \frac{\sigma_6}{(6+\mathrm{Q}_6^{*})^2}[\ \mathrm{g}_{*}^{\ 6}] = \frac{\sigma_6}{(6+\mathrm{P}_6^{*})^2} \quad \left[\mathrm{h}_{*}^{\ 6}\right] = \frac{\omega_6 - 6}{(6+\mathrm{Q}_6^{*})^2}$

In order to have (H3) and (H4) hold, $m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_{\ 6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^{\ 2}} + \frac{2\omega_6}{([m_6]_{\ 6} + [\mu_0]_{\ 6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^{\ 3}} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_{\ 6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\ 2}} + \frac{2\omega_6}{(m_6 + [\mu_0]_{\ 6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the following equation:

$(m_6 + [\mu_0]_{\ 6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_{\ 6} + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\ \mathrm{Q}_6^{*})^{\ 3}$

$= m_6 + [\mu_0]_{\ 6} + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + \mathrm{Q}^{*})(1 + \omega_6 \mathrm{Q}_6^{*}$

Now

$$\left[\mathrm{a}_{6}^{\ 2}\right] - 4\left[\mathrm{a}_{2}\right]\mathrm{e} = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6\, \mathrm{Q}_6^{*}}{(\delta_6 - b_6)(1 + \mathrm{Q}_6^{*})} + \frac{\sigma_6^{\ 2}\, \mathrm{Q}_6^{*2}}{(1 + \mathrm{Q}_6^{*})^{\ 2}}$$

$$\frac{4\,b_6\sigma_6\,(\delta_{\,6}-\mathrm{b})\mathrm{Q}^{\,*}}{\delta_{\,6}\,(1+\mathrm{Q}_{\,6}^{\,*})^{\,2}}-\frac{4\,(\delta_{\,6}-b_6)^{\,2}\,(1-\omega_6)\mathrm{Q}_{\,6}^{\,*}}{\delta_{\,6}\,(1+\omega_6\mathrm{Q}_{\,6}^{\,*})1+\mathrm{Q}_{\,6}^{\,*})}\mathrm{P}^{\,*}(0)$$

And

$\Delta=\lim_{0<a<\infty}|[h_*]_{\,6}\,ae_{\,6}^{\,mu}\;\;[\mu_0]_{\,6}-\frac{a_6}{a_6+6}$

Hence if $[\mathrm{a}_{\,6}^{\,2}>4(a_2)_{\,6}\mathrm{e}$ and

$$[a_6]=\frac{2K_6b_6}{\delta_6-b_6}+\frac{\sigma_6\,\mathrm{Q}_{\,6}^{\,*}}{6+\mathrm{Q}_{\,6}^{\,*}}>2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6}=[f_{6*}]_{\,6}+[f_{2*}]_{\,6}+[f_{6*}]\,\mathrm{P}_{\,6}^{\,*}$$

$$[a_2]_{\,6}=\left[\mathrm{f}_{\,2\,*}^{\,6}\right]\mathrm{P}_{\,6}^{\,*}-\frac{\mathrm{h}_{\,*}^{\,6}}{\mathrm{h}_{\,*}}\mathrm{P}^{\,*}(0)$$

$$e=[\,g^{6}_{*}]_{6}\,Q^{*}_{6}$$

$$\eta_6(t) = \left[h_{*}\right]_{6}\beta_{6}\left(\theta_{6}(t)\right) - \mu_{6}\left(\theta(t)\right),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_{6}\right]M_6 - [a_2]_{6}W_6 - [n(t)]M_6$$

$$w_6(t) = eM_6$$

Write

$$A=\begin{pmatrix} -[a_6]_{6} - [a_2]_{6} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{6} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(t) = e^{A,t}[X]_{6}(0) + \int_0 e^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit

case $\beta\to\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb)

local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer

that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find

that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces

(eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas**

**Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as:

arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=)

both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution

of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph);

Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual

to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb)

anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25**

**September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting

nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time

derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in

recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity.

Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current

fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks

that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent

interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across**

**Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the**

**excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb)**

**Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene**

**expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff**

**scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that**

**spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential**

**decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find**

**the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and**

**through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the**

**regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a**

**claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb)**

**action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory**

**shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP**

**Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph]**

**(or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is**

**r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices.**

**Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and**

**rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 14. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \ P_7 \ 1, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \ \infty,$$

$$\rho_7(t,0) = \int_0^{\infty} n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7\left[\frac{dQ_7}{dt}\right]_7 - x_7\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(836)(Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(837)In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(838)As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(839) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(840)But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(841)For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(842) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(843) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(844)It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(845)Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(846)The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(847)The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(848)Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(849)This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(850)Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(851)Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

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| (852)The electronic and (e&eb) structural variability among them is extraordinary. |
| (853)The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. |
| (854)The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. |
| (855) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. |
| (856)Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp |

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

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| (857)Nudge your bath tub just right and (eb) you might end up reversing time. |
| (858)Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected. |
| (859)To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse |
| (860)Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses. |
| (861)But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort. |
| (862)The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort. |
| (863)Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower |
| (864)Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower |

**Déjà vu**

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| (865)To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple |

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| of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape. |
| (866)The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says. |
| (867)Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says. |
| (868)"The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810 |

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| **Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810** |

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| (869)Wave control is usually performed by (e) spatially engineering the properties of a medium. |
| (870) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach. |
| (871) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors. |
| (872)Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source. |
| (873)They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation. |
| (874)Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption. |
| (875)The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control. |

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| **Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)** |

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| (876)Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice. |
| (877)Quantum mechanics rules the dynamics of (e) light and matter. |

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| (878) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved. |
| (879) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1]. |
| (880)Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest. |
| (881)The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (882)It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (883)Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (884)The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (885)All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (886)When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (887) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (888)To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (889). Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (890)As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (891)Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (892) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (893)Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (894)However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (895) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (896)So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |

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| (897)Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (898)The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (899) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (900)The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (901) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (902) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (903)Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (904)This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (905)First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (906) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (907)Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (908)A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (909)Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (910) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (911)The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (912) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (913) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (914) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (915)Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (916)Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (917)This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |

(918) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems.

(919)In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X.

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

(920)Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior.

(921) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques.

(922) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators

(923) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice.

(924)This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(925) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7 P_7) + \big(f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7) \int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \leq \rho_7(t, a_7) \leq \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a)e^{-\int_0^a \mu_7(\tau)dt}e^{-ra}\,da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(t)dt}\,da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^{\infty} [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t,a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7 Q_7 - c_7 P_7 Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 15. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^{\infty} \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7 (P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t,a_7) = \rho(t,a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t, 0) = \rho_7(t, 0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t, a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t, a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0^{'}Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^{\infty} \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}^{'})(\rho_0)_7Q_7$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^{\infty} \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^{\infty} u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_7}\,da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\alpha_7(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_7\,(a_7) Y_7(\gamma_7, a_7) da_7\right) u_7(0)$$

$$+w \int_0^\infty \beta_7(a_7) Y_7(\gamma_7, a_7) \int_0^a (\alpha_7)_7(\tau)_7 (Y^{-7})_7(\gamma_7, \tau_7) d\tau_7\, da_7 = 0,$$

(24)

$$\left(\int_0^\infty Y_7(\gamma_7, a_7) da_7\right) \tilde{u}_7(0)$$

$$+\left(-1 - \int_0^\infty Y_7(\gamma_7, a_7) \int_0^a (\alpha_7)_7(\tau)_7 (Y^{-7})_7(\gamma_7, \tau_7) d\tau_7\, da_7\right) w = 0.$$

Equation (20) has a solution of the form of $u_7(t, a_7) = u_7(a_7) e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) Y^{-7}\,(\gamma_7, a_7) d\tau_7$. We obtain the characteristic equation

$$f_{70_7}^l \left(\int_0^\infty Y_7(\gamma_7, a_7) da_7\right)\left(\int_0^\infty \beta_7(a_7) Y_7(\gamma_7, a_7) I_7(\gamma_7, a_7) da_7\right)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7, a_7) da\right)\left(1 + (f_{70}^l)_7 \int_0^\infty Y_7(\gamma_7, a_7) I_7(\gamma_7, a_7) da_7\right)$$

$$= 0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7) = (\rho_0)_7(0)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - (f_{70})_7 a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7)e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7}d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1+\frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^{\infty}E_7(a_7)da_7)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7 a_7)da_7=1-\int_0^{\infty}\beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\neq 0,$$

Since $\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7 a_7)\int_0^{a}\alpha_7(\tau_7)Y^{-7}(\gamma_7,\tau_7)d\tau_7 da_7>0$. Therefore, the characteristic solution is

$$\gamma_7=-(f_{70}^{'})_7(\rho_0)_7\int_0^{\infty}E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7),0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^\infty \beta_7(a_7)p_7^*(a_j)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7 \int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty} [H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)(\,[f_7^{'}]_7(z_7)\,) \int_0^{\infty} a_7 \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7) \int_0^{\infty} \beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\, da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7} da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty} \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}} da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\big(f_7(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$M_7(t) \equiv \int_0^\infty m_7(t, a_7)da_7 = P_7(t) - P_7^*,$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7)m_7(t,a_7)\,da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0)W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[\dot{f_{7*}}\right]_7$ denoting $\left[\dot{f}_7(P^*)\right]_7\left[\dot{f_{7*}}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,\quad h_7^*$ denoting $\dot{h_*}$ denoting $\dot{h_7}$ $(Q_7^*)$, and $\left[\dot{g_*}\right]_7$

Denoting $\left[\dot{g}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - \left[\dot{f}_{7*}\right]_7[p_7^*](a_7)M_7 - \left[\dot{f_{7*}}\right]_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + \left[\dot{f}_{7*}\right]_7\int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7\int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + \left[\dot{f_{7*}}\right]_7 P_7^*\right)M_7 - \left[\dot{f_{7*}}\right]_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^{\infty} ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f'_{7*}]_7 P_7^*\right) M_7 + \left(\frac{[h'_*]_7}{[h_*]^7}\, P_7^*(0) - f'_{7*}P_7^*\right) W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g'_*\right]_7 Q_7^*\, M_{.7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $^7_7$ [a $^7_7$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_7[\lambda_7]_7 = \text{Re}\lambda_7[\lambda_7]_7 = -\frac{7}{7}a_7[\text{a2}\ ]_7 < 0$.

Let ⊿=max$_{7<u<\infty}$ |h $_*$ [h $_*$] $_7$β $_7$($a_7$) − $\mu_7(a_7)$|. Then|$n_7$(t)| ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $^7_7$ <4 a $_7$ e and [a $_7$] > 1⊿,or [a $^7_7$] [a $_7$]e and (a $_7$ − $\sqrt{(\text{a}^7_7)_7 - 4(\text{a}_7)_7\,\text{e})} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m + \mu_0 + 1)^7} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7 x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7 x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [K][P_0])}{[m_7]_7 + (\mu_0)_7 + [K][P_0])_7^3\ [K]_7 [P_0]_7^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ P^*\ [f_{7*}] = \frac{\sigma Q_7^*}{7+Q_7^*}$ $[h_*] = \frac{7+\omega_7 Q_7^*}{7+Q_7^*}$

$[f_{7*}] = K_7\ [f_{7*}^7] = \frac{\sigma_7}{(7+Q_7^*)^7} [\ g_*^7] = \frac{\sigma_7}{(7+P_7^*)^7}$ $[h_*^7] = \frac{\omega_7 - 7}{(7+Q_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^7} + \frac{1\omega_7}{([m_7]_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^3} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^7} + \frac{1\omega_7}{(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_7\right]e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7 Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^7 Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7 (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\, (\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae_7^{mu}\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7 \mathrm{Q}_7^*}{1 + \mathrm{Q}_7^*} > 1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the

model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(195)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(196) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(197)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(198) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(199)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(200) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

| |
|---|
| (201)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (202)Humans are not very good at understanding interacting many-body systems. |
| (203)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (204)But what happens when we couple (e&eb) the particles strongly? |
| (205)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |
| (206)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (207)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (208) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (209) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (210)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (211) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (212)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (213) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (214) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (215)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (216)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |

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| (217) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (218) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (219)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (220)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (221)When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (222)The energy levels of these quantized orbitals take on (e) discrete values |
| (223):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (224) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (225) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (226)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (227) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (228)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (229) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (230)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (231)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (232)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (233) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (234)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (235)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (236)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (237) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |

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| (238)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (239)The remarkable thing is that solitons can become (=) "elementary," in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (240)An example of a duality, which we will use below, arises in (eb) Maxwell's theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (241)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (242)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (243)Another classic example of duality involves (e&eb) vortices. |
| (244) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (245)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (246) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (247)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (248)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (249) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (250)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (251)TIs are distinguished only by (e) their edge states. |
| (252) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (253)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (254) But the Dirac cone is not (e) the only possible signature. |
| (255) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (256)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (257)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (258)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |

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| (259) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (260). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (261)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (262)The bulk duality implies (eb) a duality between edge theories. |
| (263)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (264)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (265) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (266) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (267)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |
| (268)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (269)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (270)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (271)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (272)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (273) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (274)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (275)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (276) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (277)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |

(278)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version)

IF we define

$$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,P_7^*$$

$$[a_7]_7 = \left[f_{7*}^{7}\right] P_7^* - \frac{h_*^7}{h_*} P^*(0)$$

$$e=[g_*^7]_7\, Q_7^*$$

$$\eta_7(t) = \left[h_*\right]_7 \beta_7\left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

### 16. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(926)Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a

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| link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement. |
| (927)Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature. |
| (928) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics. |
| (929)"Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related." |
| (930)Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules. |
| (931) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (932) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (933)At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (934) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (935)And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (936)Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (937)One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (938) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (939)"The underlying physics is (=) chaos, essentially," he said. |
| (940)The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (941)But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (942) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (943)To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (944) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (945) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |

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| (946)By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (947)The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (948)The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (949)"There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |
| (950) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (951)It turns out that they are actually the driving forces behind (eb) thermalization. |
| (952)"What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (953)The study's findings have fundamental implications for (e) quantum computing. |
| (954) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (955)"It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (956)A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (957) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (958)Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (959)The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'-the basic component of a quantum computer. |
| (960)Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (961)By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (962)The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |

(963)Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device.

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(964)Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(965)As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(966) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(967) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(968)Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(969)A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(970)For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(971) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(972)The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(973)Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(974)H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(975)This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(976)Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(977)At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(978)Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(979)The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(980) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(981)However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(982) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(983)In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(984)Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(985) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(986) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(987)When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(988) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(989)Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(990)Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(991)The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(992)Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(993)Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(994)If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(995) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(996) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(997) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(998)According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(999) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(1000) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(1001) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(1002) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(1003) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(1004) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(1005) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(1006) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(1007) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(1008) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(1009) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(1010) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(1011) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(1012) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(1013) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that

ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(1014) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(1015) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(1016) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(1017) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(1018) However, when summed the entire series appears to be (=) divergent.

(1019) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(1020) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(1021) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(1022) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(1023) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(1024) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(1025) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(1026) The granular properties of space can be (=) explicitly computed.

(1027) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(1028) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(1029) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(1030) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(1031) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(1032) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(1033) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(1034) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(1035) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(1036) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(1037) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(1038) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(1039) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(1040) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(1041) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(1042) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ 1, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \infty,$$

$$\rho_8(t,0) = \int_0^\infty n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \le \rho_8(t, a_8) \le \rho_8^*(t, a_8),$$

$$0 \le Q_8(t) \le Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t, 0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t, a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8 Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t, a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a) e^{-\int_0^a \mu_8(\tau)dt} e^{-ra}\, da_8 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(t)dt}\, da_8 > 1, \qquad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8Q_8 - c_8P_8Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 17. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8e^{r_8t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8\,(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \quad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \quad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0'Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}')(\rho_0)_8 Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8 u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8 a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t,a_8)da_8$$

$$u_8(t,0) = \int_0^\infty \beta_8\,(a_8)u_8(t,a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t,a_8) = \tilde{\mathrm{u}}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_8)$ gives,

$$\tilde{\mathrm{u}}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{\mathrm{u}}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a(\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8\,(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8=0,$$

(24)

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty} Y_8(\gamma_8,a_8)\int_0^{a}(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80}^{l}{}_8\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8, a_8) = E_8(a_8)e^{-\gamma_8 a_8}$. by a simple calculation,

$$\int_0^\infty \beta_8(a_8)Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma}\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}\ da\right);$$

$$\int_0^\infty Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\ (\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1 + \frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(\,a_8)da_8 - \int_0^\infty E_8(\,a_8)e^{-\gamma_8 a_8}da_8\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1 + \frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(\,a_8)da_8) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8\ (\tau_8)Y^{-8}(\gamma_8, \tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8 \int_0^{\infty} E_8(a_8)da_8.$$

Since $f_8'(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^{\infty} \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^{\infty} p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8} da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8} da_{,8}$$

And assume that

$$[k_3]_8\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)([f_8']_8(z_8))\int_0^\infty a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8} da_{\square}$$

$-h'(z_8)\int_0^\infty \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\ da_8$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8) m_8(t,a_8)\, da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0) W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting

$[f_8]_8$ $(Q_8^*,$ $h_{*}^{'}$ denoting $h_{*}^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[g^{'}\right]_8 (P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8) m_8 - \left[f^{'}{}_{8*}\right]_8 [p_8^*](a_8) M_8 - \left[f_{8*}^{'}\right]_8 p_8^*(a_8) W_8$

$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8) m_8(t,a_8) da_8 + \frac{[h_*^8]_8}{[h^*]_8} [p^*]_8(0)\, W_{,8}$ (30)

$$\frac{dW_8}{dt} = [g_*^8]_8 [Q_8^*] M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8) da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8) da_8 - \int_0^\infty \mu(a_8) m(t,a_8) da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}{}_{8*}\right]_8 \int_0^\infty p_8^*(a_8) da_8\right) M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8) da_8 W_8$$

$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8) da_8$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f_{8*}^{\cdot}\right]_8 P_8^*\right) M_8 - \left[f_{8*}^{\cdot}\right]_8 P_8^* W_{\cdot 8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8 \beta_8(a_8) - \mu_8(a_8)) m_8(t,a_8) da_8$$

$$-\left([f_{8*}] + [f_{8*}]_8 + \left[f_{8*}^{\cdot}\right]_8 P_8^*\right) M_8 + \left(\frac{[h_*^{'}]_8}{[h_*]^8}\, P_8^*(0) - f_{8*}^{'} P_8^*\right) W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\, M_{\cdot 8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_8 \beta_8(a_8) - \mu_8(a_8)) m_8(t,a_8) da_8 = ([h_*]_8 \beta_8(\theta_8) - \mu_8(\theta_8)) M_{\cdot 8}$$

Where $0< \theta_8 = \theta_8(t) < \infty$.

And if $a_8^8 [a_8^8] 2<4 [a_8] 2<4 a_8 e$,

$\mathrm{Re}\,\lambda_8 [\lambda_8]_8 = \mathrm{Re}\lambda_8 [\lambda_8]_8 = -\frac{8}{8} a_8 [a2\ ]_8 < 0.$

Let $\Delta = \max_{8<u<\infty} \left|h_* [h_*]_8 \beta_8(a_8) - \mu_8(a_8)\right|$. Then $|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_8^8 < 4\,a_8\,e$ and $[a_8] > 1\Delta$,or $[a_8^8] \geq 4[a_8]e$ and $(a_8 - \sqrt{(a_8^8)_8 - 4(a_8)_8\,e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+8}\right)$ and $\beta_8(a_8) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8) + d\tau_8}da = \frac{1}{(m+\mu_0+1)^8} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0] \geq 0.5$. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\,x_8/(1 + x_8)$and $h_8(x_8)=(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [K][P_0])}{[m_8]_8 + (\mu_0)_8 + [K][P_0])_8^3\ [K]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)\ [P_0]_8(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ P^*\ [f_{8*}] = \frac{\sigma Q_8^*}{8+Q_8^*} \quad [h_*] = \frac{8+\omega_8 Q_8^*}{8+Q_8^*}$

$[f_{8*}] = K_8 \left[\mathrm{f}_{8*}^{\ 8}\right] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{\ 8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8} \quad \left[\mathrm{h}_*^{\ 8}\right] = \frac{\omega_8 - 8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold, $m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^{\ 8}} + \frac{1\omega_8}{([m_8]_{\ 8} + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^{\ 3}} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 8}} + \frac{1\omega_8}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_{\ 8} + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\ \mathrm{Q}_8^*)^{\ 3}$

$= m_8 + [\mu_0]_{\ 8} + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + \mathrm{Q}^*)(1 + \omega_8 \mathrm{Q}_8^*$

Now

$$\left[\mathrm{a}_{\ 1}^{\ 1}\right] - 4\left[\mathrm{a}_{\ 8}\right]\mathrm{e} = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8\, \mathrm{Q}_8^*}{(\delta_8 - b_8)(8 + \mathrm{Q}_8^*)} + \frac{\sigma_8^{\ 8}\, \mathrm{Q}_8^{*8}}{(1 + \mathrm{Q}_8^*)^{\ 8}}$$

$$\frac{4\,b_8\sigma_8\,(\delta_{\ 8}-\mathrm{b})\mathrm{Q}^{\ *}}{\delta_{\ 8}\,(1+\mathrm{Q}_{8}^{*})^{\ 8}}-\frac{4\,(\delta_{\ 8}-b_8)^{\ 8}\,(1-\omega_8)\mathrm{Q}_{\ 8}^{*}}{\delta_{\ 8}\,(1+\omega_8\mathrm{Q}_{8}^{*})(1+\mathrm{Q}_{8}^{*})}\mathrm{P}^{\ *}(0)$$

And

$$\Delta=\ \lim_{0<a<\infty}|[h_*]_{\ 8}\,ae_{\ 8}^{mu}\ \ [\mu_0]_{\ 8}-\frac{a_8}{a_8+8}$$

Hence if $[\mathrm{a}_{\ 8}^{\ 8}>4(a_8)_{\ 8}\mathrm{e}$ and

$$[a_8]=\frac{1K_8b_8}{\delta_8-b_8}+\frac{\sigma_8\,\mathrm{Q}_{\ 8}^{\ *}}{1+\mathrm{Q}_{\ 8}^{*}}>1\Delta$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\ 8}=[f_{8*}]_{\ 8}+[f_{8*}]_{\ 8}+[f_{8*}]\,\mathrm{P}_{\ 8}^{\ *}$$

$$[a_8]_{\ 8}=\left[\mathrm{f}_{8\,*}^{\ 8}\right]\mathrm{P}_{\ 8}^{\ *}-\frac{\mathrm{h}_{\ *}^{\ 8}}{\mathrm{h}_{\ *}}\mathrm{P}^{\ *}(0)$$

$$\mathrm{e}=[\,\mathrm{g}_{\ *}^{\ 8}]\ _8\,\mathrm{Q}_{\ 8}^{\ *}$$

$$\eta_8(\mathrm{t})=\left[\mathrm{h}_{\ *}\right]_{\ 8}\beta_{\ 8}\left(\theta_{\ 8}(\mathrm{t})\right)-\mu_{\ 8}\left(\theta(\mathrm{t})\right),$$

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{8}\right]M_8 - [a_8]_{\,8}W_8 - [\mathrm{n(t)}]M_8$$

$$w_8(\mathrm{t}) = \mathrm{e}M_8$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_8]_{\,8} & -[a_8]_{\,8} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{\,8} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$$

The solution of (32)is

$$X_8(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{\,8}(0) + \int_0 \mathrm{e}^{At}$$

This equation notwithstanding its importance in all the models stand out for its importance for the following models

to which it is applied:

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(164)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.

| |
|---|
| (165)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (166) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (167) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (168)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (169) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (170)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (171)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

| |
|---|
| (172)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems. |
| (173)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions. |
| (174) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field. |
| (175)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface. |
| (176)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator. |
| (177)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states. |
| (178) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions. |
| (179)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(180)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(181)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(182)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(183)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(184)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(185)Bringing them to thermal equilibrium requires (e) an external bath.

(186) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(187)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(188)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(189)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(190)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(191)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(192) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(193) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(194) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(195)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(196)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

| |
|---|
| (197) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL. |
| (198)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked. |
| (199)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice. |
| (200) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site. |
| (201)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself). |
| (202)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition. |
| (203)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (204) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (205)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (206) This is a realistic scenario for (e) all known systems. |
| (207)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (208)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (209)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (210) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (211)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (212)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (213)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (214)So in those systems, the quantum phase transition does indeed exist. |
| (215)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (216)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus |

find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity**

**avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2} \in [-0.99, 0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2} \in [2,48] M_\odot$, and (e&eb) total mass $M_{total} \leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced**

**LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 18. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \ P_9 \ 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \ \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth

rate with $\beta(a) \geq 0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h'(Q) \leq 0, \forall Q \geq 0$, and $\lim_{Q\to\infty} h(Q) = k_3 \geq 0$.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9 P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9) \int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \leq \rho_9(t, a_9) \leq \rho_9^*(t, a_9),$$

$$0 \leq Q_9(t) \leq Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9)da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9)da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \tag{10}$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \tag{$H1$}$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9Q_9 + c_9P_9Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9P_9 - \lambda_9P_9Q_9$$

$$\frac{dQ_9}{dt} = b_9Q_9 - c_9P_9Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t, 0)$$

$$= \int_0^\infty \beta_9\,(a_9)\rho_9(t, a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t, a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9\,(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9\,(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9 u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t, a_9) = \tilde{u}_9(a_9)^{e^{\gamma(t-a_9)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_9}\, da_9$. Then,

A solution of the first differential equation for $\tilde{u}(a_9)$ gives,

$$\tilde{u}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{u}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})\tau_9}\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a (\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds}\,d\tau_9\, da_9$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\,da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds + y\tau}\,\alpha_9(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty}Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90_9}^{l}\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^{a}\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9, a_9) = \int_0^a (\rho_0)_9(\tau_9) e^{\gamma_9\tau_9 + (f_{90})_9\tau + \int_0^\tau \mu_9(s_9)ds_9} d\tau$$

$$= \frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9 a_9} - 1)$$

Let

$$E_9(a_9) = e^{-(f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) da_9 = 1 - \int_0^\infty \beta_9(a_9) E_9(a_9) e^{-\gamma_9 a_9} da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) \int_0^a \alpha_9(\tau_9) Y^{-9}(\gamma_9, \tau_9) d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}')_9 (\rho_0)_9 \int_0^\infty E_9(a_9) da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$p_9^*(0) = h_9(Q_9^*) \int_0^\infty \beta_9(a_9) p_9^*(a_J) p_9^*(a_9) da_9$ (27)

$g_9(P_9^*) - b_9 = 0,$ (28)

Where $P_9^* = \int_0^\infty p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b),$

From (26),

$p_9^*(a) = p_9^*(0) e^{\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9}$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*) \int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9} da_9\right) p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9) \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(z_9) a_9} da_{,9}$$

And assume that

$$[k_3]_9 \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9 a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left([f_9^{'}]_9(z_9)\right)\int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9)\int_0^{\infty}\beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}\; da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty}\beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty}\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y_9^a} da_9}$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r$, $R\left(f_9(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$M_9(t) \equiv \int_0^\infty m_9(t, a_9)da_9 = P_9(t) - P_9^*,$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - \left[p^*\right]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9)m_9(t,a_9)\,da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0)W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $\left[\dot{f}_{9*}\right]_9$ denoting $\left[f^{'}{}_9(P^*)\right]_9 \left[\dot{f}_{9*}\right]_9$ denoting $[f_9]_9$ $(Q_9^*,$ $h_{9}^{*}$ denoting $\dot{h}_*$ denoting $\dot{h_9}$ $(Q_9^*)$, and $\left[\dot{g}_*\right]_9$

Denoting $\left[\dot{g}\right]_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9[p_9^*](a_9)M_9 - [f'_{9*}]_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9)m_9(t,a_9)da_9 + \frac{[h_*^9]_9}{[h^*]_9}[p^*]_9(0)\,W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9[Q_9^*]M_9$$

Integrating $[m_9]_9(t,a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t,a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t,a_9)da_9 - \int_0^\infty \mu(a_9)m(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f'_{9*}]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$$= m_9(t,0) - \int_0^\infty \mu_9(a_9)\,m_9(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}']_9 P_9^*\right)M_9 - [f_{9*}']_9 P_9^* W_{,9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + [f_{9*}']_9 P_9^*\right)M_9 + \left(\frac{[h_*']_9}{[h_*]^9}\,P_9^*(0) - f_{9*}' P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_{*}\right]_9 Q_9^{*} M_{.9}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9 = ([h_{*}]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9[a_9^9]2<4\ [a_9]2<4\ a_9e$,

$\text{Re}\,\lambda_9[\lambda_9]_9 = \text{Re}\lambda_9[\lambda_9]_9 = -\frac{9}{9}a_9[a2\ ]_9 < 0.$

Let $\Delta=\max_{9<u<\infty}\left|h_{*}[h_{*}]_9\beta_9(a_9) - \mu_9(a_9)\right|$. Then$|n_9(t)|$ 2Δ∀t 0. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{*}(a), \boldsymbol{Q}^{*})$ is asymptotically stable if either $a_9^9 <4\ a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ 4$[a_9]e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae\text{-}^{ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9) + d\tau_9}da = \frac{1}{(m+\mu_0+1)^9} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ 0.5. Assume$[f_9](x_9)= K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)= [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9)=(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])\,{}^{3}_{9}\ [\mathrm{K}]_{\,9}[P_0]\,{}^{3}_{9}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)\ [P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$ $[h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}\,{}^{9}_{9*}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}[\,\mathrm{g}\,{}^{9}_{*}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9}$ $[\mathrm{h}\,{}^{9}_{*}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_9 + [\mu_0]_9 + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_9 + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\, Q_9^*)^3$$

$$= m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1 + Q^*)(1 + \omega_9 Q_9^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9\, Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9\, Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9\, (\delta_9 - b)Q^*}{\delta_9\, (1 + Q_9^*)^9} - \frac{4\, (\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{mu}_9\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9\, Q_9^*}{1 + Q_9^*} > 1\Delta$$

| |
|---|
| |
| The nontrivial equilibrium is stable. |
| |
| IF we define |
| $$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$ |
| $$[a_9]_9 = \left[f^9_{9*}\right]P^*_9 - \frac{h^9_*}{h_*}P^*(0)$$ |
| $$e=[g^9_*]_9\,Q^*_9$$ |
| $$\eta_9(t) = \left[h_*\right]_9\beta_9\left(\theta_9(t)\right) - \mu_9(\theta(t)),$$ |
| The system(32)can be written as the following system: |
| $$[M(t)]\left[a_9\right]M_9 - [a_9]_9W_9 - [n(t)]M_9$$ |
| $$w_9(t) = eM_9$$ |
| Write |
| $$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \quad 0 \end{pmatrix}$$ |
| $$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$ |

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(t) = e^{A,t}[X]_9(0) + \int_0 e^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual**

**oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of**

**classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed**

**huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy**

**holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO

## Theory Of Quantum Error-Correcting Codes

### INTRODUCTION—VARIABLES USED

**A Theory of Quantum Error-Correcting Codes Emanuel Knill, Raymond Laflamme (Submitted on 26 Apr 1996**)

(1) Quantum Error Correction will be necessary for preserving coherent states against (e&eb) noise and other unwanted interactions in quantum computation and communication.
(2) They develop a general theory of quantum error correction based on (e) encoding states into larger Hilbert spaces subject to (e&eb) known interactions.
(3) Authors obtain necessary and sufficient conditions for (e) the perfect recovery of an encoded state after its degradation by (e) an interaction.
(4) The conditions depend only on (=) the behavior of the logical states.
(5) Authors use them to give a recovery operator independent definition of (e) error-correcting codes.
(6) Authors relate this definition to (e&eb) four others: The existence of a left inverse of (e) the interaction, an explicit representation of (e) the error syndrome using (e) tensor products, perfect recovery of (e) the completely entangled state, and an information theoretic identity.
(7) Two notions of fidelity and error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states.
(8) The latter is more appropriate when using (e) codes in a quantum memory or in applications of (e) quantum teleportation to communication.
(9) Authors show that the error for entangled states is bounded linearly by (e) the error for pure states.
(10) A formal definition of independent interactions for (e) qubits is given.
**(11)** This leads to (eb) lower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component. Subjects: Quantum Physics (quant-ph) Journal reference:Phys.Rev.Lett.84:2525-2528,2000 DOI: 10.1103/PhysRevLett.84.2525 Report

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| number: LA-UR 96-1300 Cite as: arXiv:quant-ph/9604034 (or arXiv:quant-ph/9604034v1 for this version) | |
| **NOTATION** | |
| **Module One**<br>The methods also allow (eb) isolation of quantum communication from (e) noise and eavesdropping (quantum privacy amplification). | |
| $G_{13}$ : Category one of noise and eavesdropping (quantum privacy amplification).<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of isolation of quantum communication<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>Quantum Error Correction will be necessary for preserving coherent states against (e&eb) noise and other unwanted interactions in quantum computation and communication | |
| $G_{16}$ : Category one of Quantum Error Correction<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of preserving coherent states against (e&eb) noise and other unwanted interactions in quantum computation and communication<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>Quantum Error Correction will be necessary for preserving coherent states against (e&eb) noise and other unwanted interactions in quantum computation and communication | |
| $G_{20}$ : Category one of **Quantum Error Correction will be necessary for preserving coherent states;** noise and other unwanted interactions in quantum computation and communication<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of noise and other unwanted interactions in quantum computation and communication **;Quantum Error Correction will be necessary for preserving coherent states** | |

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| $T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>They develop a general theory of quantum error correction based on (e) encoding states into larger Hilbert spaces subject to (e&eb) known interactions | |
| $G_{24}$ : Category one of encoding states into larger Hilbert spaces subject to (e&eb) known interactions<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of general theory of quantum error correction<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>They develop a **general theory of quantum error correction based on encoding states into larger Hilbert spaces** subject to (e&eb) known interactions | |
| $G_{28}$ : Category one of **general theory of quantum error correction based on encoding states into larger Hilbert spaces;** known interactions<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of known interactions; **general theory of quantum error correction based on encoding states into larger Hilbert spaces**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors obtain necessary and sufficient conditions for (e) the perfect recovery of an encoded state after its degradation by (e) an interaction | |
| $G_{32}$ : Category one of perfect recovery of an encoded state after its degradation by (e) an interaction<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of necessary and sufficient conditions | |

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| $T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors obtain necessary and sufficient conditions for the perfect recovery of an encoded state after its degradation by (e) an interaction | |
| $G_{36}$ : Category one of interactions<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of necessary and sufficient conditions for the perfect recovery of an encoded state after its degradation<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The conditions depend only on (=) the behavior of the logical states | |
| $G_{40}$ : Category one of conditions depend<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of behavior of the logical states<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>Authors use them to give a recovery operator independent definition of error-correcting codes | |
| $G_{44}$ : Category one of error-correcting codes<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |

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| $T_{44}$ : Category one of recovery operator independent definition<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |

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|---|---|
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |

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| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |

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| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}')^{(1)} & -(b_{15}'')^{(1)}(G,t) & -(b_{18}'')^{(2,2,)}(G_{19},t) & -(b_{22}'')^{(3,3,)}(G_{23},t) \\ -(b_{26}'')^{(4,4,4,4,)}(G_{27},t) & -(b_{30}'')^{(5,5,5,5,)}(G_{31},t) & -(b_{34}'')^{(6,6,6,6,)}(G_{35},t) & \\ -(b_{38}'')^{(7,7,)}(G_{39},t) & -(b_{42}'')^{(8,8)}(G_{43},t) & -(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}'')^{(1)}(G,t)$, $-(b_{14}'')^{(1)}(G,t)$, $-(b_{15}'')^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}'')^{(2,2,)}(G_{19},t)$, $-(b_{17}'')^{(2,2,)}(G_{19},t)$, $-(b_{18}'')^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}'')^{(3,3,)}(G_{23},t)$, $-(b_{21}'')^{(3,3,)}(G_{23},t)$, $-(b_{22}'')^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}'')^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}'')^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}'')^{(7,7,)}(G_{39},t)$, $-(b_{36}'')^{(7,7,)}(G_{39},t)$, $-(b_{38}'')^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}'')^{(8,8)}(G_{43},t)$ $-(b_{41}'')^{(8,8)}(G_{43},t)$ $-(b_{42}'')^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}'')^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}')^{(2)} & +(a_{16}'')^{(2)}(T_{17},t) & +(a_{13}'')^{(1,1,)}(T_{14},t) & +(a_{20}'')^{(3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}'')^{(7,7,7)}(T_{37},t) & +(a_{40}'')^{(8,8,8)}(T_{41},t) & +(a_{44}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}')^{(2)} & +(a_{17}'')^{(2)}(T_{17},t) & +(a_{14}'')^{(1,1,)}(T_{14},t) & +(a_{21}'')^{(3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}'')^{(7,7,7)}(T_{37},t) & +(a_{41}'')^{(8,8,8)}(T_{41},t) & +(a_{45}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}')^{(2)} & +(a_{18}'')^{(2)}(T_{17},t) & +(a_{15}'')^{(1,1,)}(T_{14},t) & +(a_{22}'')^{(3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6)}(T_{33},t) & \\ +(a_{38}'')^{(7,7,7)}(T_{37},t) & +(a_{42}'')^{(8,8,8)}(T_{41},t) & +(a_{46}'')^{(9,9)}(T_{45},t) & \end{bmatrix} G_{18} \quad (63)$$

Where $+(a_{16}'')^{(2)}(T_{17},t)$, $+(a_{17}'')^{(2)}(T_{17},t)$, $+(a_{18}'')^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}'')^{(1,1,)}(T_{14},t)$, $+(a_{14}'')^{(1,1,)}(T_{14},t)$, $+(a_{15}'')^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

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| $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | 66 |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3 | |

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| $-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)} & +(a_{22}^{''})^{(3)}(T_{21},t) & +(a_{18}^{''})^{(2,2,2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & & +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | 70 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | 71 |

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* $1, 2\ 3$

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$+(a_{13}'')^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}'')^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1)}(T_{14},t)$
*are fourth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{16}'')^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2)}(T_{17},t)$
*are fifth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{20}'')^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3)}(T_{21},t)$
*are sixth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{36}'')^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7)}(T_{37},t)$
*are seventh augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{40}'')^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}'')^{(8,8,8,8,8)}(T_{41},t)$
*are eighth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{46}'')^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}'')^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}')^{(4)}\left[-(b_{24}'')^{(4)}(G_{27},t)\right] & \left[-(b_{28}'')^{(5,5,)}(G_{31},t)\right] & \left[-(b_{32}'')^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{13}'')^{(1,1,1,1)}(G,t)\right] & \left[-(b_{16}'')^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}'')^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{36}'')^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}'')^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}'')^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}')^{(4)}\left[-(b_{25}'')^{(4)}(G_{27},t)\right] & \left[-(b_{29}'')^{(5,5,)}(G_{31},t)\right] & \left[-(b_{33}'')^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{14}'')^{(1,1,1,1)}(G,t)\right] & \left[-(b_{17}'')^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}'')^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{37}'')^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}'')^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}'')^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}')^{(4)}\left[-(b_{26}'')^{(4)}(G_{27},t)\right] & \left[-(b_{30}'')^{(5,5,)}(G_{31},t)\right] & \left[-(b_{34}'')^{(6,6,)}(G_{35},t)\right] \\ \left[-(b_{15}'')^{(1,1,1,1)}(G,t)\right] & \left[-(b_{18}'')^{(2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}'')^{(3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{38}'')^{(7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}'')^{(8,8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}'')^{(9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{26} \quad (78)$$

*Where* $-(b_{24}'')^{(4)}(G_{27},t)$, $-(b_{25}'')^{(4)}(G_{27},t)$, $-(b_{26}'')^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$
$-(b_{28}'')^{(5,5,)}(G_{31},t)$, $-(b_{29}'')^{(5,5,)}(G_{31},t)$, $-(b_{30}'')^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$
$-(b_{32}'')^{(6,6,)}(G_{35},t)$, $-(b_{33}'')^{(6,6,)}(G_{35},t)$, $-(b_{34}'')^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$
$-(b_{13}'')^{(1,1,1,1)}(G,t)$, $-(b_{14}'')^{(1,1,1,1)}(G,t)$, $-(b_{15}'')^{(1,1,1,1)}(G,t)$
*are fourth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{16}'')^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}'')^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}'')^{(2,2,2,2)}(G_{19},t)$
*are fifth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{20}'')^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}'')^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}'')^{(3,3,3,3)}(G_{23},t)$
*are sixth detrition coefficients for category* $1,2$ *and* $3$
$-(b_{36}'')^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}'')^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}'')^{(7,7,7,7,7)}(G_{39},t)$
*are seventh detrition coefficients for category* $1,2$ *and* $3$
$-(b_{40}'')^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8)}(G_{43},t)$
*are eighth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt}=(a_{28})^{(5)}G_{29}-\begin{bmatrix}(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt}=(a_{29})^{(5)}G_{28}-\begin{bmatrix}(a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt}=(a_{30})^{(5)}G_{29}-\begin{bmatrix}(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{30} \quad (81)$$

*Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3

$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3

$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3

$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3

$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3

$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt}=(b_{28})^{(5)}T_{29}-\begin{bmatrix}(b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt}=(b_{29})^{(5)}T_{28}-\begin{bmatrix}(b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{30} \quad (84)$$

where $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[\begin{array}{ccc} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{34})^{(6)}\boxed{+(a''_{34})^{(6)}(T_{33},t)} & \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{34} \quad (87)$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6)}(T_{33},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)}$ are third augmentation

| | |
|---|---|
| *coefficients for category* 1, 2 *and* 3<br>$+(a_{13}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}')^{(6)} & -(b_{32}'')^{(6)}(G_{35},t) & -(b_{28}'')^{(5,5,5)}(G_{31},t) & -(b_{24}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{13}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}')^{(6)} & -(b_{33}'')^{(6)}(G_{35},t) & -(b_{29}'')^{(5,5,5)}(G_{31},t) & -(b_{25}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{14}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}')^{(6)} & -(b_{34}'')^{(6)}(G_{35},t) & -(b_{30}'')^{(5,5,5)}(G_{31},t) & -(b_{26}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{15}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}'')^{(6)}(G_{35},t)$, $-(b_{33}'')^{(6)}(G_{35},t)$, $-(b_{34}'')^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{28}'')^{(5,5,5)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}'')^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3<br>$-(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3<br>$-(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3<br>$-(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3 | |

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}')^{(7)}+(a_{36}'')^{(7)}(T_{37},t) & +(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}+(a_{37}'')^{(7)}(T_{37},t) & +(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}+(a_{38}'')^{(7)}(T_{37},t) & +(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}'')^{(7)}(T_{37},t)$, $(a_{37}'')^{(7)}(T_{37},t)$, $(a_{38}'')^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = \quad (94)$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{array}{ccc} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{array}{ccc} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[\begin{array}{ccc} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

| | |
|---|---|
| $\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and$(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br><br>Where $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)}$, $(r_i)^{(2)}$ are positive constants and $i=16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |

| | |
|---|---|
| $\|(a_i'')^{(2)}(T_{17}',t)-(a_i'')^{(2)}(T_{17},t)\| \le (\hat{k}_{16})^{(2)}\|T_{17}-T_{17}'\|e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\|(b_i'')^{(2)}((G_{19})',t)-(b_i'')^{(2)}((G_{19}),t)\| < (\hat{k}_{16})^{(2)}\|\|(G_{19})-(G_{19})'\|\|e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and$(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},$ are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\ ,\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i')^{(2)},(b_i)^{(2)},(b_i')^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i')^{(2)}+(\hat{A}_{16})^{(2)}+(\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)}+(b_i')^{(2)}+(\hat{B}_{16})^{(2)}+(\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i')^{(3)},(a_i'')^{(3)},(b_i)^{(3)},(b_i')^{(3)},(b_i'')^{(3)}>0,\quad i,j=20,21,22$<br>The functions $(a_i'')^{(3)},(b_i'')^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (r_i)^{(3)}$:<br>$(a_i'')^{(3)}(T_{21},t)\le(p_i)^{(3)}\le(\hat{A}_{20})^{(3)}$<br>$(b_i'')^{(3)}(G_{23},t)\le\ (r_i)^{(3)}\le(b_i')^{(3)}\le(\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i'')^{(3)}(T_{21},t)=(p_i)^{(3)}$<br>$\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t)=\ (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)},(p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i=20,21,22}$ | 113 |

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|---|---|
| They satisfy Lipschitz condition:<br><br>$\|(a_i'')^{(3)}(T_{21}', t) - (a_i'')^{(3)}(T_{21}, t)\| \leq (\hat{k}_{20})^{(3)}\|T_{21} - T_{21}'\|e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\|(b_i'')^{(3)}(G_{23}', t) - (b_i'')^{(3)}(G_{23}, t)\| < (\hat{k}_{20})^{(3)}\|\|G_{23} - G_{23}'\|\|e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}', t)$ and$(a_i'')^{(3)}(T_{21}, t)$ . $(T_{21}', t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}$, $(r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i'')^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i'')^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition: | 119 |

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|---|---|
| $\|(a_i^{''})^{(4)}(T_{25}^{'},t)-(a_i^{''})^{(4)}(T_{25},t)\| \leq (\hat{k}_{24})^{(4)}\|T_{25}-T_{25}^{'}\|e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})^{'},t)-(b_i^{''})^{(4)}\big((G_{27}),t\big)\| < (\hat{k}_{24})^{(4)}\|\|(G_{27})-(G_{27})^{'}\|\|e^{-(\hat{M}_{24})^{(4)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)},(\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)}=1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}<1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)},(\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)},(\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)},(a_i^{'})^{(4)},(b_i)^{(4)},(b_i^{'})^{(4)},(p_i)^{(4)},\ (r_i)^{(4)}, i=24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)}+(a_i^{'})^{(4)}+(\hat{A}_{24})^{(4)}+(\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)}+(b_i^{'})^{(4)}+(\hat{B}_{24})^{(4)}+(\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)},(a_i^{'})^{(5)},(a_i^{''})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(b_i^{''})^{(5)}>0, \quad i,j=28,29,30$<br><br>The functions $(a_i^{''})^{(5)},(b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)},\ (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t)\leq(p_i)^{(5)}\leq(\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big)\leq\ (r_i)^{(5)}\leq(b_i^{'})^{(5)}\leq(\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \leq (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i')^{(6)} \leq (\hat{B}_{32})^{(6)}$$

$lim_{T_2 \to \infty}(a_i'')^{(6)}(T_{33}, t) = (p_i)^{(6)}$ 128

$\lim_{G \to \infty}(b_i'')^{(6)}((G_{35}), t) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

| | |
|---|---|
| Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32{,}33{,}34}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}\big((G_{35}), t\big)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32{,}33{,}34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (M) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36{,}37{,}38$<br><br>(N) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(O)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(P)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = \ (r_i)^{(7)}$ | 132 |

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| **Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}'\rvert e^{-(\hat{M}_{36})^{(7)}t}$<br><br>$\lvert(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})'\rvert\rvert e^{-(\hat{M}_{36})^{(7)}t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(Q) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(R) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i, j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |

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| $(b_i^{''})^{(8)}((G_{43}),t) \leq \ (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'},t) - (a_i^{''})^{(8)}(T_{41},t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'},t) - (b_i^{''})^{(8)}\left((G_{43}),t\right)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i^{'})^{(8)} + \ (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(b_i)^{(8)} + (b_i^{'})^{(8)} + \ (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i^{'})^{(9)}, (a_i^{''})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (b_i^{''})^{(9)} > 0, \quad i,j = 44,45,46$ | 146 A |

The functions $(a_i^{''})^{(9)}, (b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i^{'})^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i^{''})^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i^{''})^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)} |T_{45} - T_{45}^{'}| e^{-(\hat{M}_{44})^{(9)} t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}((G_{47}), t)| < (\hat{k}_{44})^{(9)} ||(G_{47}) - (G_{47})^{'}|| e^{-(\hat{M}_{44})^{(9)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

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| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 |

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| $G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| Where $s_{(13)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |

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| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a_{24}')^{(4)} + a_{24}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a_{25}')^{(4)} + (a_{25}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| By | 163 |

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| $\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{32})^{(6)}$, $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}')^{(6)} + (a_{34}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}')^{(6)} - (b_{32}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}')^{(6)} - (b_{33}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b'_{37})^{(7)} - (b''_{37})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{38})^{(7)} - (b''_{38})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |

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| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that | 168 |

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| $(G_{13}(t)-G_{13}^0)e^{-(\hat{M}_{13})^{(1)}t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{P}_{13})^{(1)}+G_{14}^0)e^{\left(-\frac{(\hat{P}_{13})^{(1)}+G_{14}^0}{G_{14}^0}\right)}+(\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0+(\hat{P}_{16})^{(6)}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1+(a_{16})^{(2)}t\right)G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)}t}-1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t)-G_{16}^0)e^{-(\hat{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | |

that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that 177

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(c) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$ 180

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| $\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$ | |
| From which it follows that<br>$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$<br>$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left((G^{(1)},T^{(1)}),(G^{(2)},T^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right|\leq\int_{0}^{t}(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_{0}^{t}\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

(186)

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_{+}$.

If instead of proving the existence of the solution on $\mathbb{R}_{+}$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)},i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

From 19 to 24 it results

$$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\geq 0$$

$$T_i\,(t)\geq T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$:

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13}<(\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt}\leq\left((\widehat{M}_{13})^{(1)}\right)_1-(a_{14}^{'})^{(1)}G_{14}$ and by integrating

$$G_{14}\leq\left((\widehat{M}_{13})^{(1)}\right)_2=G_{14}^0+2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$$

(187)

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| In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\widehat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br><br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br><br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(\left((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)}\right)\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i$ (t) $= 0$ and $T_i$ (t) $= 0$<br><br>it results<br><br>$G_i$ (t) $\geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i$ (t) $\geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for t $> 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} \} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left((\widehat{M}_{20})^{(3)}\right)_1 / (a_{21}^{'})^{(3)}$ | 217 |

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| In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}} < 1$ and to choose<br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}\}$ | 225 |

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})}, \widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\, ds_{(24)} +$$

$$\int_0^t \{(a_{24}^{'})^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} \left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)}, s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)} t\right)} > 0 \quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}^{'})^{(4)}$$

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| In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$ | 236 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br><br>It results<br><br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br><br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br><br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(5)}t\right)}>0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1,\left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if | 240 |

$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating

$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a'_{29})^{(5)}$

In the same way , one can obtain

$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a'_{30})^{(5)}$

If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively.

**Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. (241)

**Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$. (242)

**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :

Indeed let $t_5$ be so that for $t > t_5$

$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$

Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to (243)

$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results

$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.
The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b''_{30})^{(5)}\left((G_{31})(t),t\right) = (b'_{30})^{(5)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$

It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose (244)

$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have

$$\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)} \quad (245)$$

$$\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)} \quad (246)$$

In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

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| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\le$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(6)}t\right)}>0$ for t > 0 | 250 |

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a'_{33})^{(6)} G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)} \quad (257)$$

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| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\le$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\ge 0$ | 261 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(7)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{36})^{(7)})_1, ((\widehat{M}_{36})^{(7)})_2$ *and* $((\widehat{M}_{36})^{(7)})_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq ((\widehat{M}_{36})^{(7)})_1 - (a_{37}')^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq ((\widehat{M}_{36})^{(7)})_2 = G_{37}^0 + 2(a_{37})^{(7)}((\widehat{M}_{36})^{(7)})_1/(a_{37}')^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq ((\widehat{M}_{36})^{(7)})_3 = G_{38}^0 + 2(a_{38})^{(7)}((\widehat{M}_{36})^{(7)})_2/(a_{38}')^{(7)}$<br><br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}((G_{39})(t),t) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + ((\widehat{P}_{40})^{(8)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
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| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$<br>$\int_0^t\{(a_{40}')^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$<br>$G_{40}^{(2)}\left\\|(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}'')^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$<br>$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}')^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}'')^{(8)}$ and $(b_{40}'')^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(8)}$ and $(b_i'')^{(8)}, i=40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$ | 275 |

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| it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}')^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(8)}((G_{43})(t), t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty} (b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)}+\left((\hat{P}_{44})^{(9)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{44})^{(9)}\right]\leq(\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),\left((G_{47})^{(2)},(T_{47})^{(2)}\right)\right)=$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\;e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\;(G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}')^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}')^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}')^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t), t\right)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}'')^{(9)}\left((G_{47})(t), t\right) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14}, t) + (a_{14}'')^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

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| $-(\tau_2)^{(1)} \le -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \le -(\tau_1)^{(1)}$ | |
| **Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ :<br><br>By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 281 |
| **Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}^{'})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}^{'})^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}^{'})^{(1)}t}\right)$ | 286 |

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| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 291 |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(T_{17}, t) + (a_{17}'')^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}\big((G_{19}), t\big) - (b_{17}'')^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |

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| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a_{18}')^{(2)} t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b_{18}')^{(2)} t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ | 317 |

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| $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ :<br>By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and<br>By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | 320 |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,<br>and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$ | 322 |

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| $(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},$ ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$<br><br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$<br><br>$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br><br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$<br><br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br><br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$<br><br>$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br><br>**Theorem:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br><br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br><br>$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br><br>$-(\tau_2)^{(4)} \le -(b_{24}')^{(4)} + (b_{25}')^{(4)} - (b_{24}'')^{(4)}((G_{27}), t) - (b_{25}'')^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0, (v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations | 329 |

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| $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$<br>$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$ | 330 |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$ | 336 |

$\frac{(a_{26})^{(4)}T_{24}^{0}}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^{0}e^{-(R_2)^{(4)}t}$

**Definition of** $(S_1)^{(4)},(S_2)^{(4)},(R_1)^{(4)},(R_2)^{(4)}$:- 337

Where $(S_1)^{(4)}=(a_{24})^{(4)}(m_2)^{(4)}-(a_{24}^{'})^{(4)}$

$(S_2)^{(4)}=(a_{26})^{(4)}-(p_{26})^{(4)}$

$(R_1)^{(4)}=(b_{24})^{(4)}(\mu_2)^{(4)}-(b_{24}^{'})^{(4)}$

$(R_2)^{(4)}=(b_{26}^{'})^{(4)}-(r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)}\le-(a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}-(a_{28}^{''})^{(5)}(T_{29},t)+(a_{29}^{''})^{(5)}(T_{29},t)\le-(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)}\le-(b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31}),t)-(b_{29}^{''})^{(5)}((G_{31}),t)\le-(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)},(v_2)^{(5)},(u_1)^{(5)},(u_2)^{(5)},v^{(5)},u^{(5)}$ : 339

By $(v_1)^{(5)}>0$ , $(v_2)^{(5)}<0$ and respectively $(u_1)^{(5)}>0$ , $(u_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and

**Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$

**Definition of** $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-

If we define $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)}$ by

$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)}=\frac{G_{28}^{0}}{G_{29}^{0}}}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$

and analogously 341

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| $(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$, ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying | 349 |

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| $-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br><br>$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}\big((G_{39}),t\big)-(b_{37}^{''})^{(7)}\big((G_{39}),t\big) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\big(v^{(7)}\big)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\big(u^{(7)}\big)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

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| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ | 370 |

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,
and $\boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

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| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,
and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,
and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G^0_{44} e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G^0_{44} e^{(S_1)^{(9)} t}$$

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| ( $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t}\le G_{46}(t)\le$ $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a'_{46})^{(9)}t}]+G_{46}^{0}e^{-(a'_{46})^{(9)}t})$ | |
| $\boxed{T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$ | |
| $\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$ | |
| $\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b'_{46})^{(9)}t}\right]+T_{46}^{0}e^{-(b'_{46})^{(9)}t}\le T_{46}(t)\le$ $\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$ | |
| **Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:- Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a'_{44})^{(9)}$ $(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$ $(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b'_{44})^{(9)}$ $(R_2)^{(9)}=(b'_{46})^{(9)}-(r_{46})^{(9)}$ | |

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$ , we obtain

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

$$\text{For } 0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

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| $\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t)=\frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain | 398 |

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \le \frac{dv^{(3)}}{dt} \le -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \ge \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \le v^{(3)}(t) \le (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \le v^{(3)}(t) \le (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \le \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \le v^{(3)}(t) \le$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \le (\bar{v}_1)^{(3)} \le (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \le v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$, we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}, \quad \boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}, \quad \boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} =$

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$(\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

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$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq v^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $v^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)v^{(7)} - (a_{37})^{(7)} v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)} v^{(7)} - (a_{36})^{(7)}\right) \leq \frac{dv^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$v^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq v^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain 419

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \leq v^{(8)}(t) \leq (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \leq v^{(8)}(t) \leq (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \leq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \leq v^{(8)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \leq (\bar{v}_1)^{(8)} \leq \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \leq v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

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$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

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| $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |

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| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ | 436 A |

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| $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |

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| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}))\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}))\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}))\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |

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| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(c) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) +$ | 486 |

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| $(a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |

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| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{13}=\frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}=\frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | 494 |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations | 499 |

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| $G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 504 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if | 510 |

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| $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |

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| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)} (T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} ((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)} ((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)} (T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)} (T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)} ((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)} ((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)} (T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)} (T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)} ((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)} ((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and<br>$G_{36}^* = \frac{(a_{36})^{(7)} G_{37}^*}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^*)]}$ , $G_{38}^* = \frac{(a_{38})^{(7)} G_{37}^*}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^*)]}$<br>$T_{36}^* = \frac{(b_{36})^{(7)} T_{37}^*}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^*)]}$ , $T_{38}^* = \frac{(b_{38})^{(7)} T_{37}^*}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^*)]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^*$ given by $\varphi((G_{43})^*) = 0$ , $T_{41}^*$ given by $f(T_{41}^*) = 0$ and<br>$G_{40}^* = \frac{(a_{40})^{(8)} G_{41}^*}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^*)]}$ , $G_{42}^* = \frac{(a_{42})^{(8)} G_{41}^*}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^*)]}$ | 523 |

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| $T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br>**Proof:**_Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions | 531 |

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| $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |

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| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |

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| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |

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| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\Big)\right]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\Big)\right]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\Big)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\right)\Big]$$

$$\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al**

**Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal**

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{3}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and

review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 3. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For $K \subset C$ and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \|x - y\| < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ($\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $\mathrm{A}_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A}_i(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A}_i(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i\ \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N_\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $\dot{x} = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $\dot{x} = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $\dot{x} = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_s$. If $r(x) = s(x), then\ x \in M_r$. Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i\ dot \end{pmatrix} = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.7) $$P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.8) $$D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.9) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^{\infty} b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^{\infty} b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s)$, $a.0$, $s\epsilon [-\tau, 0]$, $i = 1, 2$,

(1.6) $P_i(s) = \int_0^{\infty} \varphi_i(a, s)\, da$, $s\in [-\tau, 0]$,

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon [-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 3. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^*$ on $R^+ \times K_2 \times K_1$,

($\beta_1^*$ , $\beta_2^*$ are positive constants);

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is for $i = 1, 2$,

$|b_i\,(a, P_1', P_2') - b_i\,(a, P_1'', P_2'')| \leq \beta_i\,[||P_1' - P_1''|| + ||P_2' - P_2''||]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, R^-),$$

$A_3$. $\varphi_i\,(\cdot\,, s) \in L^2\,(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^+ \times [-\tau, T] \to R^+$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\,, t) \in L^1\,(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\,([-\tau, T], R^+)$ for each $a \in R^+$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t, 0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right) ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t)= B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a, t, p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a, t, P_i(t)\right)da$

$$+\int_0^\infty b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, \; t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \; \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \; t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2\ ),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in \mathrm{C}([0,T], R^+), \quad B_i \in \mathrm{C}([0,T], R^+)\ , i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in \mathrm{C}([-\tau, T], R\ ) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $\mathrm{C}^+[-\tau, T\ ]$. Then for any

$(x, y) \in \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$\mathrm{C}_T = \mathrm{C}^+[-\tau, T\ ], \quad \mathrm{C}_T^2 = \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ].$$

Members of $\mathrm{C}_T$ will be denoted by $Z_t$ with the meaning that

$Z\ (t+s) = z_t(s)$ *for* $s \in [-\tau, T\ ]$ *and* $t \in [0\ , T\ ]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in \mathrm{C}_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2\ )$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$\mathrm{B}\ (\mathrm{t}) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^*\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left(\sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^*\right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then  is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ ,  is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|(P_t^{(1)}) - (P_t^{(2)})|$ for t $\leq t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|(P^{(1)}(t)) - (P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2 / 2\}$$

Thus

$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| B_1^{(1)}(\sigma)- B_1^{(2)}(\sigma) \right| \mathrm{do}+\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\ \ 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

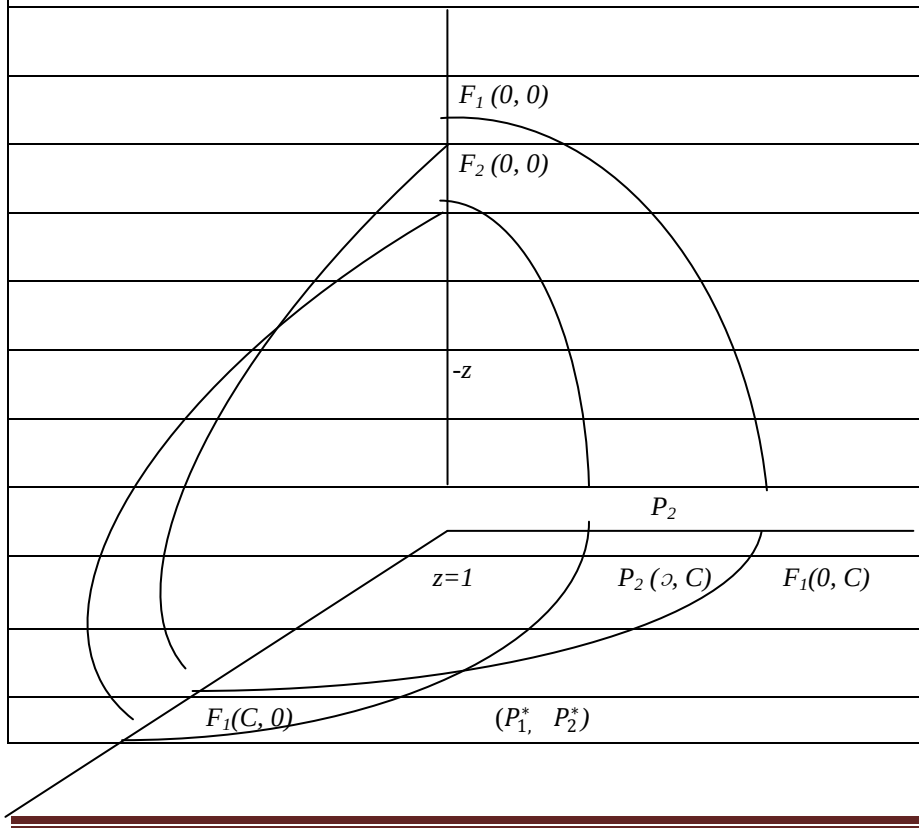
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{q}_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) q_i(a)da + \int_0^\infty p_i^*(a) \left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_i^*\right\} da ,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \; p_{22}^* = p_2^* .$$

If we integrate the first of (3.15) and use the fact that $q_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right] da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right] q_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right] da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right] q_2(a)da,$$

From the first of (3.15) we get

(3.18) $$q_i(a) = \left\{q_i(0)p_i^*(a)/p_i^*(0) \exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} e^{\lambda s} ds , \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$q_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* q_i\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} e^{\lambda s} ds , i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \|\ | - |\ A_{11} + A_{22} + \| S_1(0) \|\ |^2\right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 5. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \right\} \left\{ 1 - \frac{N(t-\tau)}{K} \right\}. \qquad (1.3)$$

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\left\{ \frac{dN(t)}{dt} = N(t) = \right\} \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\} \qquad (1.4)$$

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(vii) $b_1(0) = 0$, $m_1(0, x_2) \equiv 0, b_2(0) = 0$, $m_2(x_1, 0) \equiv 0$,

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(viii) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$,

$b_2(\delta_2) - m_2(x_1, \delta_2) < 0$, for $x_1$ 0, $x_2$ 0,

(2.5)

(ix) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha,\beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \neq 0$, $x_2(t) \neq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha,\beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i,\; j = 1, 2; i \neq j)$.

## 6. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\,(t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\,(t_2)\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\,(t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\,(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\,(t - \tau_{11}) - m_{12}Y_2\,(t - \tau_{12}),$$

$$\frac{dY_2\,(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\,(t - \tau_{21}) - m_{22}Y_2\,(t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

***THEOREM 3.1****. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \qquad \text{Where } y_1 \text{ 0}, y_1 \text{ 0},$$

(4.2)

(ix)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \qquad g_2(0) = 0, \qquad (4.3)$$

(x) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \qquad (4.4)$$

(xi) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \qquad (4.5)$$

(xii) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \qquad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{vmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{vmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 7. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{pmatrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{pmatrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau$ = max $\tau_{ij}$, $i,j$ = 1,2. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{11}\left|Y_1(\mathrm{t})\right| + b_{11}\left|Y_1(\mathrm{t}-\tau_{11})\right| + m_{12}\ Y_2(\mathrm{t}-\tau_{12})\Big|, \\ \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{22}\left|Y_2(\mathrm{t})\right| + m_{21}\left|Y_1(\mathrm{t}-\tau_{21})\right| + b_{22}\ \left|Y_2(\mathrm{t}-\tau_{21})\right|, \end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\ \left|\ Y_1(t)\ \right| = \left\{\begin{aligned} &\frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) \ge 0, \\ &\frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) < 0, \end{aligned}\right\}\ i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(\mathrm{t})}{dt} &= -m_{11}\ Z_1(\mathrm{t}) + b_{11}Z_1(\mathrm{t}-\tau_{11}) + m_{12}\ Z_2(\mathrm{t}-\tau_{12}), \\ \frac{dZ_2(\mathrm{t})}{dt} &= -m_{22}Z_2(\mathrm{t}) + m_{21}Z_1(\mathrm{t}-\tau_{21}) + b_{22}\ Z_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(\mathrm{s}) > \left|Y_1(s)\right|,\ \ s \in [-\tau, 0],\ i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(\mathrm{t})\ \text{for t} \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(\mathrm{t}) = Z_i(\mathrm{t}) - \left|Y_1(s)\right|$, $i = 1,2.$ t$\ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(\mathrm{t})}{dt} &\ge -m_{11}\ U_1(\mathrm{t}) + b_{11}U_1(\mathrm{t}-\tau_{11}) + m_{12}\ U_2(\mathrm{t}-\tau_{12}), \\ \frac{dU_2(\mathrm{t})}{dt} &\ge -m_{22}U_2(\mathrm{t}) + m_{21}U_1(\mathrm{t}-\tau_{21}) + b_{22}\ U_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(\mathrm{s}) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[\ \tau, 0]$ : It will follow from (A.7) that $U_1(\mathrm{t}) \ge U_1(0)\ \exp\ [-m_{11}\mathrm{t}]$

$$+\int_0^t \left[b_{11}U_1(\mathrm{s}-\tau_{11}) + m_{12}\ U_2(\mathrm{s}-\tau_{12})\right] \exp\left[-m_{11}(\mathrm{t}-\mathrm{s})\mathrm{ds} > 0,\right] \qquad (A.9)$$

For all t$\in [0, \tau^*]$ since $U_i(\mathrm{s}) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(\mathrm{t}) > 0$ for t $\in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(\mathrm{t}) > 0, U_2(\mathrm{t}) > 0$ for t $\in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds + \int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds + \int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0 \leq |\, Y_{1,}(t)\,| \leq Z_1(t) \to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22} m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t) \to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t) \to 0, Z_2(t) \to 0$ as $t \to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s \in [0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty s k_{ij}(s)ds < \infty;\ \int_0^\infty k_{ij}(s)ds = 1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

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| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon)$, $t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

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| **Gopalsamy** |
| **Model For Innocuously Benign Delay In A Periodic System** |
| **Abstract** |
| Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of |

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 7. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 8. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big), \qquad t > 0, \tag{2.1}$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ in $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ in $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap\left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ $\in -\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega+t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(vii) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(viii) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(ix) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(e)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(f)* *All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 9. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(g) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(h) N is monotonic nondecreasing on $K^*$.

(i) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \| x_1 \| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \| x_2 \| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1, \qquad (b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (3.3)$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad (3.4)$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \geq -\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \geq -\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x^*_1(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21})\, x^*_1(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} - {}_{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha$, $\beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha$,$\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that “In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter Ƭ:

$\frac{dx_1(\mathrm{t})}{dt} x_1(\mathrm{t})[1 - x_1(\mathrm{t}) + 0.4x_2(\mathrm{t} - \mathcal{T})/k_1(\mathrm{t})]$

$\frac{dx_2(\mathrm{t})}{dt} x_2(\mathrm{t})[1 - \{0.8\, x_1(\mathrm{t} - \mathcal{T}) + x_2(\mathrm{t})\}/k_2(\mathrm{t})]$

Where $k_1(\mathrm{t}) = 20 + \sin\ (\pi\mathrm{t})$ and $k_2$(t)=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 5. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \ldots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.3) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \left( b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left( b_j^u / a_{jj}^l \right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \geq x_i^l \quad for\ t \geq t_0,\ i = 1, 2, \ldots, n. \qquad (1.7)$$

6. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* $\boldsymbol{R}$ *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \leq x_l \leq x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1 \leq l \leq n} x_l^l; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(45) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(46) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(47) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(48) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(49) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(50) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(51) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(52) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(53) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(54) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(55) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(56) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(57) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

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|---|
| (58) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (59) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (60) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (61) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (62) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (63) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (64) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (65) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (66) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception:

"what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left|\left|f\ (t+\ \tau,x) - \ f(t,x)\right|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{ij}\ (\sigma+\tau_k) - a_{ij}\,(\sigma+\tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k), x(t+\tau_m)\big) \le v\big(x(t_0), x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$$\le [(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] \ + (\varepsilon/2x^*)$$

$$< (\varepsilon/x^*) \text{ On using (2.8), (2.10)} \qquad (2.14)$$

For $m \ge k \ge n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \le x_i(t) \le x^*, i = 1,2,\dots,n; t \ge t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \ge k \ge n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*, x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*, x^*], i = 1,2,\dots,n;$ for $t \ge t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ & $k$ $\geq n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ & $k$ $\geq n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1((t + \tau_m)) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\,(t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{aligned} &\beta_1 \leq x_i(t) \leq \beta_2, \\ &\beta_1 \leq y_i(t) \leq \beta_2, \end{aligned} \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V(x(t)y(t)) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t)y(t))$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \rho_1(t, a_1) \rho_1^*(t, a_1),$$

$$0 Q_1(t) Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1[P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ħ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** $$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{(m_1 m_2)(m_1+m_2)}{r^3},$$

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

$$t = \frac{5}{256}\frac{c^5}{G^3}\frac{r^4}{(m_1 m_2)(m_1+m_2)}.$$

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_1\right]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \quad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3) Y_3(\gamma_3, a_3) da_3\right) u_3(0)$$

$$+w \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3 = 0,$$

(24)

$$\left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right) \tilde{u}_3(0)$$

$$+\left(-1 - \int_0^\infty Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3\right) w = 0.$$

Equqtion (20) has a solution of the form of $u_3(t, a_3) = u_3(a_3) e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3) Y^{-3}\,(\gamma_3, a_3) d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l \left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right)\left(\int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) da\right)\left(1 + (f_{30}^l)_3 \int_0^\infty Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$= 0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3), 0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ \ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{.3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2[a_3^2]_3<4[a_2]_3<4a_2e$,

$\mathrm{Re}\,\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2]_3 < 0.$

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then$|n_3(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2<4a_2e$ and $[a_3] > 2\Delta$,or $[a_3^2]$ ⊠4$[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3}da = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^{3}] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

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Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1\log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 19. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \text{ 0}, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \text{ ∞},$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \quad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 20. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^\infty Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^\infty Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a (\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^\tau \mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a \mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d\, a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th]

(or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4\ [a_2]_4<1\ a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\ a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\ x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b) Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\, (\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 a e_4^{mu} [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 21. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ } 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ } \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t,a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 22. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0)$$

$$= \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(1043) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(1044) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(1045) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(1046) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(1047) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(1048) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(1049) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(1050) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(1051) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(1052) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(1053) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(1054) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(1055) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(1056) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(1057) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(1058) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(1059) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(1060) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(1061) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

## Black holes

(1062) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(1063) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(1064) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(1065) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(1066) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(1067) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(1068) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(1069) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(1070) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have**

**experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(1071) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(1072) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

**String theory**

(1073) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(1074) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(1075) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

(1076) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.**

(1077) **However, when summed the entire series appears to be (=) divergent.**

(1078) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.**

(1079) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.**

(1080) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.**

(1081) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.**

**Loop quantum gravity and spin foams**

(1082) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.**

(1083) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.**

(1084) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.**

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| (1085) | **The granular properties of space can be (=) explicitly computed.** |
| (1086) | **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (1087) | **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (1088) | **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (1089) | **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (1090) | **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (1091) | **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (1092) | **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (1093) | **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (1094) | **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (1095) | **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (1096) | **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (1097) | **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (1098) | **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (1099) | **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (1100) | **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (1101) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (1102) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (1103) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (1104) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (1105) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (1106) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (1107) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (1108) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (1109) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (1110) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (1111) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (1112) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (1113) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

(1114) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(1115) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

(1116) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.**

(1117) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016**
**DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society**

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(1118) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.**

(1119) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.**

(1120) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.**

(1121) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.**

(1122) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).**

(1123) **Bringing them to thermal equilibrium requires (e) an external bath.**

(1124) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.**

(1125) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).**

(1126) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.**

(1127) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.**

(1128) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—**

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (1129) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (1130) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (1131) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (1132) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (1133) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (1134) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (1135) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (1136) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (1137) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (1138) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (1139) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (1140) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (1141) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (1142) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (1143) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (1144) **This is a realistic scenario for (e) all known systems.** |
| (1145) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (1146) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (1147) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (1148) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (1149) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (1150) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (1151) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (1152) **So in those systems, the quantum phase transition does indeed exist.** |
| (1153) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (1154) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (1155) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (1156) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (1157) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (1158) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (1159) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (1160) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (1161) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (1162) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (1163) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (1164) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (1165) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (1166) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (1167) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (1168) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (1169) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (1170) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (1171) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (1172) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (1173) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (1174) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (1175) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (1176) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (1177) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (1178) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (1179) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (1180) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (1181) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (1182) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (1183) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (1184) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (1185) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (1186) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (1187) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (1188) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (1189) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (1190) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (1191) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (1192) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (1193) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (1194) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (1195) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (1196) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (1197) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (1198) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (1199) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (1200) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (1201) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (1202) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (1203) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (1204) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (1205) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (1206) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (1207) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (1208) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (1209) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (1210) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (1211) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (1212) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (1213) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (1214) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (1215) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (1216) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (1217) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (1218) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (1219) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (1220) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (1221) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (1222) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (1223) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (1224) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (1225) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (1226) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (1227) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (1228) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (1229) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (1230) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (1231) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (1232) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (1233) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (1234) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (1235) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (1236) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (1237) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (1238) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (1239) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (1240) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (1241) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (1242) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

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| (1243) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (1244) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (1245) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (1246) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (1247) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (1248) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (1249) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (1250) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (1251) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (1252) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (1253) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (1254) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (1255) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (1256) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |

(1257) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(1258) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(1259) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(1260) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(1261) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(1262) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(1263) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(1264) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(1265) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(1266) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(1267) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(1268) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(1269) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(1270) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(1271) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (1272) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (1273) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (1274) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (1275) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (1276) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (1277) **We know dark matter exists only from (e) its gravitational effects.** |
| (1278) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (1279) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (1280) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (1281) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (1282) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (1283) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (1284) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (1285) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (1286) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (1287) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (1288) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (1289) | **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (1290) | **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (1291) | **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (1292) | **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (1293) | **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (1294) | **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (1295) | **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (1296) | **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (1297) | **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (1298) | **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (1299) | **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (1300) | **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (1301) | **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (1302) | **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (1303) | **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (1304) | **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (1305) | **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (1306) | **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (1307) | **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (1308) | **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (1309) | **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (1310) | **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (1311) | **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (1312) | **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (1313) | **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (1314) | **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (1315) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (1316) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (1317) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (1318) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (1319) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (1320) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (1321) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (1322) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (1323) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (1324) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (1325) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (1326) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (1327) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (1328) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (1329) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (1330) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (1331) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (1332) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (1333) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (1334) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (1335) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (1336) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (1337) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (1338) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (1339) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (1340) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (1341) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (1342) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (1343) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (1344) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (1345) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (1346) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (1347) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (1348) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (1349) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (1350) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (1351) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (1352) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (1353) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (1354) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (1355) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (1356) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(193) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(194) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(195) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(196) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(197) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(198) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(199) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(200) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(201) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(202) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(203) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(204) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(205) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(206) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(207) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(208) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(209) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

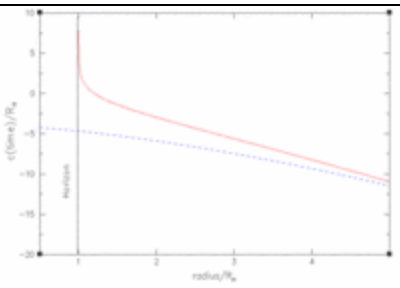
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| **(210) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(211) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(212) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(213) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(214) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(215) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(216) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(217) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(218) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(219) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(220) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(221) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(222) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(223) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(224) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(225) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(226) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(227) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(228) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(229) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(230) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(231) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(232) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(233) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(234) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(235) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(236) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(237) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(238) One of these coordinates is (=) called the "coordinate time."** |
| **(239) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(240) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(241) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(242) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(243) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(244) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(245) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(246) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(247) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(248) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(249) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(250) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(251) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(252) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(253) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(254) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(255) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(256) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(257) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(258) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(259) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(260) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(261) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(262) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(263) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(264) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(265) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(266) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(267) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(268) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(269) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(270) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(271) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(272) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(273) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(274) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(275) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(276) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(277) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(278) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(279) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(280) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(281) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(282) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(283) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(284) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(285) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(286) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(287) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(288) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6\left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,\quad h_{6}^{*}$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[f^{'}{}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[f_{2*}^{'}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right)M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6<4\,[a_2]_6<4\,a_2 e$,

$\mathrm{Re}\,\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$

Let $\Delta=\max_{2<u<\infty}\left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then$|n_6(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 <4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2]$ ⊠4$[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$ i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ P^*\ [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\ [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\ g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^2} + \frac{2\omega_6}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6} = [f_{6*}]_{\,6} + [f_{2*}]_{\,6} + [f_{6*}] \, P_6^*$$

$$[a_2]_{\,6} = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [\, g_*^6]_{\,6} \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_{\,6} \beta_{\,6} \left(\theta_{\,6}(t)\right) - \mu_{\,6}\left(\theta(t)\right),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_{\,6}\right] M_6 - [a_2]_{\,6} W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 23. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \; P_7 \; 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(1357) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(1358) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(1359) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(1360) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(1361) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(1362) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(1363) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(1364) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(1365) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(1366) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(1367) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(1368) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(1369) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(1370) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(1371) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(1372) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(1373) The electronic and (e&eb) structural variability among them is extraordinary.

(1374) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(1375) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(1376) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(1377) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(1378) Nudge your bath tub just right and (eb) you might end up reversing time.

(1379) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(1380) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(1381) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(1382) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(1383) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(1384) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(1385) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(1386) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(1387) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(1388) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(1389) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(1390) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(1391) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(1392) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(1393) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(1394) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(1395) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(1396) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(1397) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(1398) Quantum mechanics rules the dynamics of (e) light and matter.

(1399) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(1400) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(1401) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(1402) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (1403) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (1404) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (1405) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (1406) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (1407) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (1408) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (1409) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (1410) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (1411) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (1412) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (1413) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (1414) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (1415) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (1416) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (1417) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (1418) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (1419) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (1420) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (1421) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (1422) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (1423) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (1424) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (1425) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (1426) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (1427) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (1428) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (1429) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (1430) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (1431) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (1432) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (1433) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (1434) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (1435) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (1436) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (1437) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (1438) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (1439) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (1440) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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|---|
| (1441) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (1442) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (1443) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (1444) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (1445) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |
| (1446) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big(f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 24. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7) e^{-\gamma_7\tau_7} e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds} d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\, da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\,\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ \left(\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\left(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7\right) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_J) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7)da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7)m_7(t,a_7)\,da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0)W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7\left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}^{'}$ denoting $h_{7}^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7[a_7^7]2<4\,[a_7]2<4\,a_7e$,

$\mathrm{Re}\,\lambda_7\left[\lambda_7\right]_7 = \mathrm{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\,]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}\,[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⍰∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^{*}(a), \boldsymbol{Q}^{*}$) is asymptotically stable if either $a_7^7 <4\,a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⍰[$a_7$]e and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])_{\,7}^{\,3}\ [\mathrm{K}]_{\,7}[P_0]_{\,7}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^{*}\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\,7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^{\,7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\,7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a_{11}\right] - 4\left[a_{77}\right] e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7 Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^7 Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7 (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\, (\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_7\, ae^{mu}_7\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7 Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (279)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (280) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (281)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (282) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

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| (283)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (284) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (285)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (286)Humans are not very good at understanding interacting many-body systems. |
| (287)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (288)But what happens when we couple (e&eb) the particles strongly? |
| (289)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (290)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (291)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (292) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (293) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (294)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (295) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (296)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (297) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (298) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (299)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (300)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (301) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (302) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (303)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (304)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (305)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (306)The energy levels of these quantized orbitals take on (e) discrete values |
| (307):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (308) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (309) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (310)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (311) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (312)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (313) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (314)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (315)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (316)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (317) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (318)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (319)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (320)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (321) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (322)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (323)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (324)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (325)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (326)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (327)Another classic example of duality involves (e&eb) vortices. |
| (328) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (329)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (330) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (331)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (332)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (333) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (334)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (335)TIs are distinguished only by (e) their edge states. |
| (336) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (337)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (338) But the Dirac cone is not (e) the only possible signature. |
| (339) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (340)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (341)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (342)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (343) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (344). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (345)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (346)The bulk duality implies (eb) a duality between edge theories. |
| (347)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |

(348)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids.

(349) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles.

(350) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold.

(351)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables.

(352)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI.

(353)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors.

(354)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field.

(355)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions.

(356)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3.

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

(357) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$.

(358)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration.

(359)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP).

(360) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole.

(361)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process.

(362)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version)

IF we define

$$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,\mathrm{P}^{*}_{7}$$

$$[a_7]_7 = \left[f^{7}_{7*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$

$$e=[g^{7}_{*}]_7 Q^{*}_{7}$$

$$\eta_7(t) = \left[h_{*}\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 25. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(1447) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(1448) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(1449) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(1450) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(1451) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

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| (1452) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (1453) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (1454) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (1455) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (1456) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (1457) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (1458) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (1459) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (1460) "The underlying physics is (=) chaos, essentially," he said. |
| (1461) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (1462) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (1463) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (1464) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (1465) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (1466) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (1467) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (1468) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (1469) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (1470) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |

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| (1471) | "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (1472) | It turns out that they are actually the driving forces behind (eb) thermalization. |
| (1473) | "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (1474) | The study's findings have fundamental implications for (e) quantum computing. |
| (1475) | At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (1476) | "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (1477) | A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (1478) | The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (1479) | Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (1480) | The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer. |
| (1481) | Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (1482) | By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (1483) | The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (1484) | Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (1485) | Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (1486) | As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |

(1487) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(1488) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(1489) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(1490) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(1491) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(1492) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(1493) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(1494) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(1495) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(1496) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(1497) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(1498) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(1499) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as

contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(1500) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(1501) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(1502) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(1503) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(1504) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(1505) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(1506) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(1507) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(1508) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(1509) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(1510) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(1511) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(1512) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(1513) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(1514) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(1515) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(1516) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(1517) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(1518) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(1519) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(1520) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(1521) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(1522) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(1523) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background

radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(1524) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(1525) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(1526) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(1527) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(1528) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(1529) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(1530) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(1531) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(1532) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(1533) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(1534) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(1535) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(1536) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

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| (1537) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime. |
| (1538) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite. |
| (1539) However, when summed the entire series appears to be (=) divergent. |
| (1540) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime. |
| (1541) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects. |
| (1542) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing. |
| (1543) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory. |

**Loop quantum gravity and spin foams**

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| (1544) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics. |
| (1545) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity. |
| (1546) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime. |
| (1547) The granular properties of space can be (=) explicitly computed. |
| (1548) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory. |
| (1549) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks. |
| (1550) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure. |
| (1551) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit. |
| (1552) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism. |
| (1553) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism. |

**Noncommutative geometry**

(1554) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(1555) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(1556) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(1557) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(1558) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(1559) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(1560) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(1561) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(1562) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(1563) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \infty,$$

$$\rho_8(t, 0) = \int_0^\infty n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly

eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big((f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8 \; h_8(Q_8) = 1, \qquad and \;\; g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 26. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))u_8 - (\rho_0)_8(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8^{'}(P_0) = f_{80}^{'}$, and $f_8^{'}(0) = f_{80}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \quad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a \mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\left(\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)$$

$$+\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(a_8)da_8-\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^\infty E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)([f_8^{'}]_8(z_8))\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$-h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8\left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[\dot{g}\right]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - \left[\dot{f}_{8*}\right]_8[p_8^*](a_8)M_8 - \left[\dot{f}_{8*}\right]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[\dot{f}_{8*}\right]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[\dot{f}_{8*}\right]_8 P_8^*\right)M_8 - \left[\dot{f}_{8*}\right]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])_8^3\ [\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma Q_8^*}{8+Q_8^*}$ $[h_*] = \frac{8+\omega_8 Q_8^*}{8+Q_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+Q_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+P_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8-8}{(8+Q_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\,(\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, a e_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} & -[a_8]_{\,8} \\ e & 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (217)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (218)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (219) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (220) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (221)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (222) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (223)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (224)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(225)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(226)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(227) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(228)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(229)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(230)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(231) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(232)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(233)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(234)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(235)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(236)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(237)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(238)Bringing them to thermal equilibrium requires (e) an external bath.

(239) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(240)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(241)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(242)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(243)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(244)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(245) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(246) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(247) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(248)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(249)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(250) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(251)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(252)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(253) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(254)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(255)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (256)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (257) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (258)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (259) This is a realistic scenario for (e) all known systems. |
| (260)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (261)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (262)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (263) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (264)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (265)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (266)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (267)So in those systems, the quantum phase transition does indeed exist. |
| (268)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (269)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 27. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ∞; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a) e^{-\int_0^a \mu_9(\tau)dt} e^{-ra}\, da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(t)dt}\, da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9} \rho_9(t, a_9)\, da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9\,(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9)\int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2 < 4 [a_9] 2 < 4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} \left| h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9) \right|$. Then $|n_9(t)| \ 2 \Delta \forall t \ 0$. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 < 4\, a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]\ 4 [a_9] e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]\ 0.5$. Assume $[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$ and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}[\ \mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a\ {}_{9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_9 - b) Q^*}{\delta_9 (1 + Q_9^*)^9} - \frac{4\,(\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{mu}_{9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a^{9}_{9} > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[\,g^9_*]\ _9\,Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO B

## Theory Of Quantum Error-Correcting Codes-2

**INTRODUCTION—VARIABLES USED**

**A Theory of Quantum Error-Correcting Codes Emanuel Knill, Raymond Laflamme (Submitted on 26 Apr 1996**)

(1) Authors relate this definition to (e&eb) four others: The existence of a left inverse of (e) the interaction, an explicit representation of (e) the error syndrome using (e) tensor products, perfect recovery of (e) the completely entangled state, and an information theoretic identity.
(2) Two notions of fidelity and error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states.
(3) The latter is more appropriate when using (e) codes in a quantum memory or in applications of (e) quantum teleportation to communication.
(4) Authors show that the error for entangled states is bounded linearly by (e) the error for pure states.
**(5)** A formal definition of independent interactions for (e) qubits is given.
**(6)** This leads to (eb) lower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component. Subjects: Quantum Physics (quant-ph) Journal reference:Phys.Rev.Lett.84:2525-2528,2000 DOI: 10.1103/PhysRevLett.84.2525 Report number: LA-UR 96-1300 Cite as: arXiv:quant-ph/9604034 (or arXiv:quant-ph/9604034v1 for this version)

**NOTATION**

**Module One**

Authors relate this definition to (e&eb) four others: The existence of a left inverse of (e) the interaction, an explicit representation of (e) the error syndrome using (e) tensor products, perfect recovery of (e) the completely entangled state, and an information theoretic identity

$G_{13}$ : Category one of **recovery operator's independent definition of error-correcting codes;** existence of a left inverse of (e) the interaction

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of existence of a left inverse of (e) the interaction **;recovery operator's independent definition of error-correcting codes**

| | |
|---|---|
| $T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two** | |
| $G_{16}$ : Category one of **recovery operator's independent definition of error-correcting codes;** explicit representation of (e) the error syndrome<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of explicit representation of (e) the error syndrome ;**recovery operator's independent definition of error-correcting codes**<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three** | |
| $G_{20}$ : Category one of **recovery operator's independent definition of error-correcting codes;** perfect recovery of (e) the completely entangled state,<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of perfect recovery of (e) the completely entangled state,;**recovery operator's independent definition of error-correcting codes**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four** | |
| $G_{24}$ : Category one of **recovery operator's independent definition of error-correcting codes;** information theoretic identity<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of information theoretic identity **;recovery operator's independent definition of error-correcting codes**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five** | |

| | |
|---|---|
| explicit representation of the error syndrome using (e) tensor products | |
| $G_{28}$ : Category one of tensor products<br>$G_{29}$ : Category two of SAS<br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of explicit representation of the error syndrome<br>$T_{29}$ : Category two of SAS<br>$T_{30}$ : Category three of SAS | |
| **Module six**<br>perfect recovery of (e) the completely entangled state, | |
| $G_{32}$ : Category one of completely entangled state<br>$G_{33}$ : Category two of SAS<br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of perfect recovery<br>$T_{33}$ : Category two of SAS<br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br>perfect recovery of the completely entangled state, and an information theoretic identity | |
| $G_{36}$ : Category one of **perfect recovery of the completely entangled state;** information theoretic identity<br>$G_{37}$ : Category two of SAS<br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of information theoretic identity **;perfect recovery of the completely entangled state**<br>$T_{37}$ : Category two of SAS<br>$T_{38}$ : Category three of SAS | |
| **Module eight** | |
| $G_{40}$ : Category one of information theoretic identity; **perfect recovery of the completely entangled state**<br>$G_{41}$ : Category two of SAS | |

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of **perfect recovery of the completely entangled state**; information theoretic identity

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

$G_{44}$ : Category one of perfect recovery of (e) the completely entangled state; **explicit representation of the error syndrome using (e) tensor products**

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of **explicit representation of the error syndrome using (e) tensor products;** perfect recovery of (e) the completely entangled state

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

| | |
|---|---|
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14},t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14},t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |

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| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |

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| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3 | |

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| $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16}$ | 61 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}+(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{17}$ | 62 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{18}$ | 63 |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | 65 |

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

| | |
|---|---|
| $+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)} & -(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)} & -(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ & -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ & -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 | |

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}+(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t)\\ +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t)\\ +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)\end{bmatrix}G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

Where $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$

are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$

are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$

are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$

are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$

are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

Where $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ are first augmentation coefficients for category 1, 2 and 3

And $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)}$ are third augmentation coefficient for category 1, 2 and 3

| | |
|---|---|
| $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | *82* |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | *83* |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | *84* |
| *where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3<br>$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3<br>$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3<br>$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3 | |

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\begin{bmatrix}(a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\begin{bmatrix}(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{34} \quad (87)$$

$+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt}=(b_{32})^{(6)}T_{33}-\begin{bmatrix}(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

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Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

| | |
|---|---|
| Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[\begin{matrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{matrix} (a_{41}^{'})^{(8)}\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{matrix} (a_{42}^{'})^{(8)}\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{15}$ | |
| Where $\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second | |

augmentation coefficient for category 1, 2 and 3
$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3
$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3
$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3
$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}(G_{43},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}-(b_{41}^{''})^{(8)}(G_{43},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}(G_{43},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3
$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3
$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3
$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

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|---|---|
| $-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt}$<br>$= (a_{44})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt}$<br>$= (a_{45})^{(9)}G_{44}$<br>$-\begin{bmatrix} (a_{45}^{'})^{(9)}+(a_{45}^{''})^{(9)}(T_{45},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt}$<br>$= (a_{46})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3

$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b_{44}^{'})^{(9)}\left[-(b_{44}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\left[-(b_{45}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\left[-(b_{46}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{44}^{''})^{(9)}(G_{47},t)$, $-(b_{45}^{''})^{(9)}(G_{47},t)$, $-(b_{46}^{''})^{(9)}(G_{47},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

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| $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| Where we suppose | |
| $(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$<br><br>The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | *97* |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}$, $(r_i)^{(1)}, i = 13,14,15,$ | 101 |

satisfy the inequalities

$$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$$

$$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$$

Where we suppose

$(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$

The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$:

$$(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)} \quad (102)$$

$$(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)} \quad (103)$$

$$\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)} \quad (104)$$

$$\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)} \quad (105)$$

**Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : (106)

Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)| \le (\hat{k}_{16})^{(2)}|T_{17} - T_{17}'|e^{-(\hat{M}_{16})^{(2)}t} \quad (107)$$

$$|(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)| < (\hat{k}_{16})^{(2)}||(G_{19}) - (G_{19})'||e^{-(\hat{M}_{16})^{(2)}t} \quad (108)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$. It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ :

$(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants (109)

$$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$$

**Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :

There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants

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| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$ satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$<br>$\lim_{G \to \infty}(b_i^{''})^{(3)}(G_{23}, t) = (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

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| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)} (\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(4)} (T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(4)} ((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)\rvert \leq (\hat{k}_{24})^{(4)} \lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)} t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})', t) - (b_i^{''})^{(4)}((G_{27}), t)\rvert < (\hat{k}_{24})^{(4)} \lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)} t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |

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| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}(G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\rvert \leq (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}((G_{31}), t)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants | 125 |

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| $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\| \leq (\hat{k}_{32})^{(6)} \|T_{33} - T_{33}^{'}\| e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\|(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\| < (\hat{k}_{32})^{(6)} \|(G_{35}) - (G_{35})'\| e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants | 129 |

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(S) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(T) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(U)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(V)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the

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| system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(W) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(X) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}, (r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i'')^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i')^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i'')^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i'')^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}'\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |

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| $\|(b_i^{''})^{(8)}((G_{43})',t)-(b_i^{''})^{(8)}\big((G_{43}),t\big)\|<(\hat{k}_{40})^{(8)}\|(G_{43})-(G_{43})'\|e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}',t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}',t)$ and $(T_{41},t)$ are points belonging to the interval $\big[(\hat{k}_{40})^{(8)},(\hat{M}_{40})^{(8)}\big]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)}=1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\ ,\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}<1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)},(\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},(\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)},(a_i')^{(8)},(b_i)^{(8)},(b_i')^{(8)},(p_i)^{(8)},\ (r_i)^{(8)},i=40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)}+(a_i')^{(8)}+\ (\hat{A}_{40})^{(8)}+(\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}]<1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\ (b_i)^{(8)}+(b_i')^{(8)}+\ (\hat{B}_{40})^{(8)}+(\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}]<1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)},(a_i')^{(9)},(a_i^{''})^{(9)},(b_i)^{(9)},(b_i')^{(9)},(b_i^{''})^{(9)}>0,\quad i,j=44,45,46$<br><br>The functions $(a_i^{''})^{(9)},(b_i^{''})^{(9)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(9)},\ (r_i)^{(9)}$:<br><br>$(a_i^{''})^{(9)}(T_{45},t)\le(p_i)^{(9)}\le(\hat{A}_{44})^{(9)}$<br><br>$(b_i^{''})^{(9)}(G_{47},t)\le\ (r_i)^{(9)}\le(b_i')^{(9)}\le(\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i^{''})^{(9)}(T_{45},t)=(p_i)^{(9)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(9)}(G_{47},t)=\ (r_i)^{(9)}$<br><br>**Definition of** $(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)}$ :<br><br>Where $\boxed{(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}}$ are positive constants and $\boxed{i=44,45,46}$ | |
| They satisfy Lipschitz condition: | |

$$|(a_i^{''})^{(9)}(T_{45}^{'},t)-(a_i^{''})^{(9)}(T_{45},t)|\le(\hat{k}_{44})^{(9)}|T_{45}-T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'},t)-(b_i^{''})^{(9)}\big((G_{47}),t\big)|<(\hat{k}_{44})^{(9)}||(G_{47})-(G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'},t)$ and$(a_i^{''})^{(9)}(T_{45},t)$ . $(T_{45}^{'},t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)}=1$ then the function $(a_i^{''})^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)}and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i^{'})^{(9)},(b_i)^{(9)},(b_i^{'})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i^{'})^{(9)}+(\hat{A}_{44})^{(9)}+(\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)}+(b_i^{'})^{(9)}+(\hat{B}_{44})^{(9)}+(\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0),T_i(0)$ :

$$G_i(t)\le(\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le(\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0),T_i(0)$

$$G_i(t)\le(\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ G_i(0)=G_i^0>0$$

$$T_i(t)\le(\hat{Q}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}t}\ ,\ G_i(0)=G_i^0>0$$

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| $T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 B |

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| $G_i(t) \leq \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{20})^{(3)}$ , $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{24})^{(4)}$ , $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |

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| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{\mathrm{T}}_{26}(\mathrm{t}) = \mathrm{T}_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{\mathrm{T}}_{30}(\mathrm{t}) = \mathrm{T}_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |

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| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}')^{(7)} - (b_{37}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}')^{(7)} - (b_{38}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br><br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br><br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}')^{(8)} + a_{40}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}')^{(8)} + (a_{41}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}')^{(8)} + (a_{42}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}')^{(8)} - (b_{40}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}')^{(8)} - (b_{41}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}')^{(8)} - (b_{42}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br><br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$$

Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$

The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$$

$$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$$

(167)

From which it follows that

$$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 1

(168)

Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$$

(169)

From which it follows that

$$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$$

(170)

Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

(171)

$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that 172

$$(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$$

From which it follows that 174

$$(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that 175

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

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(d) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

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$$G_{36}(t) \le G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$
$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$
$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$

180

From which it follows that

$$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

181

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)} =$$
$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

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| $(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose<br>$(\widehat{\mathrm{P}}_{13})^{(1)}$ and $(\widehat{\mathrm{Q}}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G}, \tilde{T}$ : $(\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{13})^{(1)}\left\|G_{14}^{(1)} - G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$G_{13}^{(2)}\left\|(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a_{13}'')^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right\| \; e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a_{13}')^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$ | 186 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(1)}t)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results | |

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| $T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded. The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t} \leq$ | |

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| $\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+\ (a'_{16})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)\mathrm{d}\left(((G_{19})^{(1)},(T_{19})^{(1)};\ (G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a''_{16})^{(2)}$ and $(b''_{16})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(2)}$ and $(b''_i)^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a'_i)^{(2)}-(a''_i)^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}\mathrm{d}s_{(16)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{(-(b'_i)^{(2)}t)}>0$ for $t>0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1,\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16}<(\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt}\leq\left((\widehat{M}_{16})^{(2)}\right)_1-(a'_{17})^{(2)}G_{17}$ and by integrating<br><br>$G_{17}\leq\left((\widehat{M}_{16})^{(2)}\right)_2=G_{17}^0+2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a'_{17})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18}\leq\left((\widehat{M}_{16})^{(2)}\right)_3=G_{18}^0+2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a'_{18})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(2)}((G_{19})(t),t))=(b'_{17})^{(2)}$ then $T_{17}\to\infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t>t_2$<br><br>$(b_{17})^{(2)}-(b''_i)^{(2)}((G_{19})(t),t)<\varepsilon_2, T_{16}(t)>(m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt}\geq(a_{17})^{(2)}(m)^{(2)}-\varepsilon_2 T_{17}$ which leads to | 204 |

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| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1-e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t), t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left(\widetilde{(G_{23})}, \widetilde{(T_{23})}\right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}'')^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |

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| $\left\|{G_{23}}^{(1)} - {G_{23}}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)}t} \leq$ $\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right) d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a''_{20})^{(3)}$ and $(b''_{20})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ and $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(3)}$ and $(b''_i)^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(and not on $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$<br>it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(3)} - (a''_i)^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ and $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ and $G_{22}$ . indeed if<br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a'_{21})^{(3)} G_{21}$ and by integrating<br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1 / (a'_{21})^{(3)}$<br>In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2 / (a'_{22})^{(3)}$<br>If $G_{21}$ or $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ is bounded, from below, the same property holds for $G_{21}$ and $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(3)}((G_{23})(t), t)) = (b'_{21})^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b''_i)^{(3)}((G_{23})(t), t) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to | 220 |

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1-e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.
The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose

$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have

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$$\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)} \qquad 222$$

$$\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)} \qquad 223$$

In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself 224

The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric 225

$$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{24})^{(4)}\left|G_{25}^{(1)} - G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

| | |
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| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\| e^{-(\widehat{M}_{24})^{(4)}t} \leq$ $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a'_{24})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$ And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a''_{24})^{(4)}$ and $(b''_{24})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(4)}$ and $(b''_i)^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(4)} - (a''_i)^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ : **Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating $G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a'_{25})^{(4)}$ In the same way , one can obtain $G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2 / (a'_{26})^{(4)}$ If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(4)}((G_{27})(t), t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$. **Definition of** $(m)^{(4)}$ and $\varepsilon_4$ : Indeed let $t_4$ be so that for $t > t_4$ $(b_{25})^{(4)} - (b''_i)^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad (234)$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad (235)$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric (236)

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{31})}, \widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})}, \widetilde{(T_{31})}\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)}, s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

| | |
|---|---|
| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$ | 242 |

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| $(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}\,(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\,\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $G_{32}^{(2)}\|(a_{32}^{''})^{(6)}(T_{33}^{(1)}, s_{(32)}) - (a_{32}^{''})^{(6)}(T_{33}^{(2)}, s_{(32)})\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br><br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)} - (G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t} \leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)} + (a_{32}^{'})^{(6)} + (\widehat{A}_{32})^{(6)} + (\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)}, (T_{35})^{(1)};\ (G_{35})^{(2)}, (T_{35})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}\ and\ (\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i = 32{,}33{,}34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(6)} - (a_i^{''})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\}ds_{(32)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(6)}t)} > 0$ for t > 0 | 250 |
| **Definition of** $((\widehat{M}_{32})^{(6)})_1, ((\widehat{M}_{32})^{(6)})_2\ and\ ((\widehat{M}_{32})^{(6)})_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq ((\widehat{M}_{32})^{(6)})_1 - (a_{33}^{'})^{(6)}G_{33}$ and by integrating<br><br>$G_{33} \leq ((\widehat{M}_{32})^{(6)})_2 = G_{33}^0 + 2(a_{33})^{(6)}((\widehat{M}_{32})^{(6)})_1/(a_{33}^{'})^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq ((\widehat{M}_{32})^{(6)})_3 = G_{34}^0 + 2(a_{34})^{(6)}((\widehat{M}_{32})^{(6)})_2/(a_{34}^{'})^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t), t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ : | 253 |

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| Indeed let $t_6$ be so that for $t > t_6$<br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t), t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}^{'})^{(6)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$<br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left|G_{37}^{(1)} - G_{37}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}} +$ | 258 |

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| $G_{36}^{(2)}\|(a_{36}^{''})^{(7)}(T_{37}^{(1)},s_{(36)})-(a_{36}^{''})^{(7)}(T_{37}^{(2)},s_{(36)})\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}\ and\ (\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{\left(-(b_i^{'})^{(7)}t\right)}>0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1,\left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36}<(\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt}\leq\left((\widehat{M}_{36})^{(7)}\right)_1-(a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37}\leq\left((\widehat{M}_{36})^{(7)}\right)_2=G_{37}^0+2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38}\leq\left((\widehat{M}_{36})^{(7)}\right)_3=G_{38}^0+2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |

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| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$. | 264 |
| **Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br>Indeed let $t_7$ be so that for $t > t_7$<br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1-e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t),t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br>**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results | 271 |

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| $\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)}-G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\, ds_{(40)} +$ $\int_0^t \{(a_{40}^{'})^{(8)} \left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $G_{40}^{(2)} \left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\} ds_{(40)}$ | |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(\left((G_{43})^{(1)}, (T_{43})^{(1)}; (G_{43})^{(2)}, (T_{43})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)}-(a_i^{''})^{(8)}(T_{41}(s_{(40)}),s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)}G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1/(a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2/(a_{42}^{'})^{(8)}$ | 276 |

If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

**Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

**Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t),t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.
The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t),t(t),t\big) = (b_{42}^{'})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ and to choose $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[(\hat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\Big(((G_{47})^{(1)},(T_{47})^{(1)}),\big((G_{47})^{(2)},(T_{47})^{(2)}\big)\Big) =$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}$ : $\big(\widetilde{(G_{47})},\widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}),(T_{47})\big)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{44})^{(9)} \left|G_{45}^{(1)}-G_{45}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\, ds_{(44)} +$$
$$\int_0^t \{(a_{44}^{'})^{(9)} \left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\} ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)} (\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{45}^{'})^{(9)}$ then $T_{45}\to\infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t>t_9$

$$(b_{45})^{(9)}-(b_i^{''})^{(9)}\left((G_{47})(t),t\right)<\varepsilon_9, T_{44}(t)>(m)^{(9)}$$

Then $\frac{dT_{45}}{dt}\geq(a_{45})^{(9)}(m)^{(9)}-\varepsilon_9 T_{45}$ which leads to

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1-e^{-\varepsilon_9 t})+T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t}=\frac{1}{2}$ it results

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t=log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)}\leq-(a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}-(a_{13}^{''})^{(1)}(T_{14},t)+(a_{14}^{''})^{(1)}(T_{14},t)\leq-(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)}\leq-(b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G,t)-(b_{14}^{''})^{(1)}(G,t)\leq-(\tau_1)^{(1)}$

**Definition of** $(v_1)^{(1)},(v_2)^{(1)},(u_1)^{(1)},(u_2)^{(1)},v^{(1)},u^{(1)}$ : 281

By $(v_1)^{(1)}>0$, $(v_2)^{(1)}<0$ and respectively $(u_1)^{(1)}>0$, $(u_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_1)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_1)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(\bar{v}_1)^{(1)},,(\bar{v}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ : 282

By $(\bar{v}_1)^{(1)}>0$, $(\bar{v}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0$, $(\bar{u}_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(v^{(1)}\right)^2+(\sigma_2)^{(1)}v^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)},(v_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)}$ by
$(m_2)^{(1)}=(v_0)^{(1)},(m_1)^{(1)}=(v_1)^{(1)}$, $if\ (v_0)^{(1)}<(v_1)^{(1)}$

$(m_2)^{(1)}=(v_1)^{(1)},(m_1)^{(1)}=(\bar{v}_1)^{(1)}$, $if\ (v_1)^{(1)}<(v_0)^{(1)}<(\bar{v}_1)^{(1)}$,

and $\boxed{(v_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$

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| $(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | |
| and analogously<br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$ are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}')^{(1)})}\left[e^{(S_1)^{(1)} t} - e^{-(a_{15}')^{(1)} t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b_{15}')^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b_{15}')^{(1)} t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$<br>$(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |

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| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(T_{17}, t) + (a_{17}^{''})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)}, , (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ , ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$ , ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities | 310 |

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| $G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^{0}e^{(S_1)^{(2)}t}$ | |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}}G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}}G_{16}^{0}e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t}-e^{-(S_2)^{(2)}t}\right]+G_{18}^{0}e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and | 320 |

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| By $(\bar{\nu}_1)^{(3)} > 0$ , $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ , ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,<br>and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br>Then the solution of global equations satisfies the inequalities<br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le\right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a_{22}')^{(3)}t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b_{22}')^{(3)}t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

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**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(\nu_1)^{(4)}, (\nu_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, \nu^{(4)}, u^{(4)}$ :

By $(\nu_1)^{(4)} > 0$, $(\nu_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(\nu^{(4)})^2 + (\sigma_1)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

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**Definition of** $(\bar{\nu}_1)^{(4)},, (\bar{\nu}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ :

By $(\bar{\nu}_1)^{(4)} > 0$, $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}(\nu^{(4)})^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}(u^{(4)})^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$

$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$, ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,

and $\boxed{(\nu_0)^{(4)} = \frac{G^0_{24}}{G^0_{25}}}$

$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$

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and analogously

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ : | 339 |

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| By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$, $(v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b'_{30})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b'_{30})^{(5)}t}\right]+T_{30}^{0}e^{-(b'_{30})^{(5)}t}\le T_{30}(t)\le$$

$$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$$ 347

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- 348

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a'_{28})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b'_{28})^{(5)}$

$(R_2)^{(5)}=(b'_{30})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** 349

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a'_{32})^{(6)}+(a'_{33})^{(6)}-(a''_{32})^{(6)}(T_{33},t)+(a''_{33})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b'_{32})^{(6)}+(b'_{33})^{(6)}-(b''_{32})^{(6)}\big((G_{35}),t\big)-(b''_{33})^{(6)}\big((G_{35}),t\big)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : 350

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,
and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b'_{34})^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b'_{34})^{(6)} t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$<br>$(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

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| $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying<br><br>$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$<br><br>$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}\big((G_{39}), t\big) - (b''_{37})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ :<br><br>By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ :<br><br>By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$<br>and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br><br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br><br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | 362 |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{36} e^{\left((S_1)^{(7)} - (p_{36})^{(7)}\right)t} \le G_{36}(t) \le G^0_{36} e^{(S_1)^{(7)}t}$ | 364 |

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| where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$ $\frac{(a_{38})^{(7)}G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$ $\frac{(a_{38})^{(7)}T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- <br> Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ <br> $(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$ <br> $(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$ <br> $(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define <br> **Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ : <br> $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying <br> $-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$ <br> $-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br> By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(v^{(8)})^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ | 372 |

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| and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(8)},,(\bar{v}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the<br>roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br>**Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |

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| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$ $\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(v_1)^{(9)}, (v_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, v^{(9)}, u^{(9)}$ :<br>By $(v_1)^{(9)} > 0$, $(v_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$, $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(9)}, , (\bar{v}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :<br>By $(\bar{v}_1)^{(9)} > 0$, $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$, $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$<br>**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (v_0)^{(9)}$ :-<br>If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by<br>$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$ | |

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{44}(t) \leq G_{44}^0 e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{45}(t) \leq \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)} t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})} \left[ e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)} t} \right] + G_{46}^0 e^{-(S_2)^{(9)} t} \leq G_{46}(t) \leq \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a'_{46})^{(9)})} [e^{(S_1)^{(9)} t} - e^{-(a'_{46})^{(9)} t}] + G_{46}^0 e^{-(a'_{46})^{(9)} t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)} t} - e^{-(b'_{46})^{(9)} t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)} t} \leq T_{46}(t) \leq$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)} + (r_{44})^{(9)} + (R_2)^{(9)})} \left[ e^{((R_1)^{(9)} + (r_{44})^{(9)})t} - e^{-(R_2)^{(9)} t} \right] + T_{46}^0 e^{-(R_2)^{(9)} t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get 384

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, 385

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq v^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{v^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $v^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

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| In the same manner , we get<br>$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | 391 |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br>**Definition of** $u^{(2)}(t)$ :-<br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |

**Proof** : From global equations we obtain 398

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \leq \frac{dv^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \geq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \leq v^{(3)}(t) \leq (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \leq v^{(3)}(t) \leq (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \leq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \leq v^{(3)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \leq (\bar{v}_1)^{(3)} \leq (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \leq v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq v^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{v^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $v^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{dv^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)v^{(4)} - (a_{25})^{(4)}v^{(4)}$$

**Definition of** $v^{(4)}$ :- $\boxed{v^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)}\right) \leq \frac{dv^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_4)^{(4)}v^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$v^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq v^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

$$v^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

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From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)\le\frac{dv^{(5)}}{dt}\le-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(5)},(v_0)^{(5)}$ :-

For $0<\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}<(v_1)^{(5)}<(\bar{v}_1)^{(5)}$

$$v^{(5)}(t)\ge\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}{5+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}\quad,\quad\boxed{(C)^{(5)}=\frac{(v_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(v_2)^{(5)}}}$$

it follows $(v_0)^{(5)}\le v^{(5)}(t)\le(v_1)^{(5)}$

In the same manner , we get

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$$v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(5)}=\frac{(\bar{v}_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(\bar{v}_2)^{(5)}}}$$

From which we deduce $(v_0)^{(5)}\le v^{(5)}(t)\le(\bar{v}_5)^{(5)}$

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If $0<(v_1)^{(5)}<(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}<(\bar{v}_1)^{(5)}$ we find like in the previous case,

$$(v_1)^{(5)}\le\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}\le v^{(5)}(t)\le$$

$$\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(\bar{v}_1)^{(5)}$$

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If $0<(v_1)^{(5)}\le(\bar{v}_1)^{(5)}\le\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(v_1)^{(5)}\le v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(v_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(5)}(t)$ :-

$(m_2)^{(5)}\le v^{(5)}(t)\le(m_1)^{(5)}$, $\boxed{v^{(5)}(t)=\frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)}\le u^{(5)}(t)\le(\mu_1)^{(5)}$, $\boxed{u^{(5)}(t)=\frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case, 414

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain 415

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\quad \boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

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| In the same manner , we get<br><br>$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$<br><br>From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$ | 417 |
| If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,<br><br>$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$<br><br>$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$ | 418 |
| If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain<br><br>$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$<br><br>And so with the notation of the first part of condition (c) , we have<br>**Definition of** $\nu^{(7)}(t)$ :-<br><br>$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$<br>In a completely analogous way, we obtain | 419 |
| **Definition of** $u^{(7)}(t)$ :-<br><br>$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$<br><br>Now, using this result and replacing it in global equations we get easily the result stated in the theorem.<br><br>**Particular case :**<br><br>If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .<br><br>Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$. | 420 |

**Proof** : From global equations we obtain 421

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \leq v^{(8)}(t) \leq (v_1)^{(8)}$

In the same manner , we get 422

$$v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \leq v^{(8)}(t) \leq (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{v}_1)^{(8)}$ we find like in the previous case, 423

$$(v_1)^{(8)} \leq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \leq v^{(8)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \leq (\bar{v}_1)^{(8)} \leq \boxed{(v_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(v_1)^{(8)} \leq v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \leq \frac{dv^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$v^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq v^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ , $(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}')^{(1)}(r_{14})^{(1)} - (b_{14}')^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$ $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i'')^{(2)} and\ (b_i'')^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}')^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}')^{(2)}(r_{17})^{(2)} - (b_{17}')^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ $with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i'')^{(3)} and\ (b_i'')^{(3)}$ are independent on $t$ , and the conditions with the notations $(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$ $(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}')^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$ $(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ , $(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}')^{(3)}(r_{21})^{(3)} - (b_{21}')^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$ $with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following **Theorem :** If $(a_i'')^{(4)} and\ (b_i'')^{(4)}$ are independent on $t$ , and the conditions with the notations $(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$ $(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}')^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$ $(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ , $(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}')^{(4)}(r_{25})^{(4)} - (b_{25}')^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$ $with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i'')^{(5)} and\ (b_i'')^{(5)}$ are independent on $t$ , and the conditions with the notations $(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$ $(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}')^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$ $(b_{28}')^{(5)}(b_{29}')^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ , | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$ | 436 A |

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| $(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |

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| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |

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| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(d) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if | 486 |

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| $F(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) + (a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) + (a''_{28})^{(5)}(T_{29})(a''_{29})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a'_{32})^{(6)}(a'_{33})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a'_{32})^{(6)}(a''_{33})^{(6)}(T_{33}) + (a'_{33})^{(6)}(a''_{32})^{(6)}(T_{33}) + (a''_{32})^{(6)}(T_{33})(a''_{33})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a'_{36})^{(7)}(a'_{37})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a'_{36})^{(7)}(a''_{37})^{(7)}(T_{37}) + (a'_{37})^{(7)}(a''_{36})^{(7)}(T_{37}) + (a''_{36})^{(7)}(T_{37})(a''_{37})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a'_{40})^{(8)}(a'_{41})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a'_{40})^{(8)}(a''_{41})^{(8)}(T_{41}) + (a'_{41})^{(8)}(a''_{40})^{(8)}(T_{41}) + (a''_{40})^{(8)}(T_{41})(a''_{41})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a'_{44})^{(9)}(a''_{45})^{(9)}(T_{45}) + (a'_{45})^{(9)}(a''_{44})^{(9)}(T_{45}) +$ | 492 A |

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| $(a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first | 499 |

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| equations<br><br>$G_{32} = \frac{(a_{32})^{(6)} G_{33}}{[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)} G_{33}}{[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)} G_{37}}{[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)} G_{37}}{[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)] + (b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})] + (b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |

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| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |

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| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b'_{40})^{(8)}(b'_{41})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b'_{40})^{(8)}(b''_{41})^{(8)}(G_{43}) + (b'_{41})^{(8)}(b''_{40})^{(8)}(G_{43})\right] + (b''_{40})^{(8)}(G_{43})(b''_{41})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{41}$ such that $\varphi(G_{43}{}^*) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b'_{44})^{(9)}(b''_{45})^{(9)}(G_{47}) + (b'_{45})^{(9)}(b''_{44})^{(9)}(G_{47})\right] + (b''_{44})^{(9)}(G_{47})(b''_{45})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{45}$ such that $\varphi((G_{47})^*) = 0$ | |
| Finally we obtain the unique solution<br><br>$G^*_{14}$ given by $\varphi(G^*) = 0$ , $T^*_{14}$ given by $f(T^*_{14}) = 0$ and<br><br>$G^*_{13} = \frac{(a_{13})^{(1)}G^*_{14}}{[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T^*_{14})]}$ , $G^*_{15} = \frac{(a_{15})^{(1)}G^*_{14}}{[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T^*_{14})]}$<br><br>$T^*_{13} = \frac{(b_{13})^{(1)}T^*_{14}}{[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G^*)]}$ , $T^*_{15} = \frac{(b_{15})^{(1)}T^*_{14}}{[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G^*)]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $\mathrm{G}^*_{17}$ given by $\varphi((G_{19})^*) = 0$ , $\mathrm{T}^*_{17}$ given by $f(\mathrm{T}^*_{17}) = 0$ and | 512 |
| $\mathrm{G}^*_{16} = \frac{(\mathrm{a}_{16})^{(2)}\mathrm{G}^*_{17}}{[(\mathrm{a}'_{16})^{(2)} + (\mathrm{a}''_{16})^{(2)}(\mathrm{T}^*_{17})]}$ , $\mathrm{G}^*_{18} = \frac{(\mathrm{a}_{18})^{(2)}\mathrm{G}^*_{17}}{[(\mathrm{a}'_{18})^{(2)} + (\mathrm{a}''_{18})^{(2)}(\mathrm{T}^*_{17})]}$ | 513 |
| $\mathrm{T}^*_{16} = \frac{(\mathrm{b}_{16})^{(2)}\mathrm{T}^*_{17}}{[(\mathrm{b}'_{16})^{(2)} - (\mathrm{b}''_{16})^{(2)}((G_{19})^*)]}$ , $\mathrm{T}^*_{18} = \frac{(\mathrm{b}_{18})^{(2)}\mathrm{T}^*_{17}}{[(\mathrm{b}'_{18})^{(2)} - (\mathrm{b}''_{18})^{(2)}((G_{19})^*)]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G^*_{21}$ given by $\varphi((G_{23})^*) = 0$ , $T^*_{21}$ given by $f(T^*_{21}) = 0$ and<br><br>$G^*_{20} = \frac{(a_{20})^{(3)}G^*_{21}}{[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T^*_{21})]}$ , $G^*_{22} = \frac{(a_{22})^{(3)}G^*_{21}}{[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T^*_{21})]}$<br><br>$T^*_{20} = \frac{(b_{20})^{(3)}T^*_{21}}{[(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23}{}^*)]}$ , $T^*_{22} = \frac{(b_{22})^{(3)}T^*_{21}}{[(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23}{}^*)]}$ | 515 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)} (T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} ((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)} ((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)} (T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)} (T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)} ((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)} ((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)} (T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)} (T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)} ((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)} ((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and<br>$G_{36}^* = \frac{(a_{36})^{(7)} G_{37}^*}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^*)]}$ , $G_{38}^* = \frac{(a_{38})^{(7)} G_{37}^*}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^*)]}$<br>$T_{36}^* = \frac{(b_{36})^{(7)} T_{37}^*}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^*)]}$ , $T_{38}^* = \frac{(b_{38})^{(7)} T_{37}^*}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^*)]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^*$ given by $\varphi((G_{43})^*) = 0$ , $T_{41}^*$ given by $f(T_{41}^*) = 0$ and | 523 |

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| $G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523 A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 531 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |

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| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b'_{20})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T^*_{20}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b'_{21})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T^*_{21}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b'_{22})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T^*_{22}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T^*_{25}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^*) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a'_{24})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G^*_{24}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a'_{25})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G^*_{25}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a'_{26})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G^*_{26}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b'_{24})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T^*_{24}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b'_{25})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T^*_{25}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b'_{26})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T^*_{26}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T^*_{29}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^*) = s_{ij}$ | 556 |

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| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ $and$ $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ $and$ $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 571 |

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| Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |

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| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$<br><br>$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ | |

$$+\left(\left((\lambda)^{(1)}+(a_{14}^{'})^{(1)}+(p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^{*}+(a_{13})^{(1)}(q_{14})^{(1)}G_{14}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(13)}T_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$+\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}^{'})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}^{'})^{(2)}-(r_{18})^{(2)})\{\left((\lambda)^{(2)}+(a_{18}^{'})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left(\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}^{'})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}^{'})^{(3)}-(r_{22})^{(3)})\{\left((\lambda)^{(3)}+(a_{22}^{'})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

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| $+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$ $\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$ $\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$ $\left(\left((\lambda)^{(3)}\right)^{2}+\left((b'_{20})^{(3)}+(b'_{21})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$ $+\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$ $+\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$ $\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\}=0$ $+$ | |
| $(\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\}\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$ $\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$ $+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$ $\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$ $\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$ $+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$ $+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$ $\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$ $+$ | |
| $(\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$ $\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$ $\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$ | |

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b'_{34})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a'_{34})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(33)}T^{*}_{33}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^{*}_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^{*}_{33}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(32)}T^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(34)}T^{*}_{32}\right)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(37)}T^{*}_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^{*}_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^{*}_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(36)}T^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(38)}T^{*}_{36}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\left(\left((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left(\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future**. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_{i,}$ $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i Xi(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \qquad (4)$$

Where $x_i$ is the abundance of population $i$, $x= (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 4. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon\ C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For $K \subset C$ and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \|x - y\| < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}$ $(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. *If* $x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B*.

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ($\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $\mathrm{A_i}(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A_i}(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A_i}(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i\ \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N_\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $\dot{x} = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $\dot{x} = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x$,$W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $\dot{x} = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - \alpha x_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.10) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.11) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.12) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^{\infty} b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^{\infty} b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s), a.0, s\epsilon [-\tau, 0], \quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^{\infty} \varphi_i(a, s)\, da, s\in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 4. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{\ i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +}$ [$-\tau, T] \rightarrow R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +}$ [$\{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +}$ [$-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t,0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right)ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t) = B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$

$$+\int_0^\infty b_i\left(a+t,P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i\,(t)\right)da$$

$$i = 1,\ 2,\ t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*\,(t-a)] \ \le\ 1, if\ \ t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^*\,\beta_i(a)\exp[-d_i^*(t-a)]\,da + \beta_i^*\Phi_i(0)\exp[-d_i^* t],\ \ i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^*\Phi_i(0)\exp[\delta_i t],\ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i\,(t) - 0_i\,(0)| \le \beta(0)\,exp[\delta_i\,a - d_i^*(t-a)]\,da$$

$$+\int_0^\infty \varphi_i\ (a,0)|1-exp(-d_i^* t)|da$$

$$\le \beta_i^*\Phi_i\ (0)\,exp(\beta_i^* t\,)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0)d_i^* t$$

$$\le [\beta_i^*\phi_i(0)\,exp(\beta_i^* t)\,\phi_i(0)d_i^*]t\ \ for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then ▯ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , ▯ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|▯(P_t^{(1)}) - ▯(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|▯(P^{(1)}(t)) - ▯(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2 / 2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta}\, 1^{a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\ \ 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

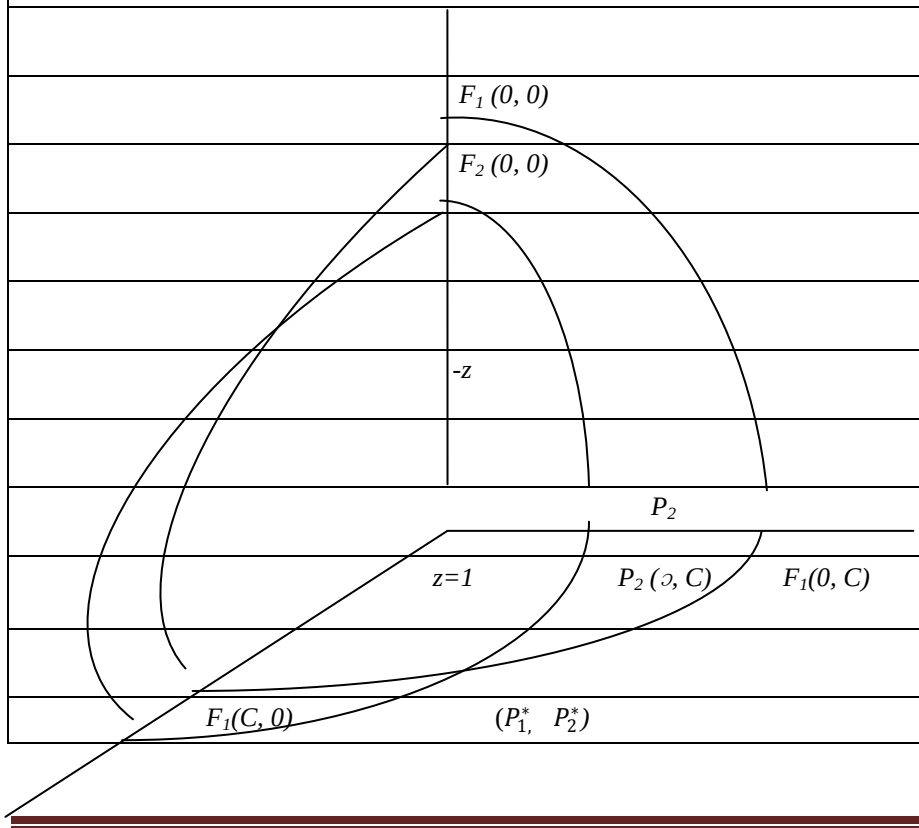
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{\varphi}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\varphi(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\varphi(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\varphi(a)da,$$

From the first of (3.15) we get

(3.18) $$\varphi(a) = \left\{\varphi(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\varphi(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp\left(-\lambda\tau\right),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \ | - | A_{11} + A_{22} + \| S_1(0) \| \ |^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 7. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(x) $b_1(0) = 0,\quad m_1(0, x_2) \equiv 0, b_2(0) = 0,\quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x^*_{1,} > 0, x^*_{2,} > 0$ we have

$$b_1(x^*_1, -m_1(x^*_1, 0) = 0,$$

$$b_2(x^*_2, -m_1(0, x^*_2) = 0,$$

(2.4)

(xi) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2 \quad (\delta_2) - m_2(x_1, \delta_2) < 0$, for $x_1$ 0, $x_2$ 0, (2.5)

(xii) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1$, $X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i, \; j = 1, 2; i \neq j \;)$.

## 8. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij} \; (i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij} \; , i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1 \; (i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ ,, $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\,(t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\,(t_2)\big) - m_1\big(x_1(t_2), x_2\,(t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\,(t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t \geq 0$.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\,(t - \tau_{11}) - m_{12}Y_2\,(t - \tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\,(t - \tau_{21}) - m_{22}Y_2\,(t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

***THEOREM 3.1***. *Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij} \geq 0$ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \qquad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(xiii)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \qquad g_2(0) = 0, \tag{4.3}$$

(xiv) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xv) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(xvi) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{vmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{vmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 8. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$N_1(s) = \phi_1(s) > 0,$

$s \in [-\tau, 0].$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \; (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt} = -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \qquad t > 0, \quad (6.1)$$

Where $m_{1J}, b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt} = -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}),$$

$$\frac{dZ_2(t)}{dt} = -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \qquad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad ||Z|| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad ||B|| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{matrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{matrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \succ \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{11}\left|Y_1(t)\right| + b_{11}\left|Y_1(t-\tau_{11})\right| + m_{12}\left|Y_2(t-\tau_{12})\right|, \\ \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{22}\left|Y_2(t)\right| + m_{21}\left|Y_1(t-\tau_{21})\right| + b_{22}\left|Y_2(t-\tau_{21})\right|, \end{aligned}\right\} \quad (A.2)$$

Where

$$\frac{d}{dt}\left|Y_1(t)\right| = \left\{\begin{aligned} &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0, \\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0, \end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(s) > \left|Y_1(s)\right|, \quad s \in [-\tau, 0], \; i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(t) \text{ for } t \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(t) = Z_i(t) - \left|Y_1(s)\right|$, $i = 1,2$. $t \ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\ge -m_{11}U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\ge -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(s) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \ge U_1(0)\exp[-m_{11}t]$

$$+\int_0^t \left[b_{11}U_1(s-\tau_{11}) + m_{12}U_2(s-\tau_{12})\right]\exp[-m_{11}(t-s)ds > 0,] \qquad (A.9)$$

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11},\tau_{12},\tau_{21},\tau_{22},)$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds+\int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds+\int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0\leq \mid Y_{1,}(t) \mid \leq Z_1(t)\to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}+\frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\to 0, Z_2(t)\to 0$ as $t\to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty sk_{ij}(s)ds< \infty;\ \int_0^\infty k_{ij}(s)ds=1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 10. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 11. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big), \qquad t > 0, \tag{2.1}$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ in $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ in $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ $\in -\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega+t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(x) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(xi) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(xii) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq 2\ max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(g)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(h) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 12. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(j) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(k) N is monotonic nondecreasing on $K^*$.

(l) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \|x_1\| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \|x_2\| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ –}\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ –}\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ –}\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(\mathrm{t} - \tau_{12})\, X_2(\mathrm{t} - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \geq -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j=1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x^*_1(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21})\, x^*_1(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} - {}_{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t}\{-a_{22}(t)\}\Big| = \max_{t}(1/K_2(t)) > \min_{t}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha,\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that “In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(t)}{dt} x_1(t)[1 - x_1(t) + 0.4x_2(t - \mathcal{T})/k_1(t)]$$

$$\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$$

Where $k_1(t) = 20 + \sin\ (\pi t)$ and $k_2(t)$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 7. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \dots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.4) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \; b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right)\Big)(1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \; x_i^l \; for \; t \; t_0, \; i = 1, 2, \ldots, n. \qquad (1.7)$$

8. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \; x_l \; x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1} x_l^l; \qquad x_* = \min_{1} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(67) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(68) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(69) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(70) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(71) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(72) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(73) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(74) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(75) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(76) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(77) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(78) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(79) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

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|---|
| (80) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (81) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (82) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (83) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (84) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (85) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (86) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (87) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (88) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception:

"what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left||f\ (t+\ \tau,x) - \ f(t,x)|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x\ \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^{*}\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{ij}\;(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^{*})/(2x^{*}). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$\upsilon\big(x(t+\tau_k),x(t+\tau_m)\big)\ \upsilon\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_{*}(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_{*})/(2x^{*})]/\mu x_{*})$$

$$[(2x_n^{*})/x_{*}]\exp[-\mu x_{*}(t+t_k-t_0)]\ +(\varepsilon/2x^{*})$$

$$<(\varepsilon/x^{*})\ \text{On using (2.8), (2.10)} \qquad (2.14)$$

For $m\ k\ n_0(\varepsilon,\beta)$ and for all $t\in\cup\subset[\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n}|log x_i\,(t+\tau_k)-log x_l(t+\tau_m)|<(\varepsilon/x^{*}) \qquad (2.15)$$

And hence using $x_{*}\ x_i(t)\ x^{*}, i=1,2,\dots,n; t\ t_0$,

$$\sum_{i=1}^{n}|x_i\,(t+\tau_k)-x_i(t+\tau_m)|<\varepsilon \text{ for all } t\in\cup\subset[\beta,\infty], m\ k\ n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t)=\{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t)\in[x_{*},x^{*}], i=1,2,\dots,n; t\in[\beta,\infty]$ follows from (2.16) and the fact that $x_i(t)\in[x_{*},x^{*}], i=1,2,\dots,n;$ for $t\ t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k)\to w_l(t)$ as $k\to\infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i=1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $x(t+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m, k \geq n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m, k \geq n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} |\frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m)| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad i = 1,2, \dots, n; t \geq t_0 \qquad (3.7)$$
$$\beta_1 \leq y_i(t) \leq \beta_2,$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V(x(t)y(t)) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t)y(t))$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ 0 $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \ \rho_1(t, a_1) \ \rho_1^*(t, a_1),$$

$$0 \ Q_1(t) \ Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ħ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** $$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{(m_1 m_2)(m_1+m_2)}{r^3},$$

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

$$t = \frac{5}{256}\frac{c^5}{G^3}\frac{r^4}{(m_1 m_2)(m_1+m_2)}.$$

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1\beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_1\right]M_1 - [a_2]_1W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_3 a_3-(f_{30})_3 a_3-\int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3,a_3)da_3\right)u_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0, \qquad (24)$$

$$\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^\infty Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^a(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^{l}\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^{l})_3\int_0^\infty Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\quad da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{.3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2\,[a_3^2]_3$<4 $[a_2]_3$<4 $a_2 e$,

Re $\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\ ]_3 < 0$.

Let $\Delta$=max$_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then$|n_3(t)|$ ⊠⊿∀t ⊠. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_3^2$ <4 $a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2]$ ⊠$[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta$=$[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)\ [P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. 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In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

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Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1 \log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 28. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \text{ 0}, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \text{ ∞},$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a) e^{-\int_0^a \mu_4(\tau)dt} e^{-ra}\, da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(t)dt}\, da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^{\infty} [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 29. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40_4}^{l}\left(\int_0^{\infty} Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a\mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\,da\right);$$

$$\int_0^{\infty}Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^{\infty}E_4(a_4)e^{-\gamma_4a_4}\,da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)([f_2^{'}]_4(z_4))\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ∼ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4[a_2]_4<1\, a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1+x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1+x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1+x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae_4^{mu} \; [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} & \\ e & 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α~|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 30. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ 0}, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ ∞},$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t,a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 31. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0) = \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(1564) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(1565) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(1566) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(1567) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(1568) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(1569) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(1570) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(1571) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(1572) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(1573) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(1574) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(1575) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(1576) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(1577) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(1578) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(1579) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(1580) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(1581) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(1582) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

## Black holes

(1583) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(1584) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(1585) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(1586) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(1587) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(1588) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(1589) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(1590) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(1591) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have**

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| **experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (1592) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (1593) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

**String theory**

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| (1594) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (1595) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (1596) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (1597) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (1598) **However, when summed the entire series appears to be (=) divergent.** |
| (1599) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (1600) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (1601) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (1602) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (1603) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (1604) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (1605) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |

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| (1606) **The granular properties of space can be (=) explicitly computed.** |
| (1607) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (1608) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (1609) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (1610) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (1611) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (1612) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (1613) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (1614) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (1615) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (1616) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (1617) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (1618) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (1619) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (1620) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (1621) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (1622) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (1623) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (1624) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (1625) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (1626) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (1627) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (1628) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (1629) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (1630) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (1631) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (1632) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (1633) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (1634) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

(1635) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(1636) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

(1637) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.**

(1638) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016**
**DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society**

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(1639) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.**

(1640) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.**

(1641) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.**

(1642) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.**

(1643) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).**

(1644) **Bringing them to thermal equilibrium requires (e) an external bath.**

(1645) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.**

(1646) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).**

(1647) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.**

(1648) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.**

(1649) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—**

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (1650) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (1651) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (1652) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (1653) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (1654) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (1655) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (1656) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (1657) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (1658) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (1659) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (1660) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (1661) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (1662) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (1663) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (1664) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (1665) **This is a realistic scenario for (e) all known systems.** |
| (1666) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (1667) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (1668) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (1669) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (1670) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (1671) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (1672) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (1673) **So in those systems, the quantum phase transition does indeed exist.** |
| (1674) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (1675) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (1676) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (1677) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (1678) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (1679) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (1680) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (1681) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (1682) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (1683) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (1684) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (1685) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (1686) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (1687) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (1688) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (1689) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (1690) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (1691) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (1692) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (1693) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (1694) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (1695) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (1696) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (1697) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (1698) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (1699) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (1700) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (1701) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (1702) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (1703) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (1704) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (1705) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (1706) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (1707) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (1708) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (1709) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (1710) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (1711) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (1712) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (1713) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (1714) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (1715) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (1716) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (1717) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (1718) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (1719) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (1720) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (1721) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (1722) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (1723) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (1724) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (1725) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (1726) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (1727) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (1728) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (1729) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (1730) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (1731) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (1732) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (1733) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (1734) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (1735) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (1736) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (1737) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (1738) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (1739) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (1740) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (1741) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (1742) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (1743) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (1744) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (1745) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (1746) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (1747) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (1748) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (1749) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (1750) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (1751) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (1752) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (1753) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (1754) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (1755) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (1756) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (1757) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (1758) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (1759) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (1760) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (1761) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (1762) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (1763) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

(1764) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.**

(1765) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.**

(1766) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."**

**A new look at the galaxy-shaping power of black holes**

(1767) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the**

(1768) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.**

(1769) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.**

**Doomed satellite missions**

(1770) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.**

(1771) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.**

(1772) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.**

(1773) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.**

(1774) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.**

(1775) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people**

(1776) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."**

(1777) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.**

(1778) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(1779) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(1780) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(1781) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(1782) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(1783) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(1784) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(1785) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(1786) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(1787) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(1788) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(1789) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(1790) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(1791) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(1792) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (1793) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (1794) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (1795) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (1796) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (1797) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (1798) **We know dark matter exists only from (e) its gravitational effects.** |
| (1799) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (1800) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (1801) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (1802) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (1803) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (1804) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (1805) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (1806) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (1807) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (1808) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (1809) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (1810) | **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (1811) | **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (1812) | **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (1813) | **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (1814) | **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (1815) | **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (1816) | **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (1817) | **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (1818) | **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (1819) | **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (1820) | **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (1821) | **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (1822) | **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (1823) | **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (1824) | **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (1825) | **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (1826) | **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (1827) | **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (1828) | **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (1829) | **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (1830) | **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (1831) | **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (1832) | **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (1833) | **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (1834) | **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (1835) | **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (1836) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (1837) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (1838) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (1839) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (1840) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (1841) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (1842) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson’s problem**

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| (1843) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking “palm tree” organization.** |
| (1844) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (1845) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (1846) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (1847) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (1848) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (1849) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (1850) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (1851) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (1852) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (1853) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (1854) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (1855) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (1856) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (1857) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (1858) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (1859) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (1860) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (1861) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (1862) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (1863) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (1864) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (1865) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (1866) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (1867) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (1868) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (1869) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (1870) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (1871) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (1872) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (1873) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (1874) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (1875) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (1876) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (1877) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(289) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(290) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(291) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(292) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(293) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(294) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(295) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(296) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(297) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(298) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(299) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(300) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(301) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(302) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(303) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(304) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(305) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

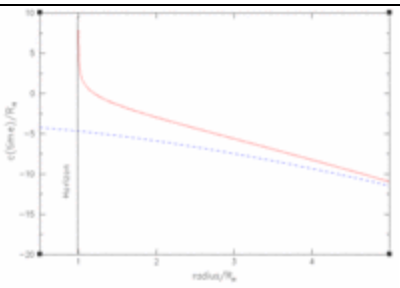
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| **(306) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(307) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(308) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(309) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(310) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(311) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(312) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(313) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(314) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(315) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(316) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(317) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(318) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(319) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(320) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(321) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(322) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(323) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(324) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(325) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(326) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(327) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(328) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(329) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(330) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(331) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(332) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(333) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(334) One of these coordinates is (=) called the "coordinate time."** |
| **(335) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(336) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(337) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(338) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(339) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(340) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(341) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(342) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(343) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(344) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(345) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(346) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(347) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(348) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(349) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(350) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(351) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(352) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(353) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(354) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(355) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(356) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(357) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(358) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(359) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(360) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(361) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(362) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(363) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(364) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(365) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(366) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(367) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(368) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(369) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(370) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(371) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(372) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(373) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(374) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(375) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(376) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(377) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(378) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(379) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(380) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(381) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(382) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(383) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(384) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6\left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,\quad h_{6}^{*}$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[f^{'}{}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[f_{2*}^{'}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right)M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$\mathrm{Re}\,\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$ i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ P^*\ [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\ [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\ g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^2} + \frac{2\omega_6}{(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6} = [f_{6*}]_{\,6} + [f_{2*}]_{\,6} + [f_{6*}] \, P_6^*$$

$$[a_2]_{\,6} = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [\, g_*^6]_{\,6} \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_{\,6} \beta_6 \left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right] M_6 - [a_2]_{\,6} W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 32. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \; P_7 \; 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(1878) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(1879) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(1880) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(1881) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(1882) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(1883) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(1884) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(1885) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(1886) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(1887) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(1888) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(1889) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(1890) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(1891) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(1892) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(1893) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(1894) The electronic and (e&eb) structural variability among them is extraordinary.

(1895) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(1896) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(1897) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(1898) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(1899) Nudge your bath tub just right and (eb) you might end up reversing time.

(1900) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(1901) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(1902) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(1903) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(1904) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(1905) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(1906) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(1907) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(1908) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(1909) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(1910) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(1911) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(1912) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(1913) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(1914) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(1915) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(1916) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(1917) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(1918) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(1919) Quantum mechanics rules the dynamics of (e) light and matter.

(1920) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(1921) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(1922) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(1923) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (1924) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (1925) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (1926) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (1927) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (1928) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (1929) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (1930) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (1931) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (1932) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (1933) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (1934) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (1935) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (1936) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (1937) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (1938) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (1939) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (1940) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (1941) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (1942) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (1943) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (1944) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (1945) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (1946) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (1947) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (1948) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (1949) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (1950) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (1951) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (1952) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (1953) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (1954) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (1955) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (1956) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (1957) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (1958) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (1959) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (1960) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (1961) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

(1962) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior.

(1963) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques.

(1964) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators

(1965) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice.

(1966) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(1967) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\,\rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\,\rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 33. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds}d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ \left(\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\left(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7\right) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_J) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7)da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7)m_7(t,a_7)\,da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0)W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7\left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}^{'}$ denoting $h_{7}^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7[a_7^7]2<4\,[a_7]2<4\,a_7e$,

$\mathrm{Re}\,\lambda_7\left[\lambda_7\right]_7 = \mathrm{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\,]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}\,[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⍰⊿∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{*}(a), \boldsymbol{Q}^{*})$ is asymptotically stable if either $a_7^7 <4\,a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⍰[$a_7$]e and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7 x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])_{\,7}^{\,3}\ [\mathrm{K}]_{\,7}[P_0]_{\,7}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$$[f_{7*}] = K_7\ \mathrm{P}^{*}\,[f_{7*}] = \frac{\sigma \mathrm{Q}_7^{*}}{7+\mathrm{Q}_7^{*}} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^{*}}{7+\mathrm{Q}_7^{*}}$$

$$[f_{7*}] = K_7\ \left[\mathrm{f}_{7*}^{\,7}\right] = \frac{\sigma_7}{(7+\mathrm{Q}_7^{*})^{7}}[\ \mathrm{g}_{*}^{\,7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^{*})^{7}} \quad \left[\mathrm{h}_{*}^{\,7}\right] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^{*})^{7}}$$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,7}\right]e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7\, Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^{\,7} Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7\, (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\, (\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae^{mu}_{\,7}\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7\, Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (363)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (364) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (365)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (366) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

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| **Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).** |

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| (367)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (368) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (369)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (370)Humans are not very good at understanding interacting many-body systems. |
| (371)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (372)But what happens when we couple (e&eb) the particles strongly? |
| (373)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (374)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (375)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (376) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (377) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (378)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (379) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (380)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (381) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (382) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (383)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (384)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (385) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (386) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (387)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (388)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (389)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (390)The energy levels of these quantized orbitals take on (e) discrete values |
| (391):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (392) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (393) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (394)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (395) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (396)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (397) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (398)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (399)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (400)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (401) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (402)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (403)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (404)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (405) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (406)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (407)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (408)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (409)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (410)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (411)Another classic example of duality involves (e&eb) vortices. |
| (412) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (413)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (414) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (415)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (416)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (417) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (418)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (419)TIs are distinguished only by (e) their edge states. |
| (420) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (421)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (422) But the Dirac cone is not (e) the only possible signature. |
| (423) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (424)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (425)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (426)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (427) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (428). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (429)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (430)The bulk duality implies (eb) a duality between edge theories. |
| (431)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |

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| (432)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (433) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (434) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (435)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |
| (436)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (437)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (438)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (439)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (440)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (441) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, TH=$\hbar\kappa/2\pi$. |
| (442)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (443)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (444) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (445)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (446)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

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| IF we define |
| $$[a_7]_{\,7} = [f_{7*}]_{\,7} + [f_{7*}]_{\,7} + [f_{7*}]\,\mathrm{P}^{*}_{7}$$ |

$$[a_7]_7 = \left[f^{7}_{7*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$

$$e=[g^{7}_{*}]_7 Q^{*}_{7}$$

$$\eta_7(t) = \left[h_{*}\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 & -[a_7]_7 \\ e & 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 34. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(1968) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(1969) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(1970) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(1971) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(1972) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

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| (1973) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (1974) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (1975) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (1976) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (1977) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (1978) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (1979) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (1980) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (1981) "The underlying physics is (=) chaos, essentially," he said. |
| (1982) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (1983) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (1984) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (1985) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (1986) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (1987) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (1988) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (1989) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (1990) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (1991) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |

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| (1992) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (1993) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (1994) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (1995) The study's findings have fundamental implications for (e) quantum computing. |
| (1996) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (1997) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (1998) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (1999) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (2000) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (2001) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'-the basic component of a quantum computer. |
| (2002) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (2003) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (2004) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (2005) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (2006) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (2007) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |

(2008) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(2009) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(2010) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(2011) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(2012) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(2013) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(2014) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(2015) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(2016) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(2017) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(2018) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(2019) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(2020) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as

contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(2021) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(2022) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(2023) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(2024) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(2025) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(2026) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(2027) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(2028) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(2029) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(2030) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(2031) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(2032) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(2033) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(2034) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(2035) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(2036) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(2037) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(2038) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(2039) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(2040) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(2041) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(2042) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(2043) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(2044) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background

radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(2045) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(2046) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(2047) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(2048) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(2049) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(2050) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(2051) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(2052) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(2053) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(2054) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(2055) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(2056) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(2057) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

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| (2058) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime. |
| (2059) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite. |
| (2060) However, when summed the entire series appears to be (=) divergent. |
| (2061) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime. |
| (2062) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects. |
| (2063) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing. |
| (2064) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory. |

**Loop quantum gravity and spin foams**

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| (2065) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics. |
| (2066) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity. |
| (2067) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime. |
| (2068) The granular properties of space can be (=) explicitly computed. |
| (2069) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory. |
| (2070) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks. |
| (2071) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure. |
| (2072) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit. |
| (2073) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism. |
| (2074) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism. |

**Noncommutative geometry**

(2075) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(2076) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(2077) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(2078) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(2079) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(2080) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(2081) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(2082) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(2083) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(2084) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ \text{0}, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \text{∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \text{∞},$$

$$\rho_8(t, 0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly

eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big((f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8$$

$$= -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0)$$

$$= \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt}$$

$$= -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8 \; h_8(Q_8) = 1, \qquad and \;\; g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 35. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))u_8 - (\rho_0)_8(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8^{'}(P_0) = f_{80}^{'}$, and $f_8^{'}(0) = f_{80}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \quad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a \mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\,(\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(a_8)da_8-\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8\,(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^\infty E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $[\dot{g}]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - [\dot{f}_{8*}]_8[p_8^*](a_8)M_8 - [\dot{f}_{8*}]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 P_8^*\right)M_8 - [\dot{f}_{8*}]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8) + d\tau_8 \mathrm{da}} = \frac{1}{(\mathrm{m}+\mu_0+1)^8} + \frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8 x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8 x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])\,{}_8^3\,[\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*} \quad [h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{\,8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\,\mathrm{g}_*^{\,8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8} \quad [\mathrm{h}_*^{\,8}] = \frac{\omega_8 - 8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^8} + \frac{1\omega_8}{([m_8]_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^3} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, a e_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (270)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (271)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (272) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (273) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (274)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (275) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (276)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (277)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(278)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(279)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(280) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(281)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(282)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(283)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(284) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(285)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(286)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(287)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(288)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(289)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(290)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(291)Bringing them to thermal equilibrium requires (e) an external bath.

(292) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(293)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(294)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(295)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(296)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(297)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(298) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(299) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(300) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(301)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(302)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(303) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(304)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(305)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(306) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(307)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(308)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (309)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (310) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (311)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (312) This is a realistic scenario for (e) all known systems. |
| (313)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (314)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (315)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (316) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (317)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (318)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (319)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (320)So in those systems, the quantum phase transition does indeed exist. |
| (321)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (322)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

### MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant $\Lambda$, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive $\Lambda$ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the $\Lambda>0$ generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins $>+0.6$ and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 36. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; \text{0}, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \text{∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \text{∞},$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ∞; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a) e^{-\int_0^a \mu_9(\tau)dt} e^{-ra}\, da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(t)dt}\, da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9} \rho_9(t, a_9)\, da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*)\int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{9}^*$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9]2<4\ [a_9]2<4\ a_9 e$,

$\mathrm{Re}\,\lambda_9[\lambda_9]_9 = \mathrm{Re}\lambda_9[\lambda_9]_9 = -\frac{9}{9}a_9[a2\ ]_9 < 0.$

Let $\Delta=\max_{9<u<\infty}\left|h_*[h_*]_9\beta_9(a_9) - \mu_9(a_9)\right|$. Then$|n_9(t)|$ ?4∀t ?0. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 <4\ a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ?4$[a_9]e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae\text{-}^{ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m+\mu_0+1)^9} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ ?0.5. Assume$[f_9](x_9)= K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)= [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\ x_9/(1 + x_9)$and $h_9(x_9)=(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\ [\mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a_{\ 9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_{\ 9} - b)Q^*}{\delta_{\ 9}(1 + Q_9^*)^9} - \frac{4\,(\delta_{\ 9} - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_{\ 9}(1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\ mu}_{\ 9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a\ {}^{9}_{9} > 4(a_9)_{\ 9}e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO C

## Theory Of Quantum Error-Correcting Codes-3

**INTRODUCTION—VARIABLES USED**

**A Theory of Quantum Error-Correcting Codes Emanuel Knill, Raymond Laflamme (Submitted on 26 Apr 1996**)

(1) Two notions of fidelity and (e&eb) error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states.
(2) The latter is more appropriate when using (e) codes in a quantum memory or in applications of (e) quantum teleportation to communication.
(3) Authors show that the error for entangled states is bounded linearly by (e) the error for pure states.
**(4)** A formal definition of independent interactions for (e) qubits is given.
**(5)** This leads to (eb) lower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component.
**(6)** Subjects: Quantum Physics (quant-ph) Journal reference:Phys.Rev.Lett.84:2525-2528,2000 DOI: 10.1103/PhysRevLett.84.2525 Report number: LA-UR 96-1300 Cite as:arXiv:quant-ph/9604034 (or arXiv:quant-ph/9604034v1 for this version)

**New class of quantum error-correcting codes for a bosonic mode Marios H. Michael, Matti Silveri, R. T. Brierley, Victor V. Albert, Juha Salmilehto, Liang Jiang, S. M. Girvin (Submitted on 29 Jan 2016 (v1), last revised 18 Jul 2016 (this version, v3))**

(7) Authors construct a new class of quantum error-correcting codes for (e) a bosonic mode which are advantageous for (e) applications in quantum memories, communication, and scalable computation.
(8) These 'binomial quantum codes' are formed from (e) a finite superposition of Fock states weighted with (e&eb) binomial coefficients.
(9) The binomial codes can exactly correct (e&eb) errors that are polynomial up to (e) a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors.
(10) For realistic continuous-time dissipative evolution, the codes can perform (eb) approximate quantum error correction to (e) any given order in the timestep between error detection measurements.
(11) Authors present an explicit approximate quantum error recovery operation based on (e) projective measurements and unitary operations.
(12) The binomial codes are tailored for (e) detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity.
(13) Authors discuss optimization of the binomial codes and demonstrate that by (e) relaxing the parity

structure, codes with (e&eb) even lower unrecoverable error rates can be achieved.

(14) The binomial codes are related to (e&eb) existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors.

(15) codes are similar in spirit to (e) 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode.

(16) The binomial quantum codes are realizable with (e&eb) current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**. Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version)

**NOTATION**

**Module One**

Two notions of fidelity and (e&eb) error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states

$G_{13}$ : Category one of **notions of fidelity;** error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states; **notions of fidelity**

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Two notions of fidelity and error for (e) imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states

$G_{16}$ : Category one of imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states

$G_{17}$ : Category two of SAS

$G_{18}$ : Category three of SAS

$T_{16}$ : Category one of notions of fidelity and error

$T_{17}$ : Category two of SAS

$T_{18}$ : Category three of SAS

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| **Module three**<br><br>Two notions of fidelity and error for imperfect recovery are introduced, one for (e) pure and the other for (e) entangled states | |
| $G_{20}$ : Category one of pure and the other for entangled states<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of notions of fidelity and error for imperfect recovery are introduced, one<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The latter is more appropriate when using (e) codes in a quantum memory or in applications of (e) quantum teleportation to communication | |
| $G_{24}$ : Category one of codes in a quantum memory or in applications of (e) quantum teleportation to communication<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of latter is more appropriate<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors show that the error for entangled states is bounded linearly by the error for pure states | |
| $G_{28}$ : Category one of error for pure states<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of error for entangled states is bounded linearly<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>error for entangled states is bounded linearly by the error for pure states leads to (eb) lower bounds on the | |

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| number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component. | |
| $G_{32}$ : Category one of error for entangled states is bounded linearly by the error for pure states<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one oflower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>error for entangled states is bounded linearly by the error for pure states leads to (eb) lower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component | |
| $G_{36}$ : Category one of error for entangled states is bounded linearly by the error for pure states<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one oflower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>error for entangled states is bounded linearly by the error for pure states leads to lower bounds on the number of qubits required to (e) correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component | |

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| $G_{40}$ : Category one ofcorrect e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of error for entangled states is bounded linearly by the error for pure states leads to lower bounds on the number of qubits<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>error for entangled states is bounded linearly by the error for pure states leads to lower bounds on the number of qubits required to correct e errors and a formal proof that the classical bounds on (e&eb) the probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component | |
| $G_{44}$ : Category one of **error for entangled states is bounded linearly by the error for pure states leads to lower bounds on the number of qubits required to correct e errors and a formal proof that the classical bounds;** probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of probability of error of e-error-correcting codes applies to (e&eb) e-error-correcting quantum codes, provided that (e) the interaction is dominated by an identity component; **error for entangled states is bounded linearly by the error for pure states leads to lower bounds on the number of qubits required to correct e errors and a formal proof that the classical bounds**<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$ | |

| | |
|---|---|
| are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}, \ (b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)} \ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |

| | |
|---|---|
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |

**Module Numbered Seven:**

**The differential system of this model is now (Seventh Module)**

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36} \quad (37)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37} \quad (38)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38} \quad (39)$$

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{36} \quad (40)$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{37} \quad (41)$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{38} \quad (42)$$

$+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor**

**Module Numbered Eight**

The differential system of this model is now

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40} \quad (43)$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41} \quad (44)$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42} \quad (45)$$

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{40} \quad (46)$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{41} \quad (47)$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{42} \quad (48)$$

**Module Numbered Nine**

The differential system of this model is now

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44} \quad (49)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45} \quad (50)$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46} \quad (51)$$

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44} \quad (52)$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45} \quad (53)$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46} \quad (54)$$

$+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor**

$-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15} \quad (60)$$

Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients

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| for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{37}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[ \begin{array}{c} (a_{16}^{'})^{(2)} +(a_{16}^{''})^{(2)}(T_{17},t) +(a_{13}^{''})^{(1,1,)}(T_{14},t) +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) +(a_{40}^{''})^{(8,8,8)}(T_{41},t) +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[ \begin{array}{c} (a_{17}^{'})^{(2)} +(a_{17}^{''})^{(2)}(T_{17},t) +(a_{14}^{''})^{(1,1,)}(T_{14},t) +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) +(a_{41}^{''})^{(8,8,8)}(T_{41},t) +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[ \begin{array}{c} (a_{18}^{'})^{(2)} +(a_{18}^{''})^{(2)}(T_{17},t) +(a_{15}^{''})^{(1,1,)}(T_{14},t) +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) +(a_{42}^{''})^{(8,8,8)}(T_{41},t) +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3 $+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | 66 |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{21}$ | 68 |

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)}\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)} & \boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}\boxed{-(b_{22}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{18}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{15}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$

*are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$

*are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$

*are sixth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$

*are seventh detrition coefficients for category* $1,2$ *and* $3$

$-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$

*are eighth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt}=(a_{28})^{(5)}G_{29}-\begin{bmatrix}(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt}=(a_{29})^{(5)}G_{28}-\begin{bmatrix}(a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

*Where* $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)}\boxed{-(b''_{28})^{(5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)}\boxed{-(b''_{29})^{(5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

*where* $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

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$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32}$$

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$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)} & +(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33}$$

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$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)} & +(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34}$$

$+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$

Eighth augmentation coefficients $+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt}=(b_{32})^{(6)}T_{33}-\begin{bmatrix}(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt}=(b_{33})^{(6)}T_{32}-\begin{bmatrix}(b_{33}^{'})^{(6)}-(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt}=(b_{34})^{(6)}T_{33}-\begin{bmatrix}(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{34} \quad (90)$$

$-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt}=(a_{36})^{(7)}G_{37}-\begin{bmatrix}(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}\boxed{+(a_{37}'')^{(7)}(T_{37},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}\boxed{+(a_{38}'')^{(7)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a_{36}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{37}'')^{(7)}(T_{37},t)}$, $\boxed{(a_{38}'')^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}')^{(7)}\boxed{-(b_{36}'')^{(7)}(G_{39},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}')^{(7)}\boxed{-(b_{37}'')^{(7)}(G_{39},t)} & \boxed{-(b_{17}'')^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}'')^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}'')^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}'')^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}'')^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}'')^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}'')^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}'')^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)} G_{41} - \begin{bmatrix} (a'_{42})^{(8)} & +(a''_{42})^{(8)}(T_{41},t) & +(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{15}$$

Where $+(a''_{40})^{(8)}(T_{41},t)$, $+(a''_{41})^{(8)}(T_{41},t)$, $+(a''_{42})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)} T_{41} - \begin{bmatrix} (b'_{40})^{(8)} & -(b''_{40})^{(8)}(G_{43},t) & -(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)} T_{40} - \begin{bmatrix} (b'_{41})^{(8)} & -(b''_{41})^{(8)}(G_{43},t) & -(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

| | |
|---|---|
| $\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and$(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br><br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},(r_i)^{(2)}}$ are positive constants and $\boxed{i=16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |

| | |
|---|---|
| $\|(a_i'')^{(2)}(T_{17}',t)-(a_i'')^{(2)}(T_{17},t)\| \le (\hat{k}_{16})^{(2)}\|T_{17}-T_{17}'\|e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\|(b_i'')^{(2)}((G_{19})',t)-(b_i'')^{(2)}((G_{19}),t)\| < (\hat{k}_{16})^{(2)}\|\|(G_{19})-(G_{19})'\|\|e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and$(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},$ are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i')^{(2)},(b_i)^{(2)},(b_i')^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i')^{(2)}+(\hat{A}_{16})^{(2)}+(\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)}+(b_i')^{(2)}+(\hat{B}_{16})^{(2)}+(\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i')^{(3)},(a_i'')^{(3)},(b_i)^{(3)},(b_i')^{(3)},(b_i'')^{(3)}>0,\quad i,j=20,21,22$<br>The functions $(a_i'')^{(3)},(b_i'')^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (r_i)^{(3)}$:<br>$(a_i'')^{(3)}(T_{21},t)\le(p_i)^{(3)}\le(\hat{A}_{20})^{(3)}$<br>$(b_i'')^{(3)}(G_{23},t)\le\ (r_i)^{(3)}\le(b_i')^{(3)}\le(\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i'')^{(3)}(T_{21},t)=(p_i)^{(3)}$<br>$\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t)=\ (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)},(p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i=20,21,22}$ | 113 |

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| They satisfy Lipschitz condition:<br><br>$\|(a_i'')^{(3)}(T_{21}', t) - (a_i'')^{(3)}(T_{21}, t)\| \leq (\hat{k}_{20})^{(3)}\|T_{21} - T_{21}'\|e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\|(b_i'')^{(3)}(G_{23}', t) - (b_i'')^{(3)}(G_{23}, t)\| < (\hat{k}_{20})^{(3)}\|\|G_{23} - G_{23}'\|\|e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}', t)$ and$(a_i'')^{(3)}(T_{21}, t)$ . $(T_{21}', t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i'')^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i'')^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition: | 119 |

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| $\|(a_i^{''})^{(4)}(T_{25}^{'},t)-(a_i^{''})^{(4)}(T_{25},t)\| \leq (\hat{k}_{24})^{(4)}\|T_{25}-T_{25}^{'}\|e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})^{'},t)-(b_i^{''})^{(4)}\big((G_{27}),t\big)\| < (\hat{k}_{24})^{(4)}\|\|(G_{27})-(G_{27})^{'}\|\|e^{-(\hat{M}_{24})^{(4)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)},(\hat{M}_{24})^{(4)}\big]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)}=1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}<1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)},(\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)},(\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)},(a_i^{'})^{(4)},(b_i)^{(4)},(b_i^{'})^{(4)},(p_i)^{(4)},\ (r_i)^{(4)}, i=24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)}+(a_i^{'})^{(4)}+(\hat{A}_{24})^{(4)}+(\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)}+(b_i^{'})^{(4)}+(\hat{B}_{24})^{(4)}+(\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)},(a_i^{'})^{(5)},(a_i^{''})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(b_i^{''})^{(5)}>0, \quad i,j=28,29,30$<br><br>The functions $(a_i^{''})^{(5)},(b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)},\ (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t)\leq(p_i)^{(5)}\leq(\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big)\leq\ (r_i)^{(5)}\leq(b_i^{'})^{(5)}\leq(\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}',t)-(a_i'')^{(5)}(T_{29},t)| \le (\hat{k}_{28})^{(5)}|T_{29}-T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})',t)-(b_i'')^{(5)}((G_{31}),t)| < (\hat{k}_{28})^{(5)}||(G_{31})-(G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}',t)$ and$(a_i'')^{(5)}(T_{29},t)$ . $(T_{29}',t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)},(\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)}=1$ then the function $(a_i'')^{(5)}(T_{29},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ ,\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}<1$$

**Definition of** $(\hat{P}_{28})^{(5)},(\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)},(\hat{A}_{28})^{(5)}and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)},(a_i')^{(5)},(b_i)^{(5)},(b_i')^{(5)},(p_i)^{(5)},\ (r_i)^{(5)},i=28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)}+(a_i')^{(5)}+(\hat{A}_{28})^{(5)}+(\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)}+(b_i')^{(5)}+(\hat{B}_{28})^{(5)}+(\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$$

Where we suppose

$(a_i)^{(6)},(a_i')^{(6)},(a_i'')^{(6)},(b_i)^{(6)},(b_i')^{(6)},(b_i'')^{(6)}>0,\quad i,j=32,33,34$ 127

The functions $(a_i'')^{(6)},(b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)},\ (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33},t)\le(p_i)^{(6)}\le(\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}),t)\le\ (r_i)^{(6)}\le(b_i')^{(6)}\le(\hat{B}_{32})^{(6)}$$

$lim_{T_2\to\infty}(a_i'')^{(6)}(T_{33},t)=(p_i)^{(6)}$ 128

$\lim_{G\to\infty}(b_i'')^{(6)}((G_{35}),t)=\ (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)},(\hat{B}_{32})^{(6)}$ :

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|---|---|
| Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32{,}33{,}34}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}\big((G_{35}), t\big)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32{,}33{,}34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (Y) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36{,}37{,}38$<br><br>(Z) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(AA)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(BB)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = \ (r_i)^{(7)}$ | 132 |

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| **Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :<br><br>Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)\rvert \leq (\hat{k}_{36})^{(7)} \lvert T_{37} - T_{37}'\rvert e^{-(\hat{M}_{36})^{(7)}t}$<br><br>$\lvert(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)\rvert < (\hat{k}_{36})^{(7)} \lvert\lvert(G_{39}) - (G_{39})'\rvert\rvert e^{-(\hat{M}_{36})^{(7)}t}$ | 133 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(CC) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(DD) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i, j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |

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| $(b_i^{''})^{(8)}((G_{43}),t) \leq \ (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'},t) - (a_i^{''})^{(8)}(T_{41},t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'},t) - (b_i^{''})^{(8)}\left((G_{43}),t\right)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i^{'})^{(8)} + \ (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(b_i)^{(8)} + (b_i^{'})^{(8)} + \ (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i^{'})^{(9)}, (a_i^{''})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (b_i^{''})^{(9)} > 0, \quad i,j = 44,45,46$ | 146 A |

The functions $(a_i^{''})^{(9)}, (b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i^{'})^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i^{''})^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i^{''})^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)} |T_{45} - T_{45}^{'}| e^{-(\hat{M}_{44})^{(9)} t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}((G_{47}), t)| < (\hat{k}_{44})^{(9)} ||(G_{47}) - (G_{47})^{'}|| e^{-(\hat{M}_{44})^{(9)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

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| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 |

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| $G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| Where $s_{(13)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |

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| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a_{24}')^{(4)} + a_{24}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a_{25}')^{(4)} + (a_{25}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
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| $\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{32})^{(6)}$, $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a_{36}')^{(7)} + a_{36}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a_{37}')^{(7)} + (a_{37}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a_{38}')^{(7)} + (a_{38}'')^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b_{36}')^{(7)} - (b_{36}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b_{37}')^{(7)} - (b_{37}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b_{38}')^{(7)} - (b_{38}'')^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a_{40}')^{(8)} + a_{40}'')^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a_{41}')^{(8)} + (a_{41}'')^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |

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| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that | 168 |

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|---|---|
| $(G_{13}(t)-G_{13}^0)e^{-(\hat{M}_{13})^{(1)}t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{P}_{13})^{(1)}+G_{14}^0)e^{\left(-\frac{(\hat{P}_{13})^{(1)}+G_{14}^0}{G_{14}^0}\right)}+(\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0+(\hat{P}_{16})^{(6)}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1+(a_{16})^{(2)}t\right)G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)}t}-1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t)-G_{16}^0)e^{-(\hat{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | |

that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that 177

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(e) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$ 180

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| $\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$ | |
| From which it follows that<br>$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$<br>$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right|\leq\int_{0}^{t}(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_{0}^{t}\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

(186)

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_{+}$.

If instead of proving the existence of the solution on $\mathbb{R}_{+}$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)},i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

From 19 to 24 it results

$$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\geq 0$$

$$T_i\,(t)\geq T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :

(187)

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13}<(\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt}\leq\left((\widehat{M}_{13})^{(1)}\right)_1-(a_{14}^{'})^{(1)}G_{14}$ and by integrating

$$G_{14}\leq\left((\widehat{M}_{13})^{(1)}\right)_2=G_{14}^0+2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$$

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| In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\widehat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br><br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br><br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i$ (t) $= 0$ and $T_i$ (t) $= 0$<br><br>it results<br><br>$G_i$ (t) $\geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i$ (t) $\geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for t $> 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

In the same way , one can obtain

$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$

If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively.

**Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. 202

**Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$. 203

**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :

Indeed let $t_2$ be so that for $t > t_2$

$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$

Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to 204

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. 205
The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose 207

$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have

$$\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)} \quad (208)$$

$$\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)} \quad (209)$$

In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself 210

The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric 211

$$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t}\}$$

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :( \widetilde{(G_{23})}, \widetilde{(T_{23})} ) = \mathcal{A}^{(3)}((G_{23}), (T_{23}))$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} \} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ *and* $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left((\widehat{M}_{20})^{(3)}\right)_1 / (a_{21}^{'})^{(3)}$ | 217 |

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| In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}} < 1$ and to choose<br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}\}$ | 225 |

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})}, \widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\, ds_{(24)} +$$

$$\int_0^t \{(a_{24}^{'})^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} \left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)}, s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)} t\right)} > 0 \quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}^{'})^{(4)}$$

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| In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$ | 236 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br><br>It results<br><br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br><br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br><br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(5)}t\right)}>0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1,\left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if | 240 |

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| $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}')^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a_{29}')^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a_{30}')^{(5)}$<br><br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b_{29}')^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$<br><br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br><br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}'')^{(5)}\left((G_{31})(t),t\right) = (b_{30}')^{(5)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br><br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br><br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |

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| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)}+$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\le$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32,33,34$ depend only on $\mathrm{T}_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br>it results<br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$<br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(6)}t\right)}>0$ for t > 0 | 250 |

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a'_{33})^{(6)} G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)} \quad (257)$$

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| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\le$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\ge 0$ | 261 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(7)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{36})^{(7)})_1, ((\widehat{M}_{36})^{(7)})_2$ *and* $((\widehat{M}_{36})^{(7)})_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq ((\widehat{M}_{36})^{(7)})_1 - (a_{37}')^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq ((\widehat{M}_{36})^{(7)})_2 = G_{37}^0 + 2(a_{37})^{(7)}((\widehat{M}_{36})^{(7)})_1/(a_{37}')^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq ((\widehat{M}_{36})^{(7)})_3 = G_{38}^0 + 2(a_{38})^{(7)}((\widehat{M}_{36})^{(7)})_2/(a_{38}')^{(7)}$<br><br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}((G_{39})(t),t) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + ((\widehat{P}_{40})^{(8)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
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| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}ds_{(40)}+$ $\int_0^t\{(a_{40}^{'})^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $G_{40}^{(2)}\left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i=40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$ | 275 |

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| it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}')^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(8)}((G_{43})(t), t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty} (b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)}+\left((\hat{P}_{44})^{(9)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{44})^{(9)}\right]\leq(\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),\left((G_{47})^{(2)},(T_{47})^{(2)}\right)\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\;e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\;(G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}')^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}')^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}')^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(9)} ((G_{47})(t), t)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}((G_{47})(t), t) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty} (b_{46}'')^{(9)} ((G_{47})(t), t) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14}, t) + (a_{14}'')^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

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| $-(\tau_2)^{(1)} \le -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \le -(\tau_1)^{(1)}$ | |
| **Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ :<br><br>By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 281 |
| **Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}^{'})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}^{'})^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}^{'})^{(1)}t}\right)$ | 286 |

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|---|---|
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(\mathrm{T}_{17}, t) + (a_{17}'')^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}\big((G_{19}), t\big) - (b_{17}'')^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |

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| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, $\boldsymbol{if}\ (\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}, \boldsymbol{if}\ (\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}\ (\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}\ (u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}, \boldsymbol{if}\ (u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}\ (\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a_{18}')^{(2)} t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b_{18}')^{(2)} t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ | 317 |

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| $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ :<br>By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and<br>By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | 320 |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,<br>and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$ | 322 |

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| $(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},$ ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$<br><br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$<br><br>$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br><br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$<br><br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br><br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$<br><br>$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br><br>**Theorem:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br><br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br><br>$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br><br>$-(\tau_2)^{(4)} \le -(b_{24}')^{(4)} + (b_{25}')^{(4)} - (b_{24}'')^{(4)}((G_{27}), t) - (b_{25}'')^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0, (v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations | 329 |

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| $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$<br>$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$ | 330 |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}^{'})^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}^{'})^{(4)}t} \le T_{26}(t) \le$ | 336 |

$\frac{(a_{26})^{(4)}T_{24}^{0}}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^{0}e^{-(R_2)^{(4)}t}$

**Definition of** $(S_1)^{(4)},(S_2)^{(4)},(R_1)^{(4)},(R_2)^{(4)}$:- 337

Where $(S_1)^{(4)}=(a_{24})^{(4)}(m_2)^{(4)}-(a_{24}^{'})^{(4)}$

$(S_2)^{(4)}=(a_{26})^{(4)}-(p_{26})^{(4)}$

$(R_1)^{(4)}=(b_{24})^{(4)}(\mu_2)^{(4)}-(b_{24}^{'})^{(4)}$

$(R_2)^{(4)}=(b_{26}^{'})^{(4)}-(r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)}\le-(a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}-(a_{28}^{''})^{(5)}(T_{29},t)+(a_{29}^{''})^{(5)}(T_{29},t)\le-(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)}\le-(b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31}),t)-(b_{29}^{''})^{(5)}((G_{31}),t)\le-(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)},(v_2)^{(5)},(u_1)^{(5)},(u_2)^{(5)},v^{(5)},u^{(5)}$ : 339

By $(v_1)^{(5)}>0$ , $(v_2)^{(5)}<0$ and respectively $(u_1)^{(5)}>0$ , $(u_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and

**Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$

**Definition of** $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-

If we define $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)}$ by

$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)}=\frac{G_{28}^{0}}{G_{29}^{0}}}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$

and analogously 341

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| $(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, $\boldsymbol{if}\ (u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$, $\boldsymbol{if}\ (u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying | 349 |

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| $-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br><br>$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)}, , (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39}),t)-(b_{37}^{''})^{(7)}((G_{39}),t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

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| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \le G_{38}(t) \le$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \le T_{38}(t) \le$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ | 370 |

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,
and $\boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

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| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}, \ \boldsymbol{if}\ (\nu_0)^{(9)} < (\nu_1)^{(9)}$$

$$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}, \boldsymbol{if}\ (\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)},$$

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}, \ \boldsymbol{if}\ (\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$$

and analogously

$$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}, \ \boldsymbol{if}\ (u_0)^{(9)} < (u_1)^{(9)}$$

$$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}, \boldsymbol{if}\ (u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)},$$

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}, \boldsymbol{if}\ (\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G^0_{44} e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G^0_{44} e^{(S_1)^{(9)} t}$$

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| $\left(\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t}\leq G_{46}(t)\leq \frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}]+G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t}\right)$ | |
| $\boxed{T_{44}^{0}e^{(R_1)^{(9)}t}\leq T_{44}(t)\leq T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$ | |
| $\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t}\leq T_{44}(t)\leq \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$ | |
| $\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t}\leq T_{46}(t)\leq$ $\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$ | |
| **Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:- Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a_{44}^{'})^{(9)}$ $(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$ $(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b_{44}^{'})^{(9)}$ $(R_2)^{(9)}=(b_{46}^{'})^{(9)}-(r_{46})^{(9)}$ | |

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$ , we obtain

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

383 384 385 386

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

$$\text{For } 0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

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| $\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t)=\frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain | 398 |

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \le \frac{dv^{(3)}}{dt} \le -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \ge \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \le v^{(3)}(t) \le (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \le v^{(3)}(t) \le (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \le \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \le v^{(3)}(t) \le$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \le (\bar{v}_1)^{(3)} \le (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \le v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$, we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}, \quad \boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}, \quad \boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} =$

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$(\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

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$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq v^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $v^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)v^{(7)} - (a_{37})^{(7)} v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)} v^{(7)} - (a_{36})^{(7)}\right) \leq \frac{dv^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$v^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq v^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain 419

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \leq \frac{dv^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \geq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \leq v^{(8)}(t) \leq (v_1)^{(8)}$

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In the same manner , we get

$$v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \leq v^{(8)}(t) \leq (\bar{v}_8)^{(8)}$

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If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \leq \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \leq v^{(8)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (\bar{v}_1)^{(8)}$$

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If $0 < (v_1)^{(8)} \leq (\bar{v}_1)^{(8)} \leq \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \leq v^{(8)}(t) \leq \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \leq (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

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| $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |

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| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ | 436 A |

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| $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |

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| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}))\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}))\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}))\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |

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| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(e) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) +$ | 486 |

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| $(a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |

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| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{13}=\frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}=\frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | 494 |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations | 499 |

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| $G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 504 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if | 510 |

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| $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |

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| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 523 |

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| $T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br>**Proof:**_Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions | 531 |

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| $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |

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| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |

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| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |

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| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ $and$ $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \quad , \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\Big)\right]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\Big)\right]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\Big)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\Big)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\Big)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al**

**Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal**

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(5) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and

review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 5. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K $\subset$ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta_0 > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \longrightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) in\ ||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\to\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\to\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$ . If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $\mathrm{A_i}(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A_i}(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A_i}(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N_\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $\dot{x} = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $\dot{x} = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $\dot{x} = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\mathrm{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.13) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.14) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.15) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s)$, $a.0$, $s \in [-\tau, 0]$, $i = 1, 2$,

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\, da$, $s \in [-\tau, 0]$,

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s \in [-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 5. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{\ i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +}$ [$-\tau, T] \rightarrow R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +}$ [$R^{\ +} \cap\ [0, T]$] And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +}$ [$-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t, 0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right) ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t)= B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

$$(2.4)\ B_i(t) = p_i(0, t), i = 1,2, t > 0.$$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

$$(2.5)\ L_i\left(a, t, p_i(t)\right) = \exp\left[-\int_o^t f_1(a+s, p_i(s))ds\right]$$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t))da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a, t, p_i(t))da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a, t, P_i(t)\right)da$

$$+\int_0^{\infty} b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, \; t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \; \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^{\infty} \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \; t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in \mathrm{C}([0,T], R^+), \quad B_i \in \mathrm{C}([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in \mathrm{C}([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $\mathrm{C}^+[-\tau, T]$. Then for any

$(x, y) \in \mathrm{C}^+[-\tau, T] \times \mathrm{C}^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$\mathrm{C}_T = \mathrm{C}^+[-\tau, T], \quad \mathrm{C}_T^2 = \mathrm{C}^+[-\tau, T] \times \mathrm{C}^+[-\tau, T].$$

Members of $\mathrm{C}_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in \mathrm{C}_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$\mathrm{B}(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Im(P_t) = \{ (P_{1t}, P_{2t}),\ (P_{1t}, P_{2t}) \}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$ , $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \ and \ \ C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{ (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for \ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for \ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Im(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right)$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then ▯ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , ▯ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|▯(P_t^{(1)}) - ▯(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|▯(P^{(1)}(t)) - ▯(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\,P_2^*(a)}{da} + f_i(\mathrm{a}, P_1^*, P_2^*)\ P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty\ P_i^*(a)da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty\ b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ \ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1 = \int_o^\infty\ b_1(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_1(s, P_1^*, P_2^*)ds\right]da,$$

$$(3.5)\ 1 = \int_o^\infty\ b_2(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_2(s, P_1^*, P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

$$(3.6)\ F_{i.}\ (P_1, P_2) = \int_o^\infty\ b_i(a, P_1, P_2)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1, P_2)ds\right]da,$$

$$i = 1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

$$(3.7)\ A_{4.}\quad F_{i.}\ (0,\ 0) > 1,\ \ i = 1, 2:$$

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(\mathrm{C},0) < F_{2.}(\mathrm{C},0),$$

$$(3.8)\qquad F_1(0.\mathrm{C}) < F_{2.}(0,\mathrm{C}),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o,$

$P_2^* > 0$ such that

$$(3,9)\ F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

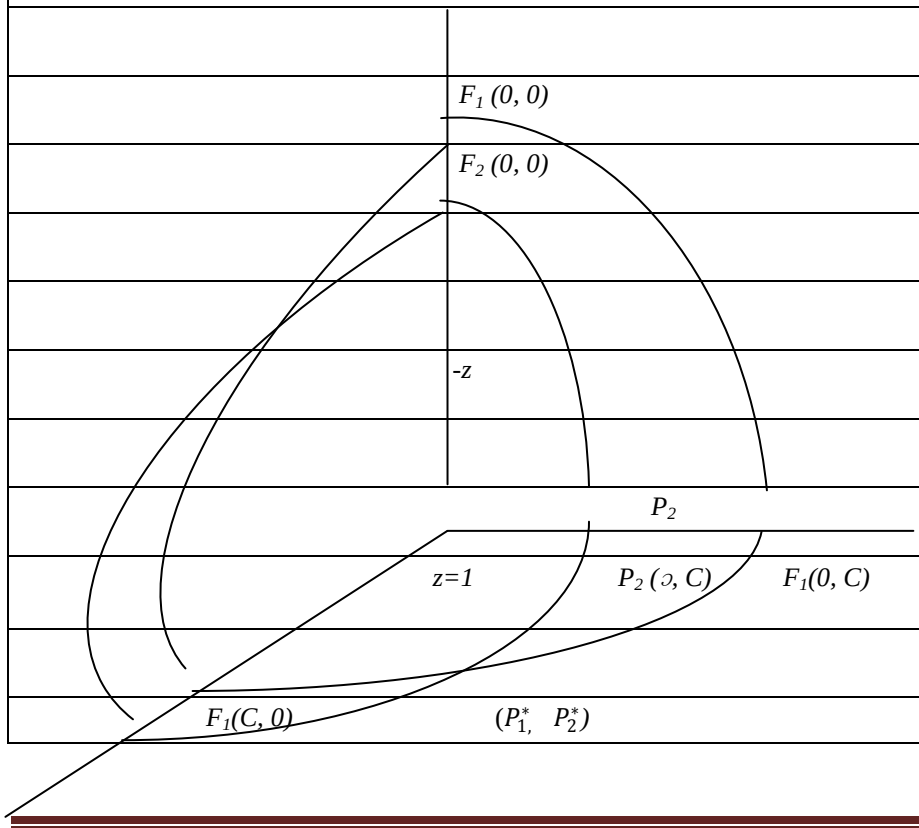
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

(3.11)
$$\begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{\eta}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)\eta(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\}da,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*,\ \exp\{-\lambda\tau\},\ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $\eta(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right]\eta(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right]\eta(a)da,$$

From the first of (3.15) we get

(3.18) $$\eta(a) = \left\{\eta(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\eta(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$\left(\lambda - A_{11}\right) p_1^* - A_{12} p_2^* \exp\left(-\lambda\tau\right) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp\left(-\lambda\tau\right) + \left(\lambda - A_{22}\right) p_2^* = g_{21}(\lambda) p_1^* \exp\left(-\lambda\tau\right) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = \left[p_i^*(0) c_i\right]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a, p_1^*, p_2^*\right) - f_i\left(a, p_1^*, p_2^*\right), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - \left(A_{11} + g_{11}(\lambda)\right) & -\left(A_{12} + g_{12}\right) e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right) e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}(\lambda)\right) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21} \exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = \left[A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)\right] \exp\left[-2\lambda\tau\right]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \ | - | A_{11} + A_{22} + \| S_1(0) \| \ |^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 9. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \right\} \left\{ 1 - \frac{N(t-\tau)}{K} \right\}. \qquad (1.3)$$

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\left\{ \frac{dN(t)}{dt} = N(t) \right\} = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\} \qquad (1.4)$$

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(xiii) $b_1(0) = 0$, $m_1(0, x_2) \equiv 0, b_2(0) = 0$, $m_2(x_1, 0) \equiv 0$,

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xiv) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$,

$b_2(\delta_2) - m_2(x_1, \delta_2) < 0$, for $x_1$ 0, $x_2$ 0,

(2.5)

(xv) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i,\; j = 1, 2; i \neq j)$.

## 10. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\ (t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\ (t_2)\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\ (t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t \geq 0$.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\ (t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\ (t - \tau_{11}) - m_{12}Y_2\ (t - \tau_{12}),$$

$$\frac{dY_2\ (t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\ (t - \tau_{21}) - m_{22}Y_2\ (t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij} \geq 0$ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \quad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(xvii)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \quad g_2(0) = 0, \tag{4.3}$$

(xviii) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xix) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(xx) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 9. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{matrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{matrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \succ \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}), \end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau$ = max $\tau_{ij}$, $i,j$ = 1,2. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{11}\left|Y_1(\mathrm{t})\right| + b_{11}\left|Y_1(\mathrm{t}-\tau_{11})\right| + m_{12}\ Y_2(\mathrm{t}-\tau_{12})\Big|, \\ \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{22}\left|Y_2(\mathrm{t})\right| + m_{21}\left|Y_1(\mathrm{t}-\tau_{21})\right| + b_{22}\ \left|Y_2(\mathrm{t}-\tau_{21})\right|, \end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\ \left|\ Y_1(t)\ \right| = \left\{\begin{aligned} &\frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) \ge 0, \\ &\frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) < 0, \end{aligned}\right\}\ i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(\mathrm{t})}{dt} &= -m_{11}\ Z_1(\mathrm{t}) + b_{11}Z_1(\mathrm{t}-\tau_{11}) + m_{12}\ Z_2(\mathrm{t}-\tau_{12}), \\ \frac{dZ_2(\mathrm{t})}{dt} &= -m_{22}Z_2(\mathrm{t}) + m_{21}Z_1(\mathrm{t}-\tau_{21}) + b_{22}\ Z_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(\mathrm{s}) > \left|Y_1(s)\right|,\ \ s \in [-\tau, 0],\ i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(\mathrm{t})\ \text{for t} \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(\mathrm{t}) = Z_i(\mathrm{t}) - \left|Y_1(s)\right|$, $i = 1,2$. t$\ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(\mathrm{t})}{dt} &\ge -m_{11}\ U_1(\mathrm{t}) + b_{11}U_1(\mathrm{t}-\tau_{11}) + m_{12}\ U_2(\mathrm{t}-\tau_{12}), \\ \frac{dU_2(\mathrm{t})}{dt} &\ge -m_{22}U_2(\mathrm{t}) + m_{21}U_1(\mathrm{t}-\tau_{21}) + b_{22}\ U_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(\mathrm{s}) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[\ \tau, 0]$ : It will follow from (A.7) that $U_1(\mathrm{t}) \ge U_1(0)\ \exp\ [-m_{11}\mathrm{t}]$

$$+\int_0^t \left[b_{11}U_1(\mathrm{s}-\tau_{11}) + m_{12}\ U_2(\mathrm{s}-\tau_{12})\right] \exp\left[-m_{11}(\mathrm{t}-\mathrm{s})\mathrm{ds} > 0,\right] \qquad (A.9)$$

For all t$\in [0, \tau^*]$ since $U_i(\mathrm{s}) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(\mathrm{t}) > 0$ for t $\in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(\mathrm{t}) > 0, U_2(\mathrm{t}) > 0$ for t $\in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds+\int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds+\int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0\leq \mid Y_{1,}(t) \mid \leq Z_1(t)\to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}+\frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\to 0, Z_2(t)\to 0$ as $t\to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty sk_{ij}(s)ds< \infty;\ \int_0^\infty k_{ij}(s)ds=1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 13. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 14. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big), \qquad t > 0, \tag{2.1}$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ in $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ in $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ ∈ $-\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega + t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(xiii) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(xiv) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t)or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(xv) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$*and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let $N: K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(i)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying $\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$ such that all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1$.*

*(j) All solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2$.*

*Then N has at least one fixed point in $K \cap (H_2/H_1)$.*

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 15. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(m) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(n) N is monotonic nondecreasing on $K^*$.

(o) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cup K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \|x_1\| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \|x_2\| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \in -\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \in -\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \in -\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j=1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x_2^*(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x_1^*(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x_1^*, x_2^*$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x_1^*(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x_2^*(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21})\, x_1^*(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x_2^*(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} -\ _{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha$, $\beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha$,$\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that “In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter Ƭ:

$\frac{dx_1(\mathrm{t})}{dt} x_1(\mathrm{t})[1 - x_1(\mathrm{t}) + 0.4x_2(\mathrm{t} - \mathcal{T})/k_1(\mathrm{t})]$

$\frac{dx_2(\mathrm{t})}{dt} x_2(\mathrm{t})[1 - \{0.8\, x_1(\mathrm{t} - \mathcal{T}) + x_2(\mathrm{t})\}/k_2(\mathrm{t})]$

Where $k_1(\mathrm{t}) = 20 + \sin\ (\pi\mathrm{t})$ and $k_2(\mathrm{t})$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 9. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \dots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.5) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \left( b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left( b_j^u / a_{jj}^l \right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \geq x_i^l \quad for\ t \geq t_0,\ i = 1, 2, \ldots, n. \qquad (1.7)$$

10. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \leq x_l \leq x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1 \leq l \leq n} x_l^l; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(89) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(90) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(91) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(92) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(93) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(94) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(95) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(96) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(97) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(98) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(99) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(100) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(101) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (102) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (103) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (104) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (105) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (106) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (107) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (108) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (109) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (110) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like

Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left||f\ (t+\ \tau,x) - \ f(t,x)|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^{*}\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{ij}\;(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^{*})/(2x^{*}). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k),x(t+\tau_m)\big)\ \ v\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_{*}(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_{*})/(2x^{*})]/\mu x_{*})$$

$$[(2x_n^{*})/x_{*}]\exp[-\mu x_{*}(t+t_k-t_0)]\ +(\varepsilon/2x^{*})$$

$$<(\varepsilon/x^{*})\ \text{On using (2.8), (2.10)} \qquad (2.14)$$

For $m\ \ k\ \ n_0(\varepsilon,\beta)$ and for all $t\in\cup\subset[\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n}|logx_i\,(t+\tau_k)-logx_l(t+\tau_m)|<(\varepsilon/x^{*}) \qquad (2.15)$$

And hence using $x_{*}\ \ x_i(t)\ \ x^{*}, i=1,2,\dots,n; t\ \ t_0$,

$$\sum_{i=1}^{n}|x_i\,(t+\tau_k)-x_i(t+\tau_m)|<\varepsilon \text{ for all } t\in\cup\subset[\beta,\infty], m\ \ k\ \ n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t)=\{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t)\ \in[x_{*},x^{*}], i=1,2,\dots,n; t\in[\beta,\infty]$ follows from (2.16) and the fact that $x_i(t)\ \in[x_{*},x^{*}], i=1,2,\dots,n;$ for $t\ \ t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k)\to w_l(t)$ as $k\to\infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i=1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ ☒ $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ ☒ $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} |\frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m)| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{l};\quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{l};\quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0,\ \ i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad \beta_1 \leq y_i(t) \leq \beta_2, \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V(x(t)y(t)) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t)y(t))$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiial Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ 0 $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \ \rho_1(t, a_1) \ \rho_1^*(t, a_1),$$

$$0 \ Q_1(t) \ Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0)$$

$$= \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1\,(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1\,(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1[P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ħ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** **d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3) Y_3(\gamma_3, a_3) da_3\right) u_3(0)$$

$$+w \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3 = 0,$$

(24)

$$\left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right) \tilde{u}_3(0)$$

$$+\left(-1 - \int_0^\infty Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3\right) w = 0.$$

Equqtion (20) has a solution of the form of $u_3(t, a_3) = u_3(a_3) e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3) Y^{-3}\,(\gamma_3, a_3) d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l \left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right)\left(\int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) da\right)\left(1 + (f_{30}^l)_3 \int_0^\infty Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$= 0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^-(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}^{'}$ denoting $h_{3}^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{,3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2\,[a_3^2]_3 < 4\,[a_2]_3 < 4\,a_2 e$,

$$\text{Re}\,\lambda_3[\lambda_3]_3 = \text{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\,]_3 < 0.$$

Let $\Delta = \max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4\,a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

**correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1 \log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 37. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \text{ 0}, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \text{ ∞},$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a) e^{-\int_0^a \mu_4(\tau)dt} e^{-ra}\, da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(t)dt}\, da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 38. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^\infty Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^\infty Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a (\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^\tau \mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a \mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d\, a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th]

(or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4[a_2]_4<1\, a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1+x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1+x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1+x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])^3_4\ [\mathrm{K}]_4[P_0]^3_4} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}^4_{2*}] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}^4_*] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}^4_*] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae_4^{mu} \quad [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 39. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ } 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ } \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t,a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 40. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0) = \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6'(P_0) = f_{60}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a \mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^\tau \mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a \mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(2085) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(2086) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(2087) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(2088) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(2089) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(2090) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(2091) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(2092) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(2093) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(2094) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(2095) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(2096) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(2097) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(2098) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

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| (2099) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.** |
| (2100) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.** |
| (2101) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.** |
| (2102) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".** |
| (2103) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.** |

**Black holes**

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| (2104) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.** |
| (2105) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.** |
| (2106) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.** |
| (2107) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.** |
| (2108) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.** |

**Astrophysical effects**

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| (2109) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.** |
| (2110) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.** |
| (2111) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.** |

**Main approaches**

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| (2112) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have** |

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| **experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (2113) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (2114) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

**String theory**

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| (2115) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (2116) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (2117) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (2118) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (2119) **However, when summed the entire series appears to be (=) divergent.** |
| (2120) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (2121) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (2122) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (2123) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (2124) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (2125) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (2126) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |

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| (2127) **The granular properties of space can be (=) explicitly computed.** |
| (2128) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (2129) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (2130) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (2131) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (2132) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (2133) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (2134) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (2135) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (2136) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (2137) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (2138) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (2139) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (2140) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (2141) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (2142) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (2143) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (2144) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (2145) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (2146) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (2147) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (2148) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (2149) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (2150) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (2151) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (2152) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (2153) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (2154) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (2155) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

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| (2156) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (2157) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (2158) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (2159) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (2160) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (2161) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (2162) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (2163) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (2164) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (2165) **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (2166) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (2167) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (2168) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (2169) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (2170) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—** |

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (2171) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (2172) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (2173) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (2174) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (2175) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (2176) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (2177) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (2178) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (2179) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (2180) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (2181) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (2182) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (2183) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (2184) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (2185) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (2186) **This is a realistic scenario for (e) all known systems.** |
| (2187) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (2188) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (2189) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (2190) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (2191) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (2192) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (2193) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (2194) **So in those systems, the quantum phase transition does indeed exist.** |
| (2195) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (2196) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (2197) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (2198) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (2199) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (2200) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (2201) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (2202) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (2203) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (2204) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (2205) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (2206) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (2207) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (2208) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (2209) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (2210) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (2211) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (2212) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (2213) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (2214) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (2215) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (2216) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (2217) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (2218) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (2219) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (2220) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (2221) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (2222) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (2223) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (2224) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (2225) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (∼104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (2226) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (2227) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (2228) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (2229) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (2230) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (2231) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (2232) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (2233) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (2234) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (2235) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (2236) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (2237) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (2238) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (2239) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (2240) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (2241) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (2242) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (2243) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (2244) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (2245) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (2246) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (2247) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (2248) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (2249) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (2250) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (2251) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (2252) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (2253) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (2254) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (2255) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (2256) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (2257) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (2258) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (2259) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (2260) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (2261) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (2262) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (2263) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (2264) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (2265) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (2266) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (2267) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (2268) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (2269) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (2270) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (2271) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (2272) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (2273) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (2274) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (2275) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (2276) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (2277) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (2278) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (2279) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (2280) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (2281) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (2282) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (2283) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (2284) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

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| (2285) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (2286) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (2287) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (2288) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (2289) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (2290) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (2291) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (2292) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (2293) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (2294) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (2295) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (2296) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (2297) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (2298) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |

(2299) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(2300) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(2301) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(2302) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(2303) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(2304) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(2305) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(2306) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(2307) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(2308) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(2309) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(2310) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(2311) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(2312) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(2313) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (2314) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (2315) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (2316) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (2317) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (2318) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (2319) **We know dark matter exists only from (e) its gravitational effects.** |
| (2320) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (2321) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (2322) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (2323) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (2324) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (2325) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (2326) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (2327) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (2328) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (2329) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (2330) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (2331) | **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (2332) | **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (2333) | **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (2334) | **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (2335) | **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (2336) | **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (2337) | **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (2338) | **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (2339) | **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (2340) | **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (2341) | **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (2342) | **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (2343) | **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (2344) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (2345) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (2346) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (2347) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (2348) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (2349) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (2350) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (2351) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (2352) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (2353) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (2354) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (2355) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (2356) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (2357) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (2358) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (2359) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (2360) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (2361) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (2362) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (2363) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson’s problem**

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| (2364) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking “palm tree” organization.** |
| (2365) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (2366) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (2367) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (2368) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (2369) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (2370) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (2371) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (2372) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (2373) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (2374) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (2375) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (2376) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (2377) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (2378) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (2379) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (2380) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (2381) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (2382) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (2383) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (2384) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (2385) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (2386) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (2387) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (2388) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (2389) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (2390) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (2391) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (2392) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (2393) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (2394) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (2395) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (2396) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (2397) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (2398) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(385)** | **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(386)** | **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(387)** | **Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(388)** | **Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(389)** | **Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(390)** | **Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(391)** | **No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(392)** | **This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(393)** | **Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(394)** | **In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(395)** | **Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

## Classical vs. relativistic black holes

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| **(396)** | **Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(397)** | **In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(398)** | **If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(399)** | **Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(400)** | **In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(401)** | **This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

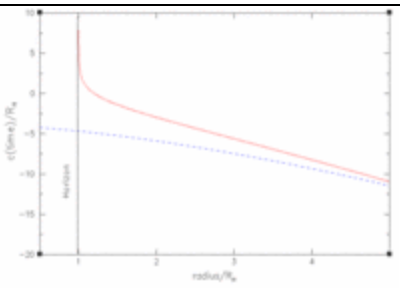
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| **(402) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(403) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(404) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(405) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(406) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(407) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(408) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(409) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(410) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(411) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(412) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(413) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(414) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(415) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(416) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(417) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(418) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(419) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(420) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(421) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(422) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(423) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(424) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(425) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(426) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(427) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(428) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(429) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(430) One of these coordinates is (=) called the "coordinate time."** |
| **(431) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(432) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(433) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(434) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(435) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(436) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(437) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(438) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(439) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(440) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(441) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(442) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(443) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(444) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(445) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(446) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(447) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(448) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(449) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(450) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(451) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(452) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(453) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(454) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(455) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(456) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(457) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(458) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(459) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(460) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(461) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(462) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(463) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(464) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(465) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(466) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(467) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(468) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(469) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(470) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(471) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(472) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(473) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(474) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(475) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(476) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(477) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(478) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(479) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(480) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[\dot{f_{6*}}\right]_6$ denoting $\left[\dot{f}_6(P^*)\right]_6\left[\dot{f_{2*}}\right]_6$ denoting $[f_2]_6$ $(Q_6^*$, $h_6^*$ denoting $\dot{h}_*$ denoting $\dot{h_6}$ $(Q_6^*)$, and $\left[\dot{g}_*\right]_6$

Denoting $\left[\dot{g}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[\dot{f}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[\dot{f_{2*}}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[\dot{f}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[\dot{f_{6*}}\right]_6 P_6^*\right)M_6 - \left[\dot{f_{2*}}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$\mathrm{Re}\,\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$ i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ P^*\ [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\ [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\ g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^2}+\frac{2\omega_6}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6}+(m_6+[\mu_0]_6+\frac{K_6b_6}{\delta_6-b_6}+\sigma_6)\,Q_6^*)^3$$

$$=m_6+[\mu_0]_6+3+\frac{K_6b_6}{\delta_6-b_6}(1+Q^*)(1+\omega_6 Q_6^*$$

Now

$$\left[a_6^2\right]-4\left[a_2\right]e=\left(\frac{2K_6b_6}{\delta_6-b_6}\right)^2+\frac{4K_6b_6\sigma_6\,Q_6^*}{(\delta_6-b_6)(1+Q_6^*)}+\frac{\sigma_6^2Q_6^{*2}}{(1+Q_6^*)^2}$$

$$\frac{4\,b_6\sigma_6\,(\delta_6-b)Q^*}{\delta_6(1+Q_6^*)^2}-\frac{4\,(\delta_6-b_6)^2(1-\omega_6)Q_6^*}{\delta_6(1+\omega_6Q_6^*)1+Q_6^*)}P^*(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_6\,ae_6^{mu}\;[\mu_0]_6-\frac{a_6}{a_6+6}$$

Hence if $[a_6^2>4(a_2)_6e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}] P_6^*$$

$$[a_2]_6 = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e=[g_*^6]_6 Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_6 \beta_6 \left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right] M_6 - [a_2]_6 W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 41. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^{\infty} \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \ P_7 \ 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \ \infty,$$

$$\rho_7(t,0) = \int_0^{\infty} n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(2399) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(2400) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(2401) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(2402) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(2403) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(2404) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(2405) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(2406) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(2407) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(2408) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(2409) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(2410) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(2411) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(2412) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(2413) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(2414) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(2415) The electronic and (e&eb) structural variability among them is extraordinary.

(2416) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(2417) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(2418) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(2419) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(2420) Nudge your bath tub just right and (eb) you might end up reversing time.

(2421) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(2422) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(2423) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(2424) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(2425) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(2426) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(2427) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(2428) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(2429) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(2430) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(2431) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(2432) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(2433) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(2434) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(2435) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(2436) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(2437) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(2438) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(2439) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(2440) Quantum mechanics rules the dynamics of (e) light and matter.

(2441) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(2442) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(2443) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(2444) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (2445) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (2446) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (2447) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (2448) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (2449) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (2450) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (2451) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (2452) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (2453) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (2454) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (2455) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (2456) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (2457) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (2458) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (2459) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (2460) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (2461) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (2462) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (2463) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (2464) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (2465) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (2466) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (2467) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (2468) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (2469) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (2470) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (2471) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (2472) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (2473) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (2474) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (2475) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (2476) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (2477) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (2478) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (2479) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (2480) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (2481) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (2482) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (2483) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (2484) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (2485) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (2486) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (2487) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |
| (2488) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\,\rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\,\rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 42. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds}d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ \left(\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\left(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7\right) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_j) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\ da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7)da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7)m_7(t,a_7)\,da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0)W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7\left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}^{'}$ denoting $h_{7}^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7[a_7^7]2<4\,[a_7]2<4\,a_7e$,

$\mathrm{Re}\,\lambda_7\left[\lambda_7\right]_7 = \mathrm{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\,]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{*}(a), \boldsymbol{Q}^{*})$ is asymptotically stable if either $a_7^7 <4\,a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⊠4$[a_7]e$ and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\ x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])_{\,7}^{\,3}\ [\mathrm{K}]_{\,7}[P_0]_{\,7}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)\ [P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^{*}\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\,7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^{\,7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\,7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,7}\right] e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7 Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^7 Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7 (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\,(\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae^{mu}_7\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7 Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (447)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (448) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (449)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (450) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

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| **Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).** |

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| (451)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (452) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (453)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (454)Humans are not very good at understanding interacting many-body systems. |
| (455)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (456)But what happens when we couple (e&eb) the particles strongly? |
| (457)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (458)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (459)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (460) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (461) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (462)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (463) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (464)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (465) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (466) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (467)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (468)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (469) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (470) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (471)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (472)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (473)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (474)The energy levels of these quantized orbitals take on (e) discrete values |
| (475):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (476) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (477) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (478)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (479) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (480)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (481) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (482)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (483)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (484)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (485) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (486)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (487)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (488)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (489) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (490)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (491)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (492)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (493)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (494)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (495)Another classic example of duality involves (e&eb) vortices. |
| (496) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (497)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (498) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (499)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (500)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (501) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (502)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (503)TIs are distinguished only by (e) their edge states. |
| (504) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (505)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (506) But the Dirac cone is not (e) the only possible signature. |
| (507) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (508)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (509)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (510)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (511) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (512). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (513)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (514)The bulk duality implies (eb) a duality between edge theories. |
| (515)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |

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| (516)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (517) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (518) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (519)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |
| (520)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (521)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (522)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (523)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (524)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (525) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (526)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (527)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (528) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (529)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (530)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

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| IF we define |
| $$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,\mathrm{P}_7^*$$ |

$$[a_7]_7 = \left[f^{7}_{7*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$

$$e=[g^{7}_{*}]_7 Q^{*}_{7}$$

$$\eta_7(t) = \left[h_{*}\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 43. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(2489) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(2490) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(2491) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(2492) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(2493) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

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| (2494) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (2495) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (2496) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (2497) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (2498) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (2499) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (2500) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (2501) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (2502) "The underlying physics is (=) chaos, essentially," he said. |
| (2503) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (2504) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (2505) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (2506) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (2507) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (2508) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (2509) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (2510) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (2511) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (2512) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |

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| (2513) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (2514) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (2515) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (2516) The study's findings have fundamental implications for (e) quantum computing. |
| (2517) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (2518) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (2519) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (2520) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (2521) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (2522) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer. |
| (2523) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (2524) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (2525) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (2526) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (2527) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (2528) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |

(2529) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(2530) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(2531) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(2532) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(2533) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(2534) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(2535) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(2536) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(2537) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(2538) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(2539) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(2540) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(2541) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as

contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(2542) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(2543) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(2544) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(2545) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(2546) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(2547) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(2548) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(2549) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(2550) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(2551) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(2552) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(2553) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(2554) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(2555) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(2556) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(2557) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(2558) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(2559) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(2560) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(2561) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(2562) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(2563) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(2564) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(2565) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background

radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(2566) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(2567) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(2568) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(2569) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(2570) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(2571) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(2572) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(2573) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(2574) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(2575) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(2576) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(2577) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(2578) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

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| (2579) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime. |
| (2580) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite. |
| (2581) However, when summed the entire series appears to be (=) divergent. |
| (2582) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime. |
| (2583) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects. |
| (2584) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing. |
| (2585) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory. |

**Loop quantum gravity and spin foams**

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| (2586) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics. |
| (2587) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity. |
| (2588) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime. |
| (2589) The granular properties of space can be (=) explicitly computed. |
| (2590) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory. |
| (2591) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks. |
| (2592) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure. |
| (2593) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit. |
| (2594) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism. |
| (2595) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism. |

**Noncommutative geometry**

(2596) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(2597) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(2598) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(2599) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(2600) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(2601) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(2602) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(2603) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(2604) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(2605) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ \text{0}, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \text{∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \text{∞},$$

$$\rho_8(t, 0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly

eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big((f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt}$$

$$= -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8 \; h_8(Q_8) = 1, \qquad and \;\; g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 44. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))u_8 - (\rho_0)_8(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8^{'}(P_0) = f_{80}^{'}$, and $f_8^{'}(0) = f_{80}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \quad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^\infty e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^\infty e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a (\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^\infty Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^\infty Y_8(\gamma_8,a_8)\int_0^a (\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a \mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\,(\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(a_8)da_8-\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^\infty E_8(a_8)da_8.$$

Since $f_8'(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $[\dot{g}]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - [\dot{f}_{8*}]_8[p_8^*](a_8)M_8 - [\dot{f}_{8*}]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 P_8^*\right)M_8 - [\dot{f}_{8*}]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8) + d\tau_8 \mathrm{da}} = \frac{1}{(\mathrm{m}+\mu_0+1)^8} + \frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8 x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8 x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])\,{}_8^3\,[\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8 - 8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^8} + \frac{1\omega_8}{([m_8]_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^3} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$$

$$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\,(\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, ae_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (323)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (324)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (325) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (326) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (327)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (328) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (329)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (330)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(331)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(332)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(333) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(334)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(335)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(336)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(337) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(338)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(339)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(340)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(341)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(342)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(343)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(344)Bringing them to thermal equilibrium requires (e) an external bath.

(345) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(346)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(347)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(348)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(349)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(350)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(351) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(352) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(353) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(354)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(355)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(356) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(357)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(358)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(359) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(360)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(361)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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|---|
| (362)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (363) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (364)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (365) This is a realistic scenario for (e) all known systems. |
| (366)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (367)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (368)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (369) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (370)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (371)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (372)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (373)So in those systems, the quantum phase transition does indeed exist. |
| (374)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (375)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

### MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2} \in [-0.99, 0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2} \in [2, 48] M_\odot$, and (e&eb) total mass $M_{total} \leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\ M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 45. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \quad P_9 \text{ 0}, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \text{ }\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \text{ }\infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9\,(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9]2<4 [a_9]2<4 a_9 e$,

$\mathrm{Re}\,\lambda_9[\lambda_9]_9 = \mathrm{Re}\lambda_9[\lambda_9]_9 = -\frac{9}{9}a_9[a2\,]_9 < 0.$

Let $\Delta=\max_{9<u<\infty}\left|h_*[h_*]_9\beta_9(a_9) - \mu_9(a_9)\right|$. Then$|n_9(t)|$ ?Δ∀t ?. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 <4 a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ?4[$a_9$]e and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m+\mu_0+1)^9} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ ?0.5. Assume$[f_9](x_9)= K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)= [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9)=(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\,[\,\mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_9\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_9 - b) Q^*}{\delta_9 (1 + Q_9^*)^9} - \frac{4\,(\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\,mu}_9 \quad [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO D

## New Class Of Quantum Error-Correcting Codes For A Bosonic Mode

### INTRODUCTION—VARIABLES USED

**New class of quantum error-correcting codes for a bosonic mode Marios H. Michael, Matti Silveri, R. T. Brierley, Victor V. Albert, Juha Salmilehto, Liang Jiang, S. M. Girvin (Submitted on 29 Jan 2016 (v1), last revised 18 Jul 2016 (this version, v3))**

(1) Authors construct a new class of quantum error-correcting codes for (e) a bosonic mode which are advantageous for (e) applications in quantum memories, communication, and scalable computation.
(2) These 'binomial quantum codes' are formed from (e) a finite superposition of Fock states weighted with (e&eb) binomial coefficients.
(3) The binomial codes can exactly correct (e&eb) errors that are polynomial up to (e) a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors.
(4) For realistic continuous-time dissipative evolution, the codes can perform (eb) approximate quantum error correction to (e) any given order in the timestep between error detection measurements.
(5) Authors present an explicit approximate quantum error recovery operation based on (e) projective measurements and unitary operations.
(6) The binomial codes are tailored for (e) detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity.
(7) Authors discuss optimization of the binomial codes and demonstrate that by (e) relaxing the parity structure, codes with (e&eb) even lower unrecoverable error rates can be achieved.
(8) The binomial codes are related to (e&eb) existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors.
**(9)** codes are similar in spirit to (e) 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode.
**(10)** The binomial quantum codes are realizable with (e&eb) current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**. Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version)

### NOTATION

| | |
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| **Module One**<br>Authors construct a new class of quantum error-correcting codes for (e) a bosonic mode which are advantageous for (e) applications in quantum memories, communication, and scalable computation | |
| $G_{13}$ : Category one of bosonic mode which are advantageous for (e) applications in quantum memories, communication, and scalable computation<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of new class of quantum error-correcting codes<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>These 'binomial quantum codes' are formed from (e) a finite superposition of Fock states weighted with (e&eb) binomial coefficients | |
| $G_{16}$ : Category one of finite superposition of Fock states weighted with (e&eb) binomial coefficients<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of'binomial quantum codes' are formed<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>These 'binomial quantum codes' are formed from a finite superposition of Fock states weighted with (e&eb) binomial coefficients | |
| $G_{20}$ : Category one of**'binomial quantum codes' are formed from a finite superposition of Fock states weighted;** binomial coefficients<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of binomial coefficients **; 'binomial quantum codes' are formed from a finite superposition of Fock states weighted**<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>The binomial codes can exactly correct (e&eb) errors that are polynomial up to (e) a specific degree in | |

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| bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors | |
| $G_{24}$ : Category one of **binomial codes can exactly correct;** errors that are polynomial up to (e) a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one oferrors that are polynomial up to (e) a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors; **binomial codes can exactly correct**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The binomial codes can exactly correct errors that are polynomial up to (e) a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors | |
| $G_{28}$ : Category one of specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of binomial codes can exactly correct errors that are polynomial<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors | |
| $G_{32}$ : Category one of amplitude damping and displacement noise as well as (=) boson addition and dephasing errors<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |

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| $T_{32}$ : Category one of binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators, including (e) amplitude damping and displacement noise as well as (=) boson addition and dephasing errors | |
| $G_{36}$ : Category one of amplitude damping and displacement noise as well as (=) boson addition and dephasing errors<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators, including amplitude damping and displacement noise as well as (=) boson addition and dephasing errors | |
| $G_{40}$ : Category one of binomial codes can exactly correct errors that are polynomial up to a specific degree in bosonic creation and annihilation operators, including amplitude damping and displacement noise<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of boson addition and dephasing errors<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>For realistic continuous-time dissipative evolution, the codes can perform approximate quantum error correction to (e) any given order in the timestep between error detection measurements | |

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| $G_{44}$ : Category one of realistic continuous-time dissipative evolution, the codes can perform<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of approximate quantum error correction to (e) any given order in the timestep between error detection measurements<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$<br>$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$<br>$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,<br>$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$<br>$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,<br>$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,<br>$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two** | |

| | |
|---|---|
| **The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}),t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}),t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}),t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |

| | |
|---|---|
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}), t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}), t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}), t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29}, t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}), t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33}, t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}, t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}), t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}), t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}), t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33}, t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}, t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}, t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}), t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}), t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}), t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37}, t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}, t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41}, t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}, t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}), t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}), t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}), t)\right]T_{42}$ | *48* |
| **Module Numbered Nine** | |

| The differential system of this model is now | |
|---|---|
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)}\ \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}\ \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |

| | |
|---|---|
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a'_{18})^{(2)}\boxed{+(a''_{18})^{(2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,)}(T_{14},t)} & \boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a''_{16})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

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| $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{20}^{'})^{(3)} +(a_{20}^{''})^{(3)}(T_{21},t) +(a_{16}^{''})^{(2,2,2)}(T_{17},t) +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[ \begin{array}{c} (a_{21}^{'})^{(3)} +(a_{21}^{''})^{(3)}(T_{21},t) +(a_{17}^{''})^{(2,2,2)}(T_{17},t) +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[ \begin{array}{c} (a_{22}^{'})^{(3)} +(a_{22}^{''})^{(3)}(T_{21},t) +(a_{18}^{''})^{(2,2,2)}(T_{17},t) +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array} \right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3 | |

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| $+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}[-(b_{20}^{''})^{(3)}(G_{23},t)] & [-(b_{16}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{13}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{44}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{20}$ | 70 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}[-(b_{21}^{''})^{(3)}(G_{23},t)] & [-(b_{17}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{14}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{45}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{21}$ | 71 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}[-(b_{22}^{''})^{(3)}(G_{23},t)] & [-(b_{18}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{15}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{46}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{22}$ | 72 |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)}[+(a_{24}^{''})^{(4)}(T_{25},t)] & [+(a_{28}^{''})^{(5,5,)}(T_{29},t)] & [+(a_{32}^{''})^{(6,6,)}(T_{33},t)] \\ [+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)] & [+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)] & [+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)] \\ [+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)] & [+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)] & [+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)] \end{bmatrix} G_{24}$ | 73 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)}[+(a_{25}^{''})^{(4)}(T_{25},t)] & [+(a_{29}^{''})^{(5,5,)}(T_{29},t)] & [+(a_{33}^{''})^{(6,6)}(T_{33},t)] \\ [+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)] & [+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)] & [+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)] \\ [+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)] & [+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)] & [+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)] \end{bmatrix} G_{25}$ | 74 |

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{25}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* $1, 2\ 3$

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}\boxed{-(b_{25}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}\boxed{-(b_{26}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

*Where* $\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$

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| $are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$ $are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$ $are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ $are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ $are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{array}{ccc} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{array}{ccc} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{30}$ | *81* |
| $Where\ \boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}\ are\ first\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ \ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation | |

coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{array}{ccc} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{32}^{''})^{(6,6,6)}(G_{35},t) & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{array}{ccc} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{33}^{''})^{(6,6,6)}(G_{35},t) & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{34}^{''})^{(6,6,6)}(G_{35},t) & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) \\ -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) \\ -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) & & \end{array}\right] T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[\begin{array}{ccc} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) \\ +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) \\ +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) \\ +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right] G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[\begin{array}{ccc} (a_{33}^{'})^{(6)} & +(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) \\ +(a_{25}^{''})^{(4,4,4,)}(T_{25},t) & +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) \\ +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) \\ +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) & & \end{array}\right] G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)}\boxed{+(a''_{34})^{(6)}(T_{33},t)} & \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{34} \quad (87)$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ *are second augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)}$ *are third augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)}$ - are fourth augmentation coefficients

$\boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)}$ - fifth augmentation coefficients

$\boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)}$ sixth augmentation coefficients

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)}$ seventh augmentation coefficients

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)}$ Eighth augmentation coefficients

$\boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)}$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)}\boxed{-(b''_{32})^{(6)}(G_{35},t)} & \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients*

$for\ category$ 1,2 $and$ 3
$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3
$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3
$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3
$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3
$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3
$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

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$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

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$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3
$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3
$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation

coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)} & +(a_{40}^{''})^{(8)}(T_{41},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)} & +(a_{41}^{''})^{(8)}(T_{41},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)} & +(a_{42}^{''})^{(8)}(T_{41},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15}$$

Where $+(a_{40}^{''})^{(8)}(T_{41},t)$, $+(a_{41}^{''})^{(8)}(T_{41},t)$, $+(a_{42}^{''})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} -$$

$$\begin{bmatrix} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

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|---|---|
| $(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lvert\lvert G - G^{'}\rvert\rvert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |

| | |
|---|---|
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t) - (a_i^{''})^{(2)}(T_{17},t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t) - (b_i^{''})^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and $(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\widehat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$ | 112 |

| | |
|---|---|
| The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded. | 117 |

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|---|---|
| **Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \leq (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:

$$(a_i^{''})^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$$

$$(b_i^{''})^{(5)}\big((G_{31}),t\big) \le (r_i)^{(5)} \le (b_i^{'})^{(5)} \le (\hat{B}_{28})^{(5)}$$

$$lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)} \qquad (123)$$

$$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$$

**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: (124)

$$|(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)| \le (\hat{k}_{28})^{(5)}|T_{29} - T_{29}^{'}|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})^{'}||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and $(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : (125)

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : (126)

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$$(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34 \qquad (127)$$

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$ 128

$\lim_{G \to \infty}(b_i^{''})^{(6)}\big((G_{35}), t\big) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}\big((G_{35}), t\big)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

Where we suppose

(EE) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(FF) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(GG)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(HH)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}$ are positive constants and $i = 36,37,38$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$

$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(II) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(JJ) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}} [(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$

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| $\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}$ are positive constants and $i = 40,41,42$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$ | |

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| Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with | |

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| $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions | 152 |

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| **Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |

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| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{16})^{(2)}$ , $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{20})^{(3)}$ , $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to$ | |

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| $\mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |

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| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br><br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br><br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{\mathrm{T}}_{38}(\mathrm{t}) = \mathrm{T}_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br><br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[ (b_{44})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[ (b_{45})^{(9)} T_{44}(s_{(44)}) - \left( (b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[ (b_{46})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{46}(s_{(44)}) \right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)}\left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$ $\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that $(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |

From which it follows that

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t} \leq \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4 — 174

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \leq G_{28}^{0}+\int_0^t\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)} =$$

$$\left(1+(a_{28})^{(5)}t\right)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 — 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 176

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6 — 177

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(f) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 178

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^0)e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^0)e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^0}{G_{37}^0}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^0+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^0+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right) \quad (180)$$

From which it follows that (181)

$$(G_{40}(t)-G_{40}^0)e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^0)e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^0}{G_{41}^0}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^0+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)} =$$

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^0+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^0)e^{-(\hat{M}_{44})^{(9)}t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^0)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^0}{G_{45}^0}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose (182)

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

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| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}^{'})^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t}\le$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i=13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br>From 19 to 24 it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0$ for t > 0 | |

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| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | 193 |

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| Equations into itself | |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}},\widetilde{T_{19}}$ : $\left(\widetilde{G_{19}},\widetilde{T_{19}}\right)=\mathcal{A}^{(2)}(G_{19},T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{16})^{(2)}\left\|G_{17}^{(1)}-G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)}+$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$G_{16}^{(2)}\left\\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)-(a_{16}^{''})^{(2)}\left(T_{17}^{(2)},s_{(16)}\right)\right\\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)}-(G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t}\le$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+(a_{16}^{'})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)},(T_{19})^{(1)};(G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(2)}-(a_i^{''})^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}ds_{(16)}\right]}\ge 0$ | 200 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)}t)} > 0$ for $t > 0$ | |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1, ((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}')^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)} ((\widehat{M}_{16})^{(2)})_1 / (a_{17}')^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)} ((\widehat{M}_{16})^{(2)})_2 / (a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t), t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}((G_{19})(t), t) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + ((\widehat{P}_{20})^{(3)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{20})^{(3)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[((\widehat{Q}_{20})^{(3)} + T_j^0)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)} + T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |

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| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})-(a_{20}'')^{(3)}(T_{21}^{(2)},s_{(20)})\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};(G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}'')^{(3)}$ and $(b_{20}'')^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(3)}$ and $(b_i'')^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i')^{(3)}-(a_i'')^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$ | 216 |

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| $T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}\,(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}\ and\ (\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$ | 228 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(4)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{24})^{(4)})_1, ((\widehat{M}_{24})^{(4)})_2$ *and* $((\widehat{M}_{24})^{(4)})_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq ((\widehat{M}_{24})^{(4)})_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq ((\widehat{M}_{24})^{(4)})_2 = G_{25}^0 + 2(a_{25})^{(4)} ((\widehat{M}_{24})^{(4)})_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq ((\widehat{M}_{24})^{(4)})_3 = G_{26}^0 + 2(a_{26})^{(4)} ((\widehat{M}_{24})^{(4)})_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}((G_{27})(t),t) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + ((\widehat{P}_{28})^{(5)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |

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| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(5)}$ and $(b_i'')^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$ | 239 |

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| it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(5)} - (a_i'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(5)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}')^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)} \left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}')^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)} \left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}')^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t), t)) = (b_{29}')^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}'')^{(5)}\left((G_{31})(t), t\right) = (b_{30}')^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |

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| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)}+\left((\hat{P}_{32})^{(6)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on | 249 |

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| $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i')^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}\ ((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33}\ \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t\ > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval [0, t]<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 260 |

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| suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\ (t) = 0\ and\ T_i\ (t) = 0$<br><br>it results<br><br>$G_i\ (t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\ (t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\ (t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |

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| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose $(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$ $\sup_i\{\max_{t\in\mathbb{R}_+} \left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+} \left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote <br> **Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results <br> $\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\, ds_{(40)} +$ <br> $\int_0^t \{(a'_{40})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$ <br> $(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$ <br> $G_{40}^{(2)}\left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t} \leq$ <br> $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)}, (T_{43})^{(1)}; (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ <br> And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 274 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results | 279 |

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{47}}), (\widetilde{T_{47}}) : \left((\widetilde{G_{47}}), (\widetilde{T_{47}})\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by

$(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \le -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \le -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \le -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G, t) - (b''_{14})^{(1)}(G, t) \le -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

are defined

Then the solution of global equations satisfies the inequalities 285

$G^0_{13}e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G^0_{13}e^{(S_1)^{(1)}t}$

where $(p_i)^{(1)}$ is defined by equation

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| $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$ $\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}')^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}')^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(T_{17}, t) + (a_{17}'')^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}((G_{19}), t) - (b_{17}'')^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |

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| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$,<br>and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ | 315 |

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| $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}^{'})^{(2)}$<br>$(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)}=(b_{18}^{'})^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)}\le-(a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}-(a_{20}^{''})^{(3)}(T_{21},t)+(a_{21}^{''})^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)}\le-(b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{21}^{''})^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$ :<br>By $(v_1)^{(3)}>0$ , $(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0$ , $(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and<br>By $(\bar{v}_1)^{(3)}>0$ , $(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0$ , $(\bar{u}_2)^{(3)}<0$ the<br>roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)}=(v_0)^{(3)},(m_1)^{(3)}=(v_1)^{(3)}$, ***if*** $(v_0)^{(3)}<(v_1)^{(3)}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)}<(v_0)^{(3)}<(\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)}=\frac{G_{20}^{0}}{G_{21}^{0}}}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)}<(v_0)^{(3)}$ | 321 |

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)},\ \boldsymbol{if}\ (u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}, \boldsymbol{if}\ (u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)},$ and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},\ \boldsymbol{if}\ (\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$

$(p_i)^{(3)}$ is defined by equation

$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ 323

$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$

$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ 324

$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ 325

$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ 326

$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$

$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ 327

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$

$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

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| $-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)} + (b_{25}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t) - (b_{25}^{''})^{(4)}((G_{27}),t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)},, (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}, (\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br><br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |

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|---|---|
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$ $\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$ $(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$ $(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$ $(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ : $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying $-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$ $-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 339 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ **Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :- If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by $(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$ $(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$, | 340 |

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| and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)},$ ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},$ ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)},$ ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{28}(t) \leq G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{29}(t) \leq \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \leq G_{30}(t) \leq$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \leq T_{30}(t) \leq$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** | 349 |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying

$$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$$

$$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}), t\big) - (b''_{33})^{(6)}\big((G_{35}), t\big) \le -(\tau_1)^{(6)}$$

**Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ : 350

By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(6)}, , (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$

**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by

$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,
and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$

and analogously 352

$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,
and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$

Then the solution of global equations satisfies the inequalities 353

$$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$$

where $(p_i)^{(6)}$ is defined by equation

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| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G_{34}^0 e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$ *(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$: $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}((G_{39}), t) - (b_{37}'')^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$: By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the | 362 |

roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$

and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (v_0)^{(7)}, (m_1)^{(7)} = (v_1)^{(7)}$, ***if*** $(v_0)^{(7)} < (v_1)^{(7)}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (\bar{v}_1)^{(7)}$, ***if*** $(v_1)^{(7)} < (v_0)^{(7)} < (\bar{v}_1)^{(7)}$,

and $\boxed{(v_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (v_0)^{(7)}$, ***if*** $(\bar{v}_1)^{(7)} < (v_0)^{(7)}$

and analogously

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$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

Then the solution of global equations satisfies the inequalities

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$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$$

where $(p_i)^{(7)}$ is defined by equation

$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$$ 365

$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le \right.$$

$$\left.\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)}t}\right)$$ 366

$$\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$ 367

$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$ 368

$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$$ 369

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| $\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | |
| **Definition of** $(S_1)^{(7)},(S_2)^{(7)},(R_1)^{(7)},(R_2)^{(7)}$:-<br>Where $(S_1)^{(7)}=(a_{36})^{(7)}(m_2)^{(7)}-(a_{36}')^{(7)}$<br>$(S_2)^{(7)}=(a_{38})^{(7)}-(p_{38})^{(7)}$<br>$(R_1)^{(7)}=(b_{36})^{(7)}(\mu_2)^{(7)}-(b_{36}')^{(7)}$<br>$(R_2)^{(7)}=(b_{38}')^{(7)}-(r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 371 |
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| **Definition of** $(\bar{\nu}_1)^{(8)},,(\bar{\nu}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{\nu}_1)^{(8)}>0,(\bar{\nu}_2)^{(8)}<0$ and respectively $(\bar{u}_1)^{(8)}>0,(\bar{u}_2)^{(8)}<0$ the<br>roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_2)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_2)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$<br>**Definition of** $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)}=(\nu_0)^{(8)},(m_1)^{(8)}=(\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)}<(\nu_1)^{(8)}$ | |

| | |
|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , $\boldsymbol{if}$ $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, $\boldsymbol{if}$ $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , $\boldsymbol{if}$ $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}$ $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
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| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\right)\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}] + G_{42}^0 e^{-(a'_{42})^{(8)} t})$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})} \left[ e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t} \right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})} [e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}] + G_{46}^0 e^{-(a'_{46})^{(9)}t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})} \left[ e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t} \right] + T_{46}^0 e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left( (a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t) \right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} \right) \le \frac{dv^{(1)}}{dt} \le -\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} \right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

| | |
|---|---|
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$ | 400 |

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

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And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$(m_2)^{(4)} \leq v^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{v^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $v^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)v^{(5)} - (a_{29})^{(5)}v^{(5)}$$

**Definition of** $v^{(5)}$ :- $\boxed{v^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)}\right) \leq \frac{dv^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$v^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq v^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get 409

$$\nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)}-(\nu_0)^{(5)}}{(\nu_0)^{(5)}-(\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \le \frac{(\nu_1)^{(5)}+(C)^{(5)}(\nu_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}} \le \nu^{(5)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \le (\bar{\nu}_1)^{(5)} \le \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \le \nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \le \nu^{(5)}(t) \le (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \le u^{(5)}(t) \le (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}\right)\le\frac{d\nu^{(6)}}{dt}\le-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)},(\nu_0)^{(6)}$ :-

For $0<\boxed{(\nu_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}<(\nu_1)^{(6)}<(\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t)\ge\frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_0)^{(6)}\right)t\right]}}\quad,\quad\boxed{(C)^{(6)}=\frac{(\nu_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)}\le\nu^{(6)}(t)\le(\nu_1)^{(6)}$

In the same manner , we get

$$\nu^{(6)}(t)\le\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(6)}=\frac{(\bar{\nu}_1)^{(6)}-(\nu_0)^{(6)}}{(\nu_0)^{(6)}-(\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)}\le\nu^{(6)}(t)\le(\bar{\nu}_1)^{(6)}$

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If $0<(\nu_1)^{(6)}<(\nu_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}<(\bar{\nu}_1)^{(6)}$ we find like in the previous case,

$$(\nu_1)^{(6)}\le\frac{(\nu_1)^{(6)}+(C)^{(6)}(\nu_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)}-(\nu_2)^{(6)}\right)t\right]}}\le\nu^{(6)}(t)\le$$

$$\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}\le(\bar{\nu}_1)^{(6)}$$

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If $0<(\nu_1)^{(6)}\le(\bar{\nu}_1)^{(6)}\le\boxed{(\nu_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(\nu_1)^{(6)}\le\nu^{(6)}(t)\le\frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}\le(\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)}\le\nu^{(6)}(t)\le(m_1)^{(6)},\quad\boxed{\nu^{(6)}(t)=\frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)}\le u^{(6)}(t)\le(\mu_1)^{(6)},\quad\boxed{u^{(6)}(t)=\frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

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**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .
Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then
$(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

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In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

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$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get

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$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \le \frac{d\nu^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \ge \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

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| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$ | 434 |

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| $with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |

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| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\,]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\,]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |

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| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(f) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:** | 488 |

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| (a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that | 494 |

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| there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)] + (b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right]+(b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right]+(b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ | |

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| Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^*$ such that $\varphi((G_{47})^*) = 0$ | |
| Finally we obtain the unique solution<br>$G_{14}^*$ given by $\varphi(G^*) = 0$ , $T_{14}^*$ given by $f(T_{14}^*) = 0$ and<br>$G_{13}^* = \frac{(a_{13})^{(1)} G_{14}^*}{[(a_{13}')^{(1)} + (a_{13}'')^{(1)}(T_{14}^*)]}$ , $G_{15}^* = \frac{(a_{15})^{(1)} G_{14}^*}{[(a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}^*)]}$<br>$T_{13}^* = \frac{(b_{13})^{(1)} T_{14}^*}{[(b_{13}')^{(1)} - (b_{13}'')^{(1)}(G^*)]}$ , $T_{15}^* = \frac{(b_{15})^{(1)} T_{14}^*}{[(b_{15}')^{(1)} - (b_{15}'')^{(1)}(G^*)]}$<br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^*$ given by $\varphi((G_{19})^*) = 0$ , $T_{17}^*$ given by $f(T_{17}^*) = 0$ and | 512 |
| $G_{16}^* = \frac{(a_{16})^{(2)} G_{17}^*}{[(a_{16}')^{(2)} + (a_{16}'')^{(2)}(T_{17}^*)]}$ , $G_{18}^* = \frac{(a_{18})^{(2)} G_{17}^*}{[(a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}^*)]}$ | 513 |
| $T_{16}^* = \frac{(b_{16})^{(2)} T_{17}^*}{[(b_{16}')^{(2)} - (b_{16}'')^{(2)}((G_{19})^*)]}$ , $T_{18}^* = \frac{(b_{18})^{(2)} T_{17}^*}{[(b_{18}')^{(2)} - (b_{18}'')^{(2)}((G_{19})^*)]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}')^{(3)} + (a_{20}'')^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}')^{(4)} + (a_{24}'')^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}')^{(4)} - (b_{24}'')^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}')^{(4)} - (b_{26}'')^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution | 518 |

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| $G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 523 A |

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| $T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}} (T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j} ((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)} G_{16}^{*} \mathbb{T}_{17}$ | 534 |

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| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)} \quad , \frac{\partial(b_i^{''})^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ *and* $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 548 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^* ) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \ , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}(\ (G_{31})^*\ ) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 586 A |

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| **Proof:** Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \quad , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$ $+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ $+$ | |

$$(\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\Big)$$

$$+\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\Big)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\Big)\} = 0$$

$$+$$

$$(\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\Big)$$

$$+\left((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\Big)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\Big)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\Big]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\Big)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\Big]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\Big)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b'_{34})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a'_{34})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(33)}T^{*}_{33}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^{*}_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^{*}_{33}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(32)}T^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(34)}T^{*}_{32}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(37)}T^{*}_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^{*}_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^{*}_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(36)}T^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(38)}T^{*}_{36}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(46)}T^{*}_{44}\right)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and**

| | |
|---|---|
| **this proves the theorem.** | |
| Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk. | |

## Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010) It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and

corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dx_i}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{6}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output,

creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so

on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 6. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\ (U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C :\ \omega\ (x) \neq \emptyset \text{ and } \omega\ (x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\ (B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\ (x)$ is continuous

**A2:** $A_i\ (x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\ (A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( Γ(A)), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *(Γ(A)).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ Γ(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in*

$||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\in C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty}\frac{1}{t}\,\text{In }|B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty}\frac{1}{t}\,\text{In }|\,(x.t)_i\,|$$

$$\le \operatorname{limsup}_{t\to\infty}\frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty}\frac{1}{t}In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x})$ =lim $_{t\to\infty}$ $\frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\,d\mu(x) - \int_c h(x)d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x)d\mu(x)|$ is

$$\leq |\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)\,d\mu_n(x)| + |\int_c h(x.t)\,d\mu_n(x) - \int_c h(x)\,d\mu(x)|$$

$$\leq \epsilon + \int_c |h(x.t) - h(x^n.t)|\,d\mu_n(x)| + |\int_c h(x^n.t)|\,d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 2\epsilon + |\int_c h(x^n.t) - h(x)\,d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 3\epsilon$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak*

compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\mu \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t c int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))c *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K⊂ ∂ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K⊂ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j \ (x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i{}_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ ($E_1$)and $\lambda_3$ ($E_1$)are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \quad i = 1, \dots, k; \; j = 1, \dots, m$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\mathcal{A}$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \; \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \quad i = 1, \dots, k$$

(11) $$\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with \; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification

doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration

dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.16) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.17) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.18) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 6. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \; P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) \, exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

| |
|---|
| $P_1(s)=(p_1(s),p_2(s-t))$, |
| $P_2(s)=(p_1(s-t),p_2(s))$, |
| For $0\leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that |
| (2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$ |
| $t > a \geq, i = 1,2,$ |
| Where |
| (2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$ |
| Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows: |
| $M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$ |
| (2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$ |
| It will now follow from (1.1),(1.3),(2.2)-2.5)that |
| (2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da+\int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$ |
| $t> 0, i = 1,2,$ |
| (2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \epsilon[-\tau, 0], i = 1,2,$ |
| (2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$ |
| $+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0) L_i(a, t, P_i(t))da$ |
| $i = 1, 2, t > 0$ |
| the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following. |
| THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of* |
| *(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).* |
| *conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are* |
| *defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] . |

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}), \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^*\Phi_i(0)\exp[B_i^*t] + \Phi_i(0)d_i^*\}t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{\sum_{j=1}^2 \|\Phi_{jt}\|_{t*}[B_j^*\exp(B_j^*t) + d_i^*] \quad t^*\right\}$$

Thus if we choose,

$$r^* = T\sum_{j=1}^2 \|\Phi_{jt}\|_{t*}[B_j^*\exp(B_j^*T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi_1(P_t^{(1)}) - \Pi_1(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi_1(P^{(1)}(t)) - \Pi_1(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+\left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\text{X}\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma,o)\left[b_1\left(\sigma+a,P_t^{(1)}(a)\right)-b_1\left(\sigma+aP_t^{(2)}(a)\right)\right],L_1\left(\sigma,a,P_t^{(1)}(a)\right)d\sigma$$

$$+\int_o^\infty \varphi(\sigma,o)b_1\left(\sigma+a,P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma,P_t^{(1)}(a)\right)-b_1\left(\sigma+a,P_t^{(2)}(a)\right)\right| \le B_1 \|P_t^{(1)}-P_t^{(2)}\|_{t^*} \text{ for } a\ \varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)|\le\beta_1\Phi_1(0)||P_t^{(1)}-P_t^{(2)}||_t^*+\Phi_1(0)\beta_1 d_1||P_t^{(1)}-P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a[B_1^{(2)}(\sigma)\{b_1(a-\sigma,P_t^{(1)}(a))-b_1(a-\sigma,P_t^{(2)}(a))\}M_1(\sigma,a,P_t^{(1)}(a))$$

$$+b_1(a,-\sigma,P_t^{(2)}(a))\{M_1(\sigma,a,P_t^{(1)}(a))-M_1(a-\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0,P_t^{(2)}(a)\right)\{M_1(\sigma,a,P_t^{(1)}(a))-M_1(a-\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$\le\int_0^a[\beta_1^*\Phi_1(0)e^{\delta 1^a}\beta_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}+\beta_1 d_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}(a-o)]do$$

$$\le\beta_1^*\Phi_1(0)e^{\delta 1^a}\|P_t^{(1)}-P_t^{(2)}\|_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left|B_1^{(1)}(a)-B_1^{(2)}(a)\right|$$

$$\le\int_o^a\left|B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right|B_1^{*d}\sigma$$

$$+\Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t^*}\left\{B_1+B_2 d_1 a+B_1^*\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le\beta_1^*\int_o^a\left|B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma)\right|do+\left(r^*+\|\Phi_t\|_{t*}\right)\|P_t^{(1)}-P_t^{(2)}\|_{t^*}g_1^{(a)}$$

Where $g_1(a)\to\beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} \{ g_1(a)+0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a))-b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^* \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\ (P_t^{(1)} - \pi\ \left(P_t^{(2)}\right) \| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*)\ P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty b_i(a,\ P_1^*,\ P_2^*) P_i^*(a) da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o) \exp\left[ - \int_o^\infty f_i(s, P_1^*, P_2^*) ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}(P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i(s,P_1,P_2)ds\right]da,$$

$$i=1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}(0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0) < F_{2.}(C,0),$$

(3.8) $$F_1(0.C) < F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$$

$$i=1,2,\quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0$ *and* , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0,\quad i=1,2,$$

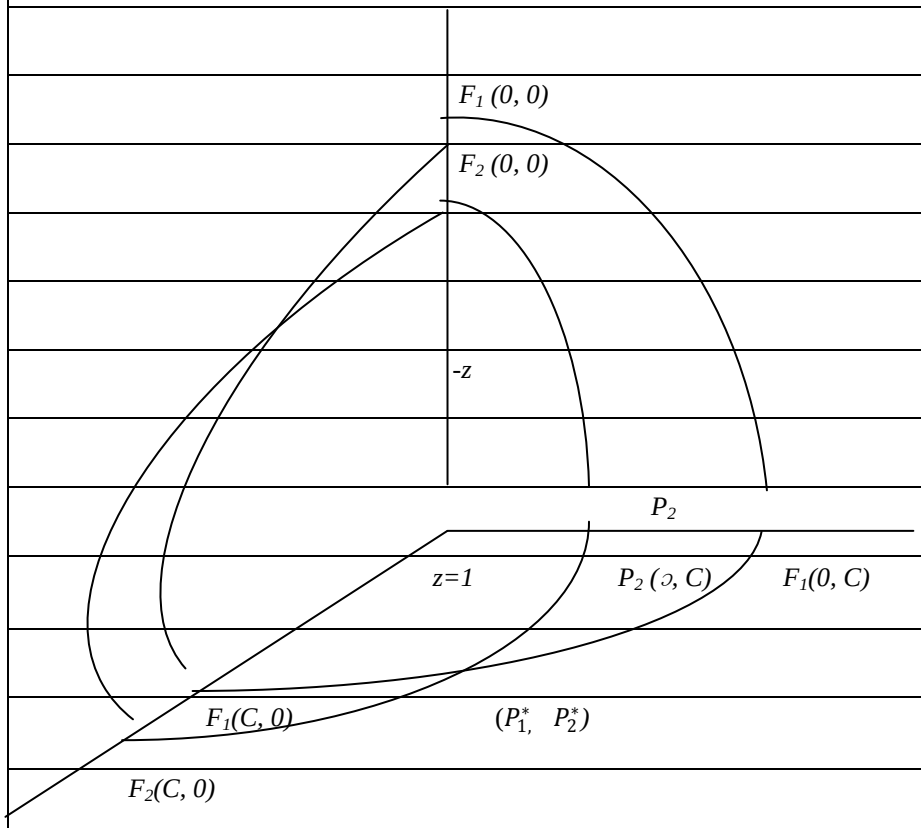
$j=1,2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p^*_1, p^*_2\right)_{\square}(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_{i\square}\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right)_{\square}(a)da + \int_0^\infty p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p^*_j}\, p^*_{i\square}\right\} da,$$

Where $i = 1, 2,$ and

$$p^*_{11} = p^*_1, \quad p^*_{12} = p^*_2 \exp\{-\lambda\tau\},$$

$$p^*_{21} = p^*_1 \exp\{-\lambda\tau\}, \quad p^*_{22} = p^*_2.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i(a, p_1^*, p_2^*)\, p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i(a, p_1^*, p_2^*)p_i^*(a)e^{-\lambda a}da, \quad i = 1, 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^*\exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^*\exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{i} = \int_0^{\infty} p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^{\infty}\left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)da,\ i, \square = 1,\ 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a,\ p_1^*, p_2^*\right) - f_i\left(a,\ p_1^*, p_2^*\right),\ i,\ = 1,\ 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - \left(A_{11} + g_{11}\langle\lambda\rangle\right) & -\left(A_{12} + g_{12}\right)e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right)e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}\langle\lambda\rangle\right)\end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21}\exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda,\tau) = \left[A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)\right]\exp\left[-2\lambda\tau\right]$

$$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ = 1,\ 2.)\ \left(see(A_1)\ and\ (A_2)\right)$;

$A_7.\ \left\{A_{11} + A_{22} + \|\,S_1(0)\,\|\right\}^2 < 4\left\{(A_{11}A_{22} - A_{12}A_{21}) - \|\,S_2(0)\,\|\right\}$,

$$\|\,S_1(0)\,\| = |\,g_{11}(0)| + |\,g_{22}(0)|\,,$$

$$\|\,S_2(0)\,\| = |\,A_{11}\,g_{12}(0)| + |\,A_{12}\,g_{21}(0)| + |\,g_{12}(0)\,g_{21}(0)|$$

$$+ |\,A_{11}\,g_{22}(0)| + |\,A_{22}\,g_{11}(0)| + |\,g_{11}(0)\,g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a,\ t) = \square(a)\exp\left[\lambda t\right] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda,\tau)$ and $F_2(\lambda,\tau)$ as follows:

$F_1(\lambda,\tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau];$

$F_2(\lambda,\tau) = \lambda S_1(\lambda) + S_2(\lambda,\tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda,\tau) - P_2(\lambda,\tau) \right|$$

$$\geq \left| F_1(\lambda,0) \right| - F_2(\lambda,\tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0,\tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0,T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda,\tau) = F_2(\lambda,\tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany

mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational.

It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection.

Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which

paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

# Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure

has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 11. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N≥0 then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \quad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) \right\} F(N(s))\, ds$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^1 f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that

"time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)\,(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.} \tag{1.5}$$

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\big(N(t-\tau)\big) - m(N(t)) \tag{1.6}$$

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\begin{cases} \frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)N(t-s)ds - m(N(t)) \end{cases} \tag{1.7}$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive

and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(xvi) $b_1(0) = 0,\quad m_1(0, x_2) \equiv 0, b_2(0) = 0,\quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xvii) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2(\delta_2) - m_2(x_1, \delta_2) < 0$, for $x_1$ 0, $x_2$ 0, (2.5)

(xviii) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha, \beta) = 0 = b_2(\beta) - m_2(\alpha, \beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\frac{dx_1(t)}{dt} = r_1 x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),$$

$$\frac{dx_2(t)}{dt} = r_2 x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),$$

(2.7)

Where $r$, $a_{ij}$ , $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i$ (0) > 0, $x_2$ (0)> 0 does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1$, $X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\frac{dX_1(t)}{dt} = \beta_{11} X_1(t) + \beta_{12} X_2(t),$$

$$\frac{dX_2(t)}{dt} = \beta_{21} X_1(t) + \beta_{22} X_2(t), \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda\,(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$$a_{11}\alpha > a_{21}\,\beta \text{ and } a_{22}\,\beta > a_{12}\,\alpha,$$

Which together imply $a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1\ (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1\ (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J\ (i,\ j = 1, 2; i \neq j)$.

## 12. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i,j=1,2)$ be a set of nonnegative constants with $\tau = \max\ \tau_{ij}\ , i,j=1,2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t-\tau_{11})\big) - m_1\big(x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t-\tau_{22})\big) - m\big(x_1(t-\tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i=1,2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s)\ \ 0, \qquad s\ \in [-\tau, 0], \qquad i=1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\{\psi_1^{(n)}(s)\}, s \in [-\tau, 0], i=1,2; (n=1,2,\dots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i=1,2 \qquad (3.3)$$

Let $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ be the solution of (3.1) corresponding to the initial condition

$$\{x_1^{(n)}(t), x_2^{(n)}\ (t)\} = \{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\}, \quad s \in [-\tau, 0],. \qquad (3.4)$$

Consider the solution $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i,j=1,2))$ for which

$$x_1^{(n)}(t^*) = 0 \ and \ x_1^{(n)}(t)(t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \qquad (3.5)$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0 \ on \ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [2\tau^*, 3\tau^*] \ etc$. Thus it will follow that so long as $(x_1^{(n)}(t), \ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t) \geq 0 \ x_2^{(n)}(t) \geq 0.$$

If we now consider the limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t), \ x_2^{(n)}(t)\right\} = \{x_1(t), x_2(t)\} \qquad (3.2)$$

With $x_1(t)$ 0,, $x_2(t)$ 0 where $(x_1(t), x_2(t))$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ 0; then there exists a $t_1 > 0$ such that at least for one of either of $x_1 \ or \ x_2$ we have

$$\lim_{t\to t_1-} x_i(t) = \infty. \qquad (3.7)$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i(t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1(x_1(t_2 - \tau_{11})) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(x_1(t_2)) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(\delta_1) - m_1(\delta_1, x(t_2 - \tau_{12})) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ 0.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1(t-\tau_{21}) - m_{22}Y_2(t-\tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i} \; at \; (\alpha, \beta). \tag{3.10}$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ 0 $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= f_1(y_1(t)) - g_1(y_1(t), y_2(t)), \\ \frac{dy_2(t)}{dt} &= f_2(y_1(t)) - y_2(t) - g_2(y_2(t)), \end{aligned}\right\} \tag{4.1}$$

Where $f_i \; and \; g_i \; i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i \; and \; g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i \; and \; g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0 \; (i = 1, 2); also \; we \; require$

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \; 0; \quad \frac{\partial g_1}{\partial y_2} \; 0, \\ &\frac{\partial f_2}{\partial y_1} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0, \end{aligned}\right\} \text{Where } y_1 \; 0, y_1 \; 0, \tag{4.2}$$

(xxi)

$$\left.\begin{aligned} f_1(0) &= 0, \quad g_1(0, y_2 \equiv 0, \\ f_2(0, y_2) &= 0 = f_2(y_1, 0); \quad g_2(0) = 0, \end{aligned}\right\} \tag{4.3}$$

(xxii) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xxiii) There exists a point $(\alpha^* \beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$\left.\begin{aligned} f_1(\alpha^*) - g_1(\alpha^*, \beta^*) &= 0, \\ f_2(\alpha^*, \beta^*) - g_2(\beta^*) &= 0, \end{aligned}\right\} \tag{4.5}$$

(xxiv) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \geq 0,$$
$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \quad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_l, a_{ij}\ (i, j = 1,2)$ are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t \geq 0$ for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all $t \geq 0$ let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l(0)$ satisfy $0 < y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0) \geq \eta_l and\ y_2(0) \geq \eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t > 0,$ ). Suppose $y_l(t)$ does not exist for all $t \geq 0$; then for some $t_l < \infty\ lim_{t \to t_l -} y_l(t) = \infty$ and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t \geq 0$; if $y_2(t)$ is not defined for all $t \geq 0$, then for some $t_3 \leq \infty$ we will have $lim_{t - t_3 -} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2) \leq f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t \geq 0$.

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\frac{dU_1(t)}{dt} = \frac{\partial f_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_2}U_2(t),$$

$$\frac{dU_2(t)}{dt} = \frac{\partial f_2}{\partial y_1}U_1(t) - \frac{\partial f_2}{\partial y_2}U_2(t) - \frac{\partial g_2}{\partial y_2}U_2(t), \qquad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

Where $f_{ij} = \frac{\partial f_1}{\partial y_l}$, $g_{ij} = \frac{\partial g_1}{\partial y_j}$, at $(\alpha^*, \beta^*)$ (4.8)

$(i,j = 1, 2)$

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}), \qquad (4.9)$$

$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}.$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$

$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \qquad \text{Evaluated at } (\alpha^*, \beta^*), \qquad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 10. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.2)$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t),$$

$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_2(t)N_1(t), \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\frac{dN_1(t)}{dt} = rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t),$$
$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_1(t)N_2(t). \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\frac{dN_1(t)}{dt} = a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t),$$
$$\frac{dN_2(t)}{dt} = a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$
$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^{\infty} \delta(a)N_2(t-a)e^{-d*a}\,da,$$
(5.6)

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$
$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)),$$
(5.7)

Where $\tau_{ij}$ $(i,j=1,2)$ are nonnegative constants with $\tau=\max\tau_{ij}, i,j=1,2$; the birth and death rate functions $f_1,f_2,g_1,g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s)=\phi_1(s)>0,\quad s\in[-\tau,0].$$

$$N_1(s)=\phi_1(s)>0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1}>0,\frac{\partial f_2}{\partial N_2}>0, f_2(N_2,N_2)>0 \text{ whenever } N_1>0,\ N_2>0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s)>0$ and $N_2(s)>0$ for $s\in[-\tau,0]$, solution of (5.7) exist for all $t\geq 0$ and remain nonnegative for all $t\geq 0$. also $(\alpha^*,\beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*,\beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state $(\alpha^*,\beta^*)$ is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt}=-m_{11}Y_1(t)+b_{11}Y_1(t-\tau_{11})-m_{12}Y_2(t-\tau_{12}),$$
$$\frac{dY_2(t)}{dt}=-m_{22}Y_1(t)+m_{21}Y_1(t-\tau_{21})-b_{22}Y_2(t-\tau_{22}),\quad t>0, \quad (6.1)$$

Where$m_{1J}, b_{i1}$ $(i,j=1,2)$ are positive constants while $\tau_{1J}(i,j=1,2)$ are non negative with $\tau=\max(\tau_{1J})(i,j=1,2.$as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt}=-m_{11}Z_1(t)+b_{11}Z_1(t-\tau_{11})-m_{12}Z_2(t-\tau_{12}),$$

$t > 0,$ (6.2)

$$\frac{dZ_2(t)}{dt} = -m_{22}\, Z_1(t) + m_{21} Z_1(t-\tau_{21}) - b_{22} Z_2(t-\tau_{22}),$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \ \ for\ s \le 0 \ \ for\ which\ Y_i(s)\ is\ defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad ||Z|| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad ||B|| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_j [b_{jj} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jj} | ]$ $B=(b_{1j})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \le [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \le \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm ‖B‖ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \ or \ -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix}\lambda+m_{11} & 0\\ 0 & \lambda+22\end{bmatrix}-b_{11}^{e-\lambda T_{11}}\begin{pmatrix}-m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}}\end{pmatrix}$$

$=\lambda I-A-B(\lambda,T_{ij})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\boldsymbol{\alpha}<0.$

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\boldsymbol{\alpha}>0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\boldsymbol{\alpha}$ in (6.8)and define

$G(z-\boldsymbol{\alpha},T_{IJ})=M(z)=(z-a)I-A-B(z-\boldsymbol{\alpha},T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$Re(\lambda)<-a$

is the following :

$-\mu(aI+A)>\|B_1(a,T_{ij})\|,$

Where

$$B_1(a,T_{ij})=\begin{pmatrix}b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}}\\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}}\end{pmatrix}$$

Since $\mu(\boldsymbol{\alpha}\, I+A)=\boldsymbol{\alpha}+\boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\boldsymbol{\alpha}+\mu(A)+\|B_I(\boldsymbol{\alpha},T_{IJ})\|<0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a}f(\mathfrak{a},T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as \; \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$$f(0, T_{Ij}) < 0 \;\; for \; \mathfrak{a} \in [0, \mathfrak{a}^{*})$$

And hence

$$f(\mathfrak{a}, T_{Ij}) = 0$$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root ofmax $b_{11}^{e-\lambda T}{}_{11} - m_{12}^{e-\lambda T}{}_{21} + \mathfrak{a}$-min $m_{11}$ =0

$b_{22}^{e-\lambda T}{}_{22} - m_{21}^{e-\lambda T}{}_{12}$ $m_{22}$

Where the maximum in (6.18) is taken over $b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij} (i, = j\ 1,2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}. \qquad (6.19)$$

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^{*}$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^{*}$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more

species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows

from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement "all animal populations have the reproductive capacity to increase without limit, yet none di" and in his concluding remarks writes " this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated" More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution "Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist." In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self – regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_{I}(x) - m_{I}(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_{2}(x,y) - m_{2}(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^*\ y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^*\ y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta\, xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of

predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned}\frac{d|Y_1(t)|}{dt} &\le -m_{11}\,|Y_1(t)| + b_{11}|Y_1(t-\tau_{11})| + m_{12}\ Y_2(t-\tau_{12})|,\\ \frac{d|Y_1(t)|}{dt} &\le -m_{22}\,|Y_2(t)| + m_{21}|Y_1(t-\tau_{21})| + b_{22}\ |Y_2(t-\tau_{21})|,\end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\,|\,Y_1(t)\,| = \left\{\begin{aligned}&\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0,\\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0,\end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned}\frac{dZ_1(t)}{dt} &= -m_{11}\ Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}\ Z_2(t-\tau_{12}),\\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}\ Z_2(t-\tau_{22}),\end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$Z_1(s) > |Y_1(s)|, \quad s \in [-\tau, 0], i = 1,2.$ (A.5)

Then

$|Y_1(t)| < Z_1(t)$ for $t \geq 0$ and $i = 1,2.$ (A.6)

PROOF. Let $U_1(t) = Z_i(t) - |Y_1(s)|, \quad i = 1,2.$ $t \geq -\tau.$ Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\geq -m_{11}\ U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}\ U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\geq -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}\ U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad \text{(A.7)}$$

With

$U_1(s) = Z_i(s) - |Y_i(s)| > 0, \quad i = 1,2, s \in [-\tau, 0].$ (A.8)

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \geq U_1(0) \exp[-m_{11}t]$

$+\int_0^t [b_{11}U_1(s-\tau_{11}) + m_{12}\ U_2(s-\tau_{12})] \exp[-m_{11}(t-s)ds > 0,]$ (A.9)

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t)$

$+\int_{1-T}^t b_{11}\, Z_1(s)ds + \int_{1-T}^t m_{12}\, Z_2(s)ds$

$+\int_{1-T}^t m_{21}\, Z_1(s)ds + \int_{1-T}^t b_{22}\, Z_{22}(s)ds$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$0 \leq |Y_{1,}(t)| \leq Z_1(t) \to 0$ as $t \to \infty$,

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$= -[Z_1(t)\{m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{m_{22} - (b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t)$,$Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\rightarrow 0$ as $t \rightarrow \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\rightarrow 0$,$Z_2(t)\rightarrow 0$ as $t\rightarrow \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i,j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$\int_0^\infty sk_{ij}(s)ds< \infty$; $\int_0^\infty k_{ij}(s)ds=1$, I,j=1,2.

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i,j = 1,2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where $d|Y_1(t)||/dt$ is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \ Z_i(t) \to \infty \leq as\ t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{Bmatrix} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{Bmatrix}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)+b_{11}\int_0^{\infty} k_{11}(u)\left(\int_{t-u}^{t} Z_1(s)ds\right)du$

$+m_{12}\int_0^{\infty} k_{12}(u)\left(\int_{t-u}^{t} Z_2(s)ds\right)du$

$+m_{21}\int_0^{\infty} k_{12}(u)\left(\int_{t-u}^{t} Z_1(s)ds\right)du$

$+b_{22}\int_0^{\infty} k_{22}(u)\left(\int_{t-u}^{t} Z_2(s)ds\right)du$

Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12))

$$\left(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]\right.$$

$< 0$(using (A.18)) for $Z_1(t) > 0\ and\ Z_2(t) > 0.$

By Lemma 6 below the result will follow

Lemma 6. Let $H(t)=H(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose

$$m_{11} > b_{11} + m_{21} \text{ and } m_{22} > b_{22} | m_{12}$$

Then

$$Z_1(t) + Z_2(t) \to 0 \text{ast} \to \infty.$$

PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence

$\{t_n\} \to \infty$ as n $\to \infty$ such that

$$[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon \qquad (A.22)$$

For some sufficiently small positive number $\varepsilon$ and hence

$$\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, \ldots.$$

By the continuity and differentiability of H with respect to t,

$$\mathrm{H}(t_n - \varepsilon) - \mathrm{H}(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$$

If $\varepsilon$ is small we will have

$$= \frac{dH(t^*)}{dt}\varepsilon/2$$

So that

$$\mathrm{H}(t_n) \leq \mathrm{H}(t_n - \varepsilon) - \varepsilon^2/2$$

$$\leq H(t_{n-1}) - \varepsilon^2/2$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2)$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2):$$

$$\leq H(t_0) - \mathrm{n}(\varepsilon^2/2): \mathrm{t}_0 = 0.$$

(A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$)

We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$

Where A and B are defined as in (6.3).

We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems

**Gopalsamy**

# Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 16. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific

interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 17. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0, \qquad \textbf{(2.1)}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

($H_2$) $f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

($H_3$) There exist positive constants $b_{il}$, $b_i^u$, $a_{il}$, $a_i^u$ ($l = 1, 2$) such that

$$b_{il} - a_{il} x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u / a_i^u, \qquad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} | h(t)|, h \in B_\omega^\tau \right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b_l = \min(b_{1l} / a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1 in\ \bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \; (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{matrix} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{matrix} \quad \text{for t> } 0, \qquad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t-s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+ \int_0^1 G_{12}(m, 0, t-s) g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) ds,$$

$$N_2(t, x_1, x_2) = \int_0^\omega G_{22}(m, \omega, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds \qquad \textbf{(2.6)}$$

$$+ \int_0^1 G_{21}(m, 0, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds,$$

For $t > 0$, with

$$g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) = mx_1(s) + x_1(s) f_1\left(s, x_1(s), x_2(s - \tau_{12})\right),$$

**(2.7)**

$$g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) = mx_2(s) + x_2(s) f_2\left(s, x_1(s - \tau_{21}) x_2, (s)\right),$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

**(2.8)**

$$G_{ij}(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2,$$

$$N(x_1,x_2)(t) = \begin{cases} N_1(n\omega+t,x_1,x_2) \\ N_2(n\omega+t,x_1,x_2) \end{cases} for\ t \in [-\tau,0], \qquad \textbf{(2.9)}$$

*m* being a suitably selected positive constant (more will be said below) while *n* is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ [illegible] and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ [illegible] then $N(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ [illegible] $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$, $t$ [illegible] $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = (x_1(t),x_2(t)) \quad and\ (x_1,x_2) \in K \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1,x_2)$ assuming that $N(x_1,x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1,x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1,x_2)(t)$. It will follow from (2.9) and the assumed periodicity of $N(x_1,x_2)$ $that\ N(x_1,x_2)$ is continuous

For $t$ [illegible] $-\tau$. Now we will verify the periodicity of $N(x_1,x_2)$ in $t$. A simple computation leads to

$$N_1(t+w,x_1,x_2) - N_1(t,x_1,x_2) = e^{-mt}F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m,\omega,t-s)\,g_1(m,s,x_1(s),x_2(s-\tau_{12}))ds$$

$$+\int_0^{\omega} G_{12}(m,0,t-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_{\omega}^{\omega+t} G_{12}(m,0,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$-\int_0^{\omega} G_{11}(m,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_0^{t} G_{12}(m,0,-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B,$ thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m,t,x_1(t),x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m,t,x_1(t-\tau_{21}),x_2(t)\right),$$

For any arbitrary real constant $m$ and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

| | |
|---|---|
| (xvi) | Either $x_1, x_2$ or both $x_1\ and\ x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1\ and\ x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1\ and\ x_2$ cannot be periodic. |
| (xvii) | While $(x_1, x_2)\ \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0\ or\ dx_2(t^*)/dt < 0,$ which will imply that the values of $x_1(t^*)\ or\ x_2(t^*)$ cannot be repeated from below. |
| (xviii) | Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}\ or\ 0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above. |

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1) and\ \lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}.$ *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq \alpha_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq \alpha_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\},$

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda\, x_2\,(t) f_2(t, x_1(t-\tau_{21}), x_2\,(t)) + x_2\,(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^{\omega} G_{11}\,(m, \omega, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^{\omega} G_{12}\,(m, 0, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds. \quad \textbf{(2.14)}$$

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for *m* for the case $N_1(t, x_1, x_2) \geq 0$ and then choose *m* = max $(m_1, m_2)$. Thus we have

$$N: K \cap (G_2/G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\,\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \rightarrow B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} \; for \;\; t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\, t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \rightarrow K$ be completely continuous and which satisfies the following conditions:*

***(k)*** *There exists* $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ *and a* $\mu$ *satisfying*

$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1$.

*(l)* *All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2$.

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that* $f_1, f_2$, *of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max(b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min(b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 18. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1, x2 \leq C*$ *where* $C*$ *some positive constant is. Assume the following:*

(p) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(q) N is monotonic nondecreasing on $K^*$.

(r) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda(1 + \eta) N(x_1, x_2)$. Then N has at most one fixed point different from $(0,0)$ in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \geq \|x_1\| \geq b_1^u/a_1^u,\ b_{2l}/a_{2l} \geq \|x_2\| \geq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2 / G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2 / G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \quad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \quad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2(t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \geq 0 \text{ and } 0 \not\equiv x_2(s) - y_2(s) \geq 0;$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2) - N_2(t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$N(x_1,\ x_2)\ -N\ (y_1,y_2)$ for $(x_1-y_1,\ x_2-y_2)\in\ K^*-\{(0,0)\}$.

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0<\lambda<1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$N_1\ (t,\ \lambda x_1,\ \lambda x_2)-\lambda N_1\ (t,x_1,\ x_2)$

$=\lambda\int_0^{\omega}\ G_{11}\ (m,\omega,t-s)g_1\big(s,x_1(s),\ x_2(s-\tau_{12})\big)H_1(s,\lambda,\ x_1,\ x_2)ds$

$+\lambda\int_0^{t}\ G_{11}\ (m,0,t-s)g_1\big(s,x_1(s),\ x_2(s-\tau_{12})\big)H_1(s,\lambda,\ x_1,\ x_2)ds$

**(3.6)**

Where

$$H_1(s,\lambda,\ x_1,\ x_2)=\frac{m\ x_1(s)+x_1(s)f_1\big(s,\lambda\ x_1(s),\lambda\ x_2(s-\tau_{12})\big)}{m\ x_1(s)+x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big)}-1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s,\lambda,\ x_1,\ x_2)\ \eta_1$ for $0<\ \lambda<1, s>0, (x_1,\ x_2)\ \in\ K^*$, **(3.8)**

And hence

$N_1(t,\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta_1)N_1(t,x_1,\ x_2)$.

A similar argument will show that there exists an $\eta_2>0$ such that

$N_2(t,\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta_2)N_2(t,x_1,\ x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ }?\text{ }-\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ }?\text{ }-\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ }?\text{ }-\tau, \tag{4.1}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(\mathrm{t} - \tau_{12})\, X_2(\mathrm{t} - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \tag{4.2}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t} - \tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t - \tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(t))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \tag{4.4}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(t-\tau_{12})|X_2(t-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(t-\tau_{21})|X_1(t-\tau_{12})| - b_{22}x_2^*(t)|X_2(t)|, \end{aligned} \tag{4.5}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \tag{4.6}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ of $(2.1) - (2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12}, \tag{4.7}$$

*Where the constants* $b_{ij}$ $(i, j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x_2^*(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x_1^*(u)|X_{1,}(u)|\mathrm{du}. \quad \textbf{(4.8)}$$

Since $x_1^*, x_2^*$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x_1^*(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x_2^*(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}. Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(\,|b_{11}| - b_{21})\, x_1^*(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x_2^*(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - v(t_m) \leq -\int_{t_m-\varepsilon}^{t_m} \frac{dv(T)}{dT} d\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$v(t_m - \varepsilon) - v(t_m)$ $\varepsilon^2/2$ and hence

$v(t_m)$ $v(t_m - \varepsilon) - (\varepsilon^2/2)$

$v(t_m - 1)$-$(\varepsilon^2/2)$

$v(t_m - 2)$-$(\varepsilon^2/2)$

$v(t_0)$-m $(\varepsilon^2/2)$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v$ $for$ $t$ 0.

this contradiction shows that $|X_{1,}(t)| + |X_2(t)| \to 0$ as $t \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1(t, x_1(t), x_2(t - T_{12}))1 \quad \left\{ \frac{x_1(t) + ax_2(t - T_{12})}{K_1(t)} \right\}$$

$$f_2(t, x_1(t - {}_{21})x_2(t)) = 1 - \left\{ \frac{\beta x_1(t - T_{21}) + x_2(t)}{K_2(t)} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left| \text{axm}_{t>T}\{-a_{11}(t)\} \right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}(-a_{21}(t)) \Big| = \min_{t} , \beta/K_2(t)$$

$$\max_{t T}\{-a_{22}(t)\} \Big| = \max_{t T}(1/K_2(t)) > \min_{t T}(-a_{12}(t)) \Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha,\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a

constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_1\left(\mathrm{t},x_1(\mathrm{t}),x_2(\mathrm{t}-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_2(\mathrm{t},x_1(\mathrm{t}-T_{21}),x_2(\mathrm{t}))$$

Since $\left(x_1(\mathrm{t}),x_2(\mathrm{t})\right)\rightarrow\left(\mathrm{x}_1^*(\mathrm{t}),\mathrm{x}_2^*(\mathrm{t})\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(t)}{dt}x_1(t)[1-x_1(t)+0.4x_2(t-\mathcal{T})/k_1(t)]$$

| |
|---|
| $\frac{dx_2(\mathrm{t})}{dt} x_2(\mathrm{t})[1-\{0.8\,x_1(\mathrm{t}-\mathcal{T})+x_2(\mathrm{t})\}/k_2(\mathrm{t})]$ |
| Where $k_1(\mathrm{t})=20+\sin(\pi\mathrm{t})$ and $k_2(\mathrm{t})$=2.5+sin (πt) |
| @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ @@@@@@@@@@@@@@@@@@ |
| There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of |

"differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction,

projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction

rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 11. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

t>$t_0$;

$t_0 \in (-\infty,\infty); i = 1,2,\dots,n,$ (1.1)

where $b_1, a_{ij}$ $(i,j = 1,2,\dots.,n)$ are nonnegative almost periodic functions defined for $t \in (-\infty,\infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i,j = 1,2,\dots,n)$ by the following:

$$\inf_{t\in R} b_l(t) = b_i^l; \qquad \sup_{t\in R} b_l\,(t) = b_l^u, \qquad i,j = 1,2,\dots,n.$$

$$\inf_{t\in R} a_{lj}\,(t) = a_{lj}^l; \qquad \sup_{t\in R} a_{lj}\,(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u \left(b_j^u \,/\, a_{lj}^l\right). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l\left\{b_l^u - a_{ll}^l x_i\right\}; \qquad t > t_0, i = 1, 2, \dots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \dots, n; t \geq t_0. \tag{1.5}$$

(1.6) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \dots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \leq \left( b_i^l - \sum_{\substack{j=1 \\ J \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \leq x_i^l \ for\ t \geq t_0,\ i = 1, 2, \dots, n. \tag{1.7}$$

12. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \dots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R}_n | x_* \leq x_i \leq x^*; i = 1, 2, \dots, n\} \quad (1.8)$$

*Where*

$$x_* = \min_{1 \leq i \leq n} x_i^l ; \qquad x^* = \min_{1 \leq i \leq n} x_i^u \quad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(111) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(112) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(113) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(114) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(115) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(116) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of

Southampton, UK. “This is a cool, new idea, and I’m up for trying to see it in experiments,” he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(117) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team’s work. Einstein’s theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth’s surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(118) Because of gravity’s effect on (e&eb) space-time, Pikovski’s team realised that variance in a molecule’s position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(119) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(120) “In most situations decoherence is due to (e) something external; here it’s as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself,” adds Pikovski.

**A practical limit**

(121) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(122) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(123) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

(124) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below).

(125) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints)

(126) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain.

(127) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy.

(128) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**.

(129) If gravity’s effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy.

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| (130) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (131) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (132) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for

any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left|\left|f\ (t+\ \tau, x) - \ f(t,x)\right|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau, x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty, \infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\, t)dt \not\equiv 0 \quad for\ x\ \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x\ \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k, x)\}$ is uniformly convergent on $\boldsymbol{R} \times S, \{g(t+, x)\}$ is also uniformly convergent on $\boldsymbol{R} \times S$ then the module of $g(t+, x)$ is contained in the module of $f(t+, x)$.*

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters $b_1, a_{1J}$ $(i, j = 1,2, \dots, n)$ of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number $\mu$ such that*

$$\inf_{1\ \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J\neq 1}}^{n} (\sup_{t\ \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i$$

$$= 1, 2, \dots, n. \qquad (2.3)$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l\left\{b_l(t) - \sum_{j=1}^{n} a_{lj}(t)x_j\right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \qquad (2.5)$$

Uniformly in $(x_1,\dots,x_n) \in S$ for $t \in (-\infty,\infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0),\dots,n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined for $t \in [\beta,\infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta,\infty), i = 1,2,\dots,n$ and the convergence is uniform on all compact subsets of $[\beta,\infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta,\infty)$.

Let $U$ be any compact subset of $\beta,\infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon,\beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon,\beta)$

and all $t \in (-\infty,\infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \quad (2.6)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \quad (2.7)$$

$$(2nx^*/x_*)\exp[-\mu x_*(\beta+\tau_k-t_0)] < (\varepsilon/2x^*) \quad (2.8)$$

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1,(s) - \log x_1(s+\tau_m+\tau_k)|, \quad x \in t_0, \quad (2.9)$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1,(t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \in v(s)$$

$$\in (1/x^*)\sum_{i=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)|. \quad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s) = \sum_{i=1}^{n} \{sign[x_i(s) - x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[\log x_i(s)]}{ds} - \frac{d[\log x_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\sum_{l=1}^{n} |b_l(s) - b_l(s + \tau_m - \tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s) - x_i(s + \tau_m - \tau_k)| + \sum_{\substack{j=1 \\ j\neq l}}^{n} a_{ij}(s)|x_j(s) - x_j(s + \tau_m - \tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(s)|a_{ij}(s + \tau_m - \tau_k)|\, x_j(s + \tau_m - \tau_k) \qquad (2.11)$$

$$-\mu x_* v\big(x(s), x(s + \tau_m - \tau_k)\big) + \sum_{l=1}^{n} |b_l(s) - b_i(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{iJ}(s) - a_{iJ}(s + \tau_m - \tau_k)\,|. \qquad (2.12)$$

Let us set $s - \tau_k = \sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n} |b_l(s) - b_1(s + \tau_m - \tau_k)| + x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{ij}(s) - a_{iJ}(s + \tau_m - \tau_k)\,|.$$

$$= \sum_{l=1}^{n} |b_l(\sigma + \tau_k) - b_i(\sigma + \tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{iJ}(\sigma + \tau_k) - a_{iJ}(\sigma + \tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t - \tau_k)$ together with (2.13) leads to

$$v\big(x(t + \tau_k), x(t + \tau_m)\big) \le v\big(x(t_0), x(t_0 + \tau_m - \tau_k)\big) \exp[-\mu x_*(t + t_k - t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$[(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$ On using (2.8), (2.10) (2.14)

For $m \geq n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \leq x_i(t) \leq x^*, i = 1,2,\dots,n; t \geq t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \geq n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*,x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*,x^*], i = 1,2,\dots,n;$ for $t \geq t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n$.

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t),\dots,x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt}\,x_i(t+\tau_k) - \frac{d}{dt}\,x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\,(t+\tau_k, x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$- f_l(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k))| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\,(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k))$$

$$- f_l\big(t + \tau_m, x_1(t + \tau_m), \dots, x_n(t + \tau_m)\big)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t} + \tau_k) = x_1(t + \tau_k), \dots,\ x_n(t + \tau_k)\ \in \mathrm{S}$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ ? $k$ ? $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\big)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t + \tau_k)\ \in S$ and $x_l(t + \tau_m)\ \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ ? $k$ ? $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x(t + \tau_m), \dots, x_n(t + \tau_m)\big)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has

been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \geq n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ( (t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k \to \infty} \frac{d}{dt} x_1(t + \tau_k) \quad exists\ for\ i = 1, 2, \dots, n$$

Uniformly on all compact subsets of $(-\infty, \infty)$. Thus we have

$$\lim_{k \to \infty} \lim_{k \to \infty} \frac{x_i(t + \tau_k + h) - x_i(t + \tau_k)}{h}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{x_l(t + \tau_k + h) - x_l(t + \tau_k)}{h} \qquad (2.22)$$

$$= \lim_{h \to 0} \frac{w_1(t + h) - w_1(t)}{h}, \quad i = 1, 2, \dots, n,$$

Showing that (since the left side of (2.22) exists) $w_i(i = 1, 2, \dots, n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt} = \lim_{k\to\infty}[\, f_i\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n\,(t + \tau_k)\right)$$

$$-f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)$$

$$+ - f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)]$$

$$= -f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big); \qquad i = 1, 2, \dots, n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1, f_2, \dots, f_n$ on $R \times S$ and hence $w(t) = \big(w_1(t), \dots, w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t) = \big(w_1(t), \dots, w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t, x_1, \dots, x_n), f_2(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n)\}$ for $(x_1, x_2 \dots, x_n) \in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t + \tau_k, x_1, \dots, x_n), f_2(t, + \tau_k, x_1, \dots, x_n), \dots, f_n(t, + \tau_k, x_1, \dots, x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1\,(t, + \tau_k)\ \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l\,(t, + \tau_k) \to w_i(t + \tau\,), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ ☒$k$ ☐ *n0(ε)* then

$$\sum_{l=1}^{n} |b_l\,(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t+\tau_k) - a_{ij}(t+\tau_m)| < \left(\frac{\varepsilon}{4x^*}\right)\mu x_* \qquad (2.25)$$

$$n(2x^*/x_*)\exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s+\tau_m-\tau_k)\big)$ for $s \in [t, t+\tau_k]$ defined by

$$v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s+\tau_m-\tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{i=1}^{n}\{\text{sign}[w_l(s) - w_i(s+\tau_m-\tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds}\cdot\frac{1}{w_l(s)} - \frac{dw_l(s+\tau_m-\tau_k)}{ds}\,\frac{1}{w_l(s+\tau_m-\tau_k)}\right\}$$

$$\leq -\mu\sum_{l=1}^{n}|w_l(s) - w_l(s+\tau_m-\tau_k)| + \sum_{l=1}^{n}|b_i(s) - b_l(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{l=1}^{n}\sum_{J=1}^{n}\left|a_{lJ}(s) - a_{lJ}(s+\tau_m-\tau_k)\right| \qquad (2.28)$$

$$\leq -\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s) - b_l(s+\tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s) - a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k = \sigma$ then we have from (2.29),

$$D^{+}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^{*}\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\,(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)\right| \qquad (2.30)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_{*})/(4x^{*})]\ +[(\varepsilon\mu x_{*})/(4x^{*})]$$

(2.31)

Integrating (2.31) over $(t,t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called “shrinking operations”), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang’s model or to

Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during 0≤t<1. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$ e^{-\mu x^* T_k} v\big(w(t), w(t+T_m-T_k)\big) + (\varepsilon/2x^*)$

$ e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t+T_m-T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$<(\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} | w_1(t+T_k) - w_1(t+T_m)| < \varepsilon \; for \; m \; k \; n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty \; as \; k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\; as\; k \to \infty \; on$ **R** follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T >$ $0. Then\; if$ (1.2), (1.3)$and$ (2.3)$hold\; the\; system$ (1.1)$will\; have\; a\; periodic\; solution\; w(t) =$ $\{w, (t), \ldots\ldots, w_n(t)\} of\; period\; T\; such\; that$

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots) and\; note\; that$

$$x_1(t+KT) - w_1(t) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) - w_1(t) \to 0 \; as \; \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \ldots, w_n(t)\} of (1.1) is\; globally\; asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \ldots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \ldots, n$ is defined for all t□0 and satisfies

$$\lim_{t \to \infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \ldots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{J=1 \\ J\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu$ $Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= g|g:**R**→ $\boldsymbol{R}$ $and$ $g(t)\lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}(t) y_1(t)\right\},$$

*i=1,2,…,n; t>*$t_0 \in R$,

Where

$$B_i \in H(b_i); (A_{ij}), \qquad i, j = 1,2, \dots, n,$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{ij}^{u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\dots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\dots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\dots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\dots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\dots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \quad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{aligned} &\beta_1 \leq x_i(t) \leq \beta_2, \\ &\beta_1 \leq y_i(t) \leq \beta_2, \end{aligned} \quad i = 1,2, \dots, n; t \geq t_0 \tag{3.7}$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \tag{3.8}$$

It is easy to see that

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1} \sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\ \big(x(t), y(t)\big) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\beta_1 V\big(x(t), y(t)\big) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+ \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t), y(t)) \leq V(x(t_0), y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right] \frac{1}{\mu\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} |x_1(t) - y_1(t)| \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} |x_1(t_0) - y_1(t_0)|$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)|\right.$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying “The cause is by cause,” which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics

would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} \mid a_{ij}(t) - A_{ij}(t) \mid \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Presciential Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of**

**dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t, a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ 0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ \infty,$$

$$\rho_1(t, 0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1 \left[\frac{dQ_2}{dt}\right]_1 - x_2 \left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3 \left[\frac{dQ_3}{dt}\right]_1 - x_4 \left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1P_1) + \big(f_2)_1(Q_1)\big)(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \le \rho_1(t, a_1) \le \rho_1^*(t, a_1),$$

$$0 \le Q_1(t) \le Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\, \rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\, \rho_1^*(t, a_1) da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt} = -b_1 Q_1^* + g_1(P_1^*)Q_1^*, \quad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*(t, a_1) da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \quad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a) e^{-\int_0^a \mu_1(\tau)dt} e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t, a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1(P_0)a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t, a_1) = \rho(t, a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1\,(t, a_1)da_1 = \int_0^\infty \rho_1(t, a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1 N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0'Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}')(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t,a_1) = \tilde{\mathrm{u}}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_1}\,da_1$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_1)$ gives,

$$\tilde{\mathrm{u}}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{\mathrm{u}}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})}\tau_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\int_0^a(\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1+\int_0^{\tau}\mu_1(s_1)ds}d\tau_1\,da_1$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}\int_0^a e^{f_{10}\tau+\int_0^{\tau}\mu(s)ds+y\tau}\alpha_1(\tau)d\tau\,da\right)w=0. \ (23)$$

Define $Y(\gamma,a)=e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta\,(a)Y(\gamma,a)da\right)u(0)$$

$$+w\int_0^{\infty}\beta(a)Y\,(\gamma,a)\int_0^a\alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da=0, (24)$$

$$\left(\int_0^{\infty}Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w=0.$$

Equation (20) has a solution of the form of $u(t,a)=u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I(y,a)=\int_0^a\rho_0(\tau)Y^{-1}\,(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^{l}\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^{l}\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right)=0. \quad (25)$$

Since

$$\rho_0(a)=\rho_0(0)e^{-\int_0^a \mu(\tau)d\tau-f_{10}a}$$

$$I(\gamma,a)=\int_0^a \rho_0(\tau)e^{\gamma\tau+f_{10}\tau+\int_0^\tau \mu(s)ds}d\tau$$

$$=\frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a)=e^{-f_{10}a-\int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1,a_1)=E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma}\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^{1}]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\left(\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)$$

$$+\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1+\frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(a_1)da_1-\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1+\frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(a_1)da_1)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1=1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\int_0^a \alpha_1(\tau_1)Y^{-1}(\gamma_1,\tau_1)d\tau_1 da_1>0$. Therefore, the characteristic solution is

$$\gamma_1=-(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(a_1)da_1.$$

Since $f_1'(P_0)_1>0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1),0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey

population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*)\int_0^\infty \beta_1(a_1)p_1^*(a_J)p_1^*(a_1)da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0)e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1 \int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty} [H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)\left( [f_2^{'}]_1(z_1) \right) \int_0^{\infty} a_1 \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_1$$

$-h^{'}(z_1) \int_0^{\infty} \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\ da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$$H_1(0) = 1 - \int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^{-}(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^{\infty} \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y_1^a}} da_1$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R(f_1(P^*)) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$$M_1(t) \equiv \int_0^\infty m_1(t, a_1) da_1 = P_1(t) - P_1^*,$$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - [p^*]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1) m_1(t, a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1} p_1^*(0) W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $[f_{1*}^{'}]_1$ denoting $[f^{'}{}_1(P^*)]_1 [f_{2*}^{'}]_1$ denoting $[f_2]_1$ $(Q_1^*$, $h_*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $[g_*^{'}]_1$

Denoting $[g^{'}]_1(P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - [f'_{1*}]_1[p_1^*](a_1)M_1 - [f'_{2*}]_1 p_1^*(a_1)W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1)m_1(t,a_1)da_1 + \frac{[h_*^1]_1}{[h^*]_1}[p^*]_1(0)\,W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1[Q_1^*]M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1)da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1)da_1 - \int_0^\infty \mu(a_1)m(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 \int_0^\infty p_1^*(a_1)da_1\right)M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1)da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\,m_1(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 - [f'_{2*}]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^\infty ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 + \left(\frac{[h'_*]_1}{[h_*]^1}\,P_1^*(0) - f'_{2*}P_1^*\right)W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_{*}\right]_1 Q_1^* M_{.1}$$

By the mean value theorem,

$$\int_0^{\infty}([h_{"*}]_1\beta_1(a_1)-\mu_1(a_1))m_1(t,a_1)da_1=([h_*]_1\beta_1(\theta_1)-\mu_1(\theta_1))M_{,1}$$

Where $0<\ \theta_1=\theta_1(t)<\infty$.

And if $a_1^2\,[a_1^2]_1<4\,[a_2]_1<4\,a_2 e$,

$\mathrm{Re}\,\lambda_1\left[\lambda_1\right]_1=\mathrm{Re}\lambda_2\left[\lambda_2\right]_1=-\frac{1}{2}a_1[a2\ ]_1<0.$

Let $\Delta=\max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1)-\mu_1(a_1)\right|$. Then$|n_1(t)|$ ⊿∀t ⊿. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a),\boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2<4\,a_2 e$ and $[a_1]>2\Delta$,or $[a_1^2]$ ⊿$[a_2]e$ and $(a_1-\sqrt{(a_1^2)_1-4(a_2)_1 e})>2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+1}\right)$ and $\beta_1(a_1)=ae^{-ma}$ Then $\Delta=[\mu_0]+1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_1(\tau_1)+d\tau_1 \mathrm{da}} = \frac{1}{(\mathrm{m}+\mu_0+1)^2} + \frac{2}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)$=$(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [\mathrm{K}][P_0])}{[m_1]_1 + (\mu_0)_1 + [\mathrm{K}][P_0])_1^3\ [\mathrm{K}]_1[P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$ i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*}$ $[h_*] = \frac{1+\omega_1 \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*}$

$[f_{1*}] = K_1\ [\mathrm{f}_{2*}^1] = \frac{\sigma_1}{(1+\mathrm{Q}_1^*)^2}[\ \mathrm{g}_*^1] = \frac{\sigma_1}{(1+\mathrm{P}_1^*)^2}$ $[\mathrm{h}_*^1] = \frac{\omega_1 - 1}{(1+\mathrm{Q}_1^*)^2}$

## AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

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| **Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model** |
| **Gravitational quantum wave equation:** |
| **Equation Time-dependent equation (Wikipedia)** |
| **(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]** |
| **(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.** |
| **(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).** |
| **(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2** |
| **(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.** |
| **(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.** |
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |

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| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |

**(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]**

**Time-dependent Schrödinger equation**

**(Single non-relativistic particle)**

**(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)**

**(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").**

**(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.**

**(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.**

**(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.**

**(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).**

**(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)**

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in**

**atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⚿ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38) d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac**

**{(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1+[\mu_0]_1+1+\tfrac{K_1 b_1}{\delta_1-b_1}+(m_1+[\mu_0]_1+\tfrac{K_1 b_1}{\delta_1-b_1}+\sigma_1)\,Q_1^*)^3$$

$=m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1+ Q^*)(1 + \omega_1 Q_1^*$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_1 a e^{mu}_1 [\mu_0]_1 - \frac{a_1}{a_1+1}$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}] P_1^*$$

$$[a_2]_1 = \left[f_{2*}^1\right] P_1^* - \frac{h_*^1}{h_*} P^*(0)$$

$$e=[\,g^{1}_{*}]_{\,1}\,Q^{*}_{1}$$

$$\eta_1(t) = \left[h_{*}\right]_{1}\beta_{1}\left(\theta_{1}(t)\right) - \mu_{1}\left(\theta(t)\right),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_{1}\right]M_1 - [a_2]_{1}W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_{1} - [a_2]_{1} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{1} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(t) = e^{A,t}[X]_{1}(0) + \int_{0} e^{At}$$

**Presciential Dynamics And Contential Staticity Of Accentuation Attrition Models**

**(Module Three)**

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and

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Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 0, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_3(Q_3) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \quad (4)$$

$$\rho_3(t_3, 0) = h_3(Q_3) \int_0^\infty \beta_3(a_3)\rho_3(t, a_3)da, \quad (5)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \leq \rho_3(t, a_3) \leq \rho_3^*(t, a_3),$$

$$0 \leq Q_3(t) \leq Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \quad (7)$$

$$\rho_3^*(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*(t, a_3)da_3 \quad (8)$$

$$\frac{dQ_3^*}{dt} = -b_3 Q_3^* + g_3(P_3^*)Q_3^*, \quad (9)$$

With $P_3^*(t) = \int_0^\infty \rho_3^*(t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_3(a) e^{-\int_0^a \mu_3(\tau)dt} e^{-ra}\, da_3 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(t)dt}\, da_3 > 1, \qquad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^\infty [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t, a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^\infty \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 \quad < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^{\infty}(\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da \quad = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t, a_3) = \rho(t, a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^{\infty} u_3\,(t, a_3)da_3 = \int_0^{\infty} \rho_3(t, a_3)da_3 - \int_0^{\infty} (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0'Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}')(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3 u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^\infty \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3 a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho \mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^\infty u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t,a_3) = \tilde{\mathrm{u}}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\,da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3, a_3)da_3\right)u_3(0)$$

$$+w\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0,$$

(24)

$$\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^{\infty}Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^{l}\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^{l})_3\int_0^{\infty}Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3)=(\rho_0)_3(0)e^{-\int_0^{a}\mu_3(\tau_3)d\tau_3-(f_{30})_3a_3}$$

$$I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)e^{\gamma_3\tau_3+(f_{30})_3\tau+\int_0^{\tau}\mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3) d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(a_3)da_3 - \int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1 + \frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^\infty E_3(a_3)da_3) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\,(\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(\,a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(\,p^*(a),Q^*\,)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(\,P_3^*) + [f_2]_3(Q_3^*)\,)P_3^* = 0, \qquad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^{\infty}\ \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \qquad (27)$$

$$g_3(P_3^*) - b_3 = 0, \qquad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty} \beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\quad da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r$, $R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - [p^*]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3) M - [f_{2*}^3]_3 p_3^*(\mathrm{a}) W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $[f_{3*}']_3$ denoting $[f'_3(P^*)]_3 [f_{2*}']_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*3}$ denoting $h_*'$ denoting $h_3'$ $(Q_3^*)$, and $[g_*']_3$

Denoting $[g']_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - [f'_{3*}]_3 [p_3^*](a_3) M_3 - [f_{2*}']_3 p_3^*(a_3) W_3$$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t, a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3)da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3)da_3 - \int_0^\infty \mu(a_3)m(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f'_{3*}]_3 \int_0^\infty p_3^*(a_3)da_3\right)M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3)da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3)da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 - [\dot{f}_{2*}]_3 P_3^* W_{\cdot 3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 + \left(\frac{[h'_*]_3}{[h_*]^3}\, P_3^*(0) - f'_{2*}P_3^*\right)W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3\, Q_3^*\, M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2 [a_3^2]_3 < 4 [a_2]_3 < 4 a_2 e$,

$\mathrm{Re}\,\lambda_3 [\lambda_3]_3 = \mathrm{Re}\lambda_2 [\lambda_2]_3 = -\frac{3}{2} a_3 [a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty} \left| h_* [h_*]_3 \beta_3(a_3) - \mu_3(a_3) \right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4 a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0] \geq 0.5$. Assume $[f_3](x_3) = K_3 x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3) = [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3) = (1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])_{\,3}^{\,3}\ [\mathrm{K}]_{\,3}[P_0]_{\,3}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$ i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\ [f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}_{2*}^{\,3}] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}_*^{\,3}] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}_*^{\,3}] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the

following equation:

$(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_{\,3} + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\ \mathrm{Q}_3^{*})^{\,3}$

$=m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3+3}$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}] P_3^*$$

$$[a_2]_3 = \left[f_{2*}^3\right] P_3^* - \frac{h_*^3}{h_*} P^*(0)$$

$$e=[\,g_{*}^{3}]_{\,3}\,Q_{3}^{*}$$

$$\eta_3(t) = \left[h_{*}\right]_{\,3}\,\beta_{\,3}\left(\theta_{\,3}(t)\right) - \mu_{\,3}\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_{\,3}\right]M_3 - [a_2]_{\,3}W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_{\,3} - [a_2]_{\,3} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{\,3} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(t) = e^{A,t}\,[X]_{\,3}(0) + \int_0 e^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's**

**1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin**

**bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and**

**Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for**

**(e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole**

**merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of**

**(e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain**

**bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating**

**accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared**

**Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard**

**Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total**

**energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of**

**information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local**

**environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any ℓ-local Hamiltonian, where ℓ≥2, can be achieved using decoupling schemes of length at most O(nℓ−1logn). Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle**

**problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 46. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \; 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \; \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4P_4) + \big(f_2)_4(Q_4)\big)(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \le \rho_4(t, a_4) \le \rho_4^*(t, a_4),$$

$$0 \le Q_4(t) \le Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\, \rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\, \rho_4^*(t, a_4) da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4) da_4$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \tag{10}$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \tag{$H1$}$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4}\rho_4(t,a_4)\,da_4$$

$$\frac{dQ_4}{dt} = -b_4Q_4 + c_4P_4Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4P_4 - \lambda_4P_4Q_4$$

$$\frac{dQ_4}{dt} = b_4Q_4 - c_4P_4Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 47. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0'Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h'_{a_4})(\rho_0)_4 Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4 u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4 a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\, o\,)_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t,a_4)da_4$$

$$u_4(t,0) = \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t,a_4) = \tilde{\mathrm{u}}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_4}\, da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})\tau_4}\,(\alpha_4)_4(\tau_4)e^{\gamma_4 \tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4} e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds} d\tau_4\, da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\, da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4 - \int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds + y\tau}\alpha_4(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_4\,(a_4) Y_4(\gamma_4, a_4) da_4\right) u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4=0,$$

(24)

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^{a}\mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$= \frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4 a_4} - 1)$$

Let

$$E_4(a_4) = e^{-(f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4) d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4, a_4) = E_4(a_4)e^{-\gamma_4 a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma}\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\ (\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1 + \frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4 - \int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1 + \frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)da_4 = 1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4 \neq 0,$$

Since $\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^{a}\alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4 > 0.$ Therefore, the characteristic solution is

$$\gamma_4 = -(f_{40}^{'})_4(\rho_0)_4\int_0^{\infty}E_4(a_4)da_4.$$

Since $f_4^{'}(P_0)_4 > 0,$ we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^{\infty}\beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^*=\int_0^\infty p_4^*(a_4)\,da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^\infty \beta_4(a_4)\,e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}da_{,4}$$

And assume that

$$[k_3]_4\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4a_4}\,da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)([f_2^{'}]_4(z_4)) \int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_4$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} \quad da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} da_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r, R\left(f_4(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$$M_4(t) \equiv \int_0^\infty m_4(t, a_4) da_4 = P_4(t) - P_4^*,$$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - [p^*]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^\infty \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0) W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M,_4$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $[f_{4*}']_4$ denoting $[f'_4(P^*)]_4 [f_{2*}']_4$ denoting $[f_2]_4$ $(Q_4^*$, $h_{*}_4$ denoting $h_*'$ denoting $h_4'$ $(Q_4^*)$, and $[g_*']_4$

Denoting $[g']_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4 + [m_a]_4 = -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4 - [f'_{4*}]_4 [p_4^*](a_4)M_4 - [f_{2*}']_4 p_4^*(a_4)W_4$$

$$m_4(t,0) = [h_*]_4 \int_0^\infty \beta_4(a_4) m_4(t, a_4) da_4 + \frac{[h_*^4]_4}{[h^*]_4} [p^*]_4(0)\, W,_4 \qquad (30)$$

$$\frac{dW_4}{dt} = [g_*^4]_4 [Q_4^*] M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the

universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in

(eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way" point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in (eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing

by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to (e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a

complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed,

authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending

well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite

as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L) mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the

possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=)

the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as:

arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the

Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the

sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th]

(or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th]

(or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory**

**techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number: MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond

last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly, by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in

(e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~

(1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum**

**physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement: Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f^{'}{}_{4*}\right]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 - \left[f_{2*}^{'}\right]_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 + \left(\frac{[h_*^{'}]_4}{[h_*]^4} P_4^*(0) - f_{2*}^{'} P_4^*\right) W_4,$$

(34)

$$\frac{dW_4(t)}{dt}\left[g^{'}{}_*\right]_4 Q_4^* M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4 = ([h_*]_4 \beta_4(\theta_4) - \mu_4(\theta_4)) M_{,4}$$

Where $0 < \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2 [a_4^2]_4 < 4 [a_2]_4 < 1\, a_2 e$,

$\mathrm{Re}\,\lambda_4 [\lambda_4]_4 = \mathrm{Re}\lambda_2 [\lambda_2]_4 = -\frac{4}{2} a_4 [a2\ ]_4 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_* [h_*]_4 \beta_4(a_4) - \mu_4(a_4)\right|$. Then $|n_4(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 < 4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2] \geq 4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4) + d\tau_4 da} = \frac{4}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ 0.5. Assume$[f_4](x_4)= K_4 x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4 x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [K][P_0])}{[m_4]_4 + (\mu_0)_4 + [K][P_0])\,{}_4^3\,[K]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\, P^*\, [f_{2*}] = \frac{\sigma Q_4^*}{4+Q_4^*}$ $[h_*] = \frac{4+\omega_4 Q_4^*}{4+Q_4^*}$

$[f_{4*}] = K_4\, [f_{2*}^4] = \frac{\sigma_4}{(4+Q_4^*)^2}[\, g_*^4] = \frac{\sigma_4}{(4+P_4^*)^2}$ $[h_*^4] = \frac{\omega_4 - 4}{(4+Q_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma)^2}+\frac{2\omega_4}{([m_4]_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma_4)^3}<1$$

And

$$\frac{1}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^2}+\frac{2\omega_4}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_4+[\mu_0]_4+1+\tfrac{K_4b_4}{\delta_4-b_4}+(m_4+[\mu_0]_4+\tfrac{K_4b_4}{\delta_4-b_4}+\sigma_4)\,Q_4^*)^3$$

$$=m_4+[\mu_0]_4+3+\tfrac{K_4b_4}{\delta_4-b_4}(4+Q^*)(1+\omega_4Q_4^*$$

Now

$$\left[a_4^2\right]-1\left[a_2\right]e=\left(\frac{2K_4b_4}{\delta_4-b_4}\right)^2+\frac{1K_4b_4\sigma_4\,Q_4^*}{(\delta_4-b_4)(1+Q_4^*)}+\frac{\sigma_4^2Q_4^{*2}}{(1+Q_4^*)^2}$$

$$\frac{1\,b_4\sigma_4\,(\delta_4-b)Q^*}{\delta_4(1+Q_4^*)^2}-\frac{1\,(\delta_4-b_4)^2(1-\omega_4)Q_4^*}{\delta_4(1+\omega_4Q_4^*)(1+Q_4^*)}P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae^{mu}_4 \ [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a^2_4 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q^*_4}{1 + Q^*_4} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P^*_4$$

$$[a_2]_4 = \left[f^4_{2*}\right] P^*_4 - \frac{h^4_*}{h_*} P^*(0)$$

$$e = [g^4_*]_4 Q^*_4$$

$$\eta_4(t) = \left[h_*\right]_4 \beta_4 \left(\theta_4(t)\right) - \mu_4(\theta(t)),$$

The system(34)can be written as the following system:

$$[M(t)]\left[a_4\right] M_4 - [a_2]_4 W_4 - [n(t)] M_4$$

$$w_4(t) = eM_4$$

Write

$$A = \begin{pmatrix} -[a_4]_4 - [a_2]_4 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_4 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}[X]_4(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum**

**systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th)**

**Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December 1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP**

**Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal**

**deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105**

**DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum**

**mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the**

**general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories $\Sigma i a_i \partial F/\partial a_i - 2F = 8\pi i b_1 u$. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr $<\phi 2>$ and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled**

**with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and**

**Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession**

**due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the**

**system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics.**

**Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc] for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures,**

**especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark**

**matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3**

**[astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley**

**National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero. Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon,"**

**Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of**

**Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two**

**dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show**

**up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of**

**polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 48. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \ P_6 \ 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \ \infty,$$

$$\rho_6(t,0) = \int_0^{\infty} n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ 0 $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6)\int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \ \rho_6(t, a_6) \ \rho_6^*(t, a_6),$$

$$0 \ Q_6(t) \ Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\,\rho_6^*, \quad (7)$$

$$\rho_6^*(t,0) = \int_0^\infty \beta_6(a_6)\,\rho_6^*(t,a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6\,Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t,a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt}e^{-ra}\,da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\,da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6 \; h_6(Q_6) = 1, \qquad and \;\; g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 49. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t,0) = \int_0^\infty \beta_6(a_6)\rho_6(t,a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^\infty u_6(t, a_6)da_6 = \int_0^\infty \rho_6(t, a_6)da_6 - \int_0^\infty (\rho_0)_6(a_6)da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6'(P_0) = f_{60}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t,0) = \rho_6(t,0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0'Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}')(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6 u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^\infty \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6 a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\, o\,)_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho o]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t,a_6)da_6$$

$$u_6(t,0) = \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t,a_6) = \tilde{\mathrm{u}}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_6}\, da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})}\tau_6\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6 - \int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds + y\tau}\alpha_6(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_6\,(a_6)Y_6(\gamma_6, a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6=0,$$

(24)

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60}^{l}{}_6\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^{a}(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$= \frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6 a_6} - 1)$$

Let

$$E_6(a_6) = e^{-(f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6) d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(2606) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(2607) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(2608) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(2609) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(2610) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(2611) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(2612) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(2613) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(2614) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(2615) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort**

**of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(2616) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(2617) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(2618) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(2619) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(2620) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(2621) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(2622) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(2623) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(2624) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

### Black holes

(2625) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(2626) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(2627) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(2628) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(2629) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(2630) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(2631) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(2632) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(2633) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(2634) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(2635) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

## String theory

(2636) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(2637) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(2638) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

| |
|---|
| (2639) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (2640) **However, when summed the entire series appears to be (=) divergent.** |
| (2641) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (2642) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (2643) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (2644) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

| |
|---|
| (2645) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (2646) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (2647) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (2648) **The granular properties of space can be (=) explicitly computed.** |
| (2649) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (2650) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (2651) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (2652) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (2653) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (2654) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

(2655) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.**

(2656) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.**

(2657) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.**

(2658) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.**

(2659) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.**

(2660) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.**

**Others**

(2661) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays**

(2662) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.**

(2663) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.**

(2664) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.**

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(2665) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.**

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| (2666) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (2667) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (2668) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (2669) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (2670) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (2671) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (2672) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (2673) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (2674) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (2675) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (2676) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |
| (2677) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (2678) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (2679) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (2680) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption**

**expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (2681) | **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (2682) | **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (2683) | **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (2684) | **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (2685) | **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (2686) | **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (2687) | **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (2688) | **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (2689) | **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (2690) | **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (2691) | **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (2692) | **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (2693) | **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (2694) | **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (2695) | **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (2696) | **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only** |

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| **be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (2697) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (2698) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (2699) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (2700) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (2701) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (2702) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (2703) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (2704) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (2705) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (2706) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (2707) **This is a realistic scenario for (e) all known systems.** |
| (2708) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (2709) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (2710) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (2711) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (2712) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (2713) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |

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| (2714) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (2715) **So in those systems, the quantum phase transition does indeed exist.** |
| (2716) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (2717) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (2718) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (2719) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (2720) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (2721) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (2722) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (2723) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (2724) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (2725) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (2726) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (2727) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (2728) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |

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| (2729) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (2730) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (2731) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (2732) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (2733) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (2734) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (2735) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (2736) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (2737) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (2738) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (2739) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (2740) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (2741) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (2742) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (2743) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (2744) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (2745) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (2746) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (∼104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (2747) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (2748) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (2749) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (2750) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (2751) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (2752) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (2753) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (2754) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (2755) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (2756) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (2757) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |

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| (2758) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (2759) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (2760) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (2761) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (2762) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (2763) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (2764) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (2765) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (2766) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (2767) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (2768) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (2769) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (2770) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (2771) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (2772) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (2773) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (2774) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (2775) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (2776) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |

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| (2777) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (2778) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (2779) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (2780) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (2781) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (2782) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (2783) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (2784) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (2785) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (2786) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (2787) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (2788) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (2789) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (2790) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (2791) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (2792) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (2793) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (2794) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (2795) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (2796) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (2797) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (2798) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (2799) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (2800) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (2801) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (2802) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (2803) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (2804) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (2805) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (2806) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (2807) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (2808) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (2809) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (2810) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |

(2811) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.**

**Doomed satellite missions**

(2812) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.**

(2813) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.**

(2814) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.**

(2815) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.**

(2816) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.**

(2817) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people**

(2818) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."**

(2819) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.**

(2820) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(2821) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(2822) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

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| (2823) However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking. |
| (2824) Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses. |
| (2825) X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well. |
| (2826) The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma. |
| (2827) These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6. |
| (2828) Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus. |
| (2829) A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate. |
| (2830) Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure. |

**Dark matter guides growth of supermassive black holes February 18, 2015**

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| (2831) This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy. |
| (2832) Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole. |
| (2833) But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy. |
| (2834) A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole. |
| (2835) It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth. |
| (2836) "There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA). |
| (2837) This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies. |
| (2838) However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated. |

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| (2839) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (2840) **We know dark matter exists only from (e) its gravitational effects.** |
| (2841) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (2842) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (2843) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (2844) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (2845) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (2846) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (2847) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (2848) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (2849) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (2850) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (2851) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (2852) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (2853) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (2854) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the** |

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| **underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (2855) **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (2856) **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (2857) **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (2858) **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (2859) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (2860) **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (2861) **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (2862) **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (2863) **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (2864) **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (2865) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (2866) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (2867) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |

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| (2868) | **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (2869) | **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (2870) | **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (2871) | **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (2872) | **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (2873) | **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (2874) | **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (2875) | **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (2876) | **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (2877) | **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |
| (2878) | **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (2879) | **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (2880) | **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (2881) | **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |

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| (2882) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (2883) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (2884) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (2885) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (2886) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (2887) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (2888) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (2889) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (2890) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (2891) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (2892) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (2893) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (2894) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (2895) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e)** |

**electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.**

**Spheres on trees**

(2896) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.**

(2897) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.**

(2898) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.**

(2899) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.**

(2900) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.**

(2901) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.**

(2902) **As a result, it provides (eb) an important clue for finding the global minimum.**

(2903) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.**

(2904) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.**

**Thompson's problem 2**

(2905) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society**

(2906) **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.**

(2907) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.**

**Implications for Thomson's and Smale's problems**

(2908) **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.**

(2909) **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry**

**community have helped understand (eb) a deep mathematical problem. Often it is the other way around."**

(2910) **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.**

(2911) **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."**

(2912) **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.**

(2913) **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.**

(2914) **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.**

(2915) **"For larger numbers, there will likely be (=) multiple funnels.**

(2916) **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."**

(2917) **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.**

(2918) **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.**

(2919) **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org**

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

**(481)** **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.**

**(482)** **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.**

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| **(483) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(484) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(485) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(486) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(487) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(488) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(489) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(490) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(491) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(492) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(493) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(494) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(495) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(496) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(497) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |
| **(498) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(499) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(500) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(501) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |

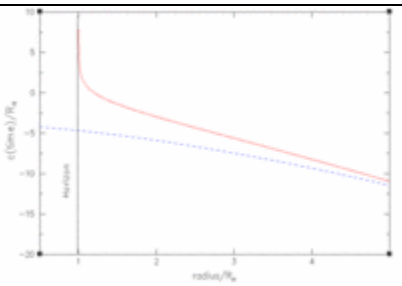
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| **(502) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

## Stationary black holes

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| **(503) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(504) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(505) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(506) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(507) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(508) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(509) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(510) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(511) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(512) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(513) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(514) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(515) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(516) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |
| **(517) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(518) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(519) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(520) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(521) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |

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| **(522) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(523) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(524) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(525) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(526) One of these coordinates is (=) called the "coordinate time."** |
| **(527) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(528) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(529) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(530) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(531) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |
| **(532) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(533) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(534) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(535) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(536) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |

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| **(537) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(538) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(539) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(540) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(541) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(542) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(543) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(544) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(545) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(546) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(547) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(548) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(549) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(550) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(551) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(552) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(553) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(554) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |

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| **(555) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(556) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(557) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(558) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(559) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(560) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(561) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(562) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(563) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(564) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(565) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(566) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(567) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(568) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(569) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(570) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(571) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(572) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |

**(573) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.**

**(574) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.**

**(575) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.**

**(576) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.**

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1 + \frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(a_6)da_6 - \int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6 = 1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6 \neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6 > 0$. Therefore, the characteristic solution is

$$\gamma_6 = -(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6^{'}(P_0)_6 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6),0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_6+(\mu_6(a_6)+[f_6]_6(P_6^*)+[f_2]_6(Q_6^*))P_6^*=0, \qquad (26)$$

$p_6^*(0) = h_6(Q_6^*) \int_0^\infty \beta_6(a_6) p_6^*(a_J) p_6^*(a_6) da_6$ (27)

$g_6(P_6^*) - b_6 = 0,$ (28)

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b),$

From (26),

$p_6^*(a) = p_6^*(0) e^{\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*) \int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6} da_6\right) p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6) \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6} da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty} [H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,) \int_0^{\infty} a_6 \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6} da_6$$

$-h^{'}(z_6) \int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-y_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0) W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6 \left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,$ $h_{*}_{6}$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - [f'_{6*}]_6[p_6^*](a_6)M_6 - [f_{2*}']_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f'_{6*}]_6 \int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6 \int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\, m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 - [f_{2*}']_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^\infty ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 + \left(\frac{[h_*']_6}{[h_*]^6}\,P_6^*(0) - f_{2*}' P_6^*\right)W_6, \qquad (36)$$

$$\frac{dW_6(t)}{dt}[g^{'}{}_{*}]_6 Q_6^* M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^\infty ([h_{*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$$\text{Re}\,\lambda_6[\lambda_6]_6 = \text{Re}\lambda_2[\lambda_2]_6 = -\frac{6}{2}a_6[a2\,]_6 < 0.$$

Let $\Delta = \max_{2<u<\infty}\left|h_*[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)$=(1+$\omega_6 x_6$)/$(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [\mathrm{K}][P_0])}{[m_6]_{\,6} + (\mu_0)_{\,6} + [\mathrm{K}][P_0])\,{}^{3}_{6}\ [\mathrm{K}]_{\,6}[P_0]\,{}^{3}_{6}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_{\,6}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$ $[h_*] = \frac{6+\omega_6 \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$

$[f_{6*}] = K_6\ [\mathrm{f}\,{}^{6}_{2*}] = \frac{\sigma_6}{(6+\mathrm{Q}_6^*)^2}[\ \mathrm{g}\,{}^{6}_{*}] = \frac{\sigma_6}{(6+\mathrm{P}_6^*)^2}$ $[\mathrm{h}\,{}^{6}_{*}] = \frac{\omega_6 - 6}{(6+\mathrm{Q}_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^{\,2}} + \frac{2\omega_6}{([m_6]_{\,6} + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^{\,3}} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,2}} + \frac{2\omega_6}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}]P^*_6$$

$$[a_2]_6 = \left[f^6_{2*}\right]P^*_6 - \frac{h^6_*}{h_*}P^*(0)$$

$$e=[g^6_*]_6 Q^*_6$$

$$\eta_6(t) = \left[h_*\right]_6 \beta_6\left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right]M_6 - [a_2]_6 W_6 - [n(t)]M_6$$

$$w_6(t) = eM_6$$

Write

$$A=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(t) = e^{A,t}[X]_6(0) + \int_0 e^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case $\beta\to\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes

(eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal

anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case ωq=−2/3 authors derive analytic expressions of (e) critical physical quantities, while (e) for

cases $\omega_q \neq -2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb)

this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence,

authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the

dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R~1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They

also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual

field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2

[hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the

recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black**

**holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the

imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves

(e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise

mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top.

Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of

quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity.

**Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions.

The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques

are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity**

**QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP.**

**L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-**

**atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A,**

**75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the**

**sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain**

**are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized**

**ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study**

**the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor**

**connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they**

**interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold**

**dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct (“extreme”) types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, ‘the field’. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb)**

**that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ⁎This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum**

**interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are**

**in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-**

**0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x.**

**Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space**

**time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics**

**(math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 50. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \quad P_7 \; 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3 \left[\frac{dQ_3}{dt}\right]_7 - x_4 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5 \left[\frac{dQ_7}{dt}\right]_7 - x_6 \left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(2920) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(2921) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(2922) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(2923) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(2924) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(2925) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(2926) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(2927) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(2928) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine,

which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(2929) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(2930) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(2931) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(2932) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(2933) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(2934) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(2935) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(2936) The electronic and (e&eb) structural variability among them is extraordinary.

(2937) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(2938) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(2939) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(2940) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(2941) Nudge your bath tub just right and (eb) you might end up reversing time.

(2942) Don’t worry – you won’t end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a “time mirror”, which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

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| (2943) | To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse |
| (2944) | Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses. |
| (2945) | But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort. |
| (2946) | The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort. |
| (2947) | Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower |
| (2948) | Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower |

**Déjà vu**

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| (2949) | To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape. |
| (2950) | The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says. |
| (2951) | Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says. |
| (2952) | "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810 |

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

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| (2953) | Wave control is usually performed by (e) spatially engineering the properties of a medium. |
| (2954) | Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach. |

(2955) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(2956) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(2957) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(2958) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(2959) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(2960) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(2961) Quantum mechanics rules the dynamics of (e) light and matter.

(2962) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(2963) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(2964) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(2965) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit.

(2966) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations.

(2967) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5].

(2968) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates.

(2969) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable.

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| (2970) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (2971) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (2972) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (2973) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (2974) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (2975) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (2976) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (2977) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (2978) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (2979) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (2980) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (2981) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (2982) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (2983) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (2984) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (2985) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (2986) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (2987) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (2988) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |

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| (2989) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (2990) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (2991) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (2992) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (2993) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (2994) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (2995) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (2996) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (2997) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (2998) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (2999) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (3000) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (3001) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (3002) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (3003) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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|---|
| (3004) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (3005) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (3006) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (3007) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |

(3008) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(3009) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ 0 $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ∞; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \leq \rho_7(t, a_7) \leq \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7\, Q_7^* + g_7(P_7^*) Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \quad (10)$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \qquad (H1)$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 51. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7e^{r_7t - \int_0^t (f_7)_7(P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau}\, e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7(P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7 N_7 + o_7(N_7), \quad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \quad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7(0) \quad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7 Q_7$$





The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{\mathrm{u}}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_7)$ gives,

$$\tilde{\mathrm{u}}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{\mathrm{u}}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a(\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds+y\tau}\alpha_7(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^a(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty} Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty} Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)e^{\gamma_7\tau_7+(f_{70})_7\tau+\int_0^{\tau}\mu_7(s_7)ds_7}d\tau$$

$$=\frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7a_7}-1)$$

Let

$$E_7(a_7)=e^{-(f_{70})_7a_7-\int_0^{a}\mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)da_7 = 1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)\ \int_0^a \alpha_7\ (\tau_7)Y^{-7}(\gamma_7, \tau_7)d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7(\rho_0)_7 \int_0^{\infty} E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^{\infty} \beta_7(a_7)p_7^*(a_J)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^{\infty} p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7']_7(z_7))\int_0^\infty a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}da_{7}$$

$-h'(z_7)\int_0^{\infty}\beta_7(a_7)\,e^{\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7-[f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^-(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\,e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^\infty m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - [p^*]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7) M - [f_{7*}^7]_7 p_7^*(\mathrm{a}) W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M,_7$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}']_7$ denoting $[f'_7(P^*)]_7 [f_{7*}']_7$ denoting $[f_7]_7$ $(Q_7^*$, $h_{*7}$ denoting $h_*'$ denoting $h_7'$ $(Q_7^*)$, and $[g_*']_7$

Denoting $[g']_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7 [p_7^*](a_7) M_7 - [f_{7*}']_7 p_7^*(a_7) W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7) m_7(t, a_7) da_7 + \frac{[h_*^7]_7}{[h^*]_7} [p^*]_7(0)\, W,_7 \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7 [Q_7^*] M_7$$

Integrating $[m_7]_7(t, a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t, a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t, a_7)da_7 - \int_0^\infty \mu(a_7)m(t, a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$= m_7(t, 0) - \int_0^\infty \mu_7(a_7)\, m_7(t, a_7)da_7$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [\dot{f_{7*}}]_7 P_7^*\right)M_7 - [\dot{f_{7*}}]_7 P_7^* W_{\cdot 7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\, P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}[g'_*]_7\, Q_7^*\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $\frac{7}{7}$ [a $\frac{7}{7}$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_7 [\lambda_7]_7 = \text{Re}\lambda_7 [\lambda_7]_7 = -\frac{7}{7} a_7$[a2 ] $_7 < 0$.

Let ⊿=max$_{7<u<\infty}$ $|h_* [h_*]_7 \beta_7(a_7) - \mu_7(a_7)|$. Then$|n_7(t)|$ ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $\frac{7}{7}$ <4 a $_7$ e and [a $_7$] $> 1⊿$,or [a $\frac{7}{7}$] [a $_7$]e and (a $_7 - \sqrt{(\text{a}\frac{7}{7})_7 - 4(\text{a}_7)_7 \text{e}} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\text{a}}{\text{a}+7}\right)$ and $\beta_7(a_7) = \text{ae}^{-\text{ma}}$ Then ⊿=$[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_7(a_7)\text{e}^{-\int_0^{\text{a}} \mu_7(\tau_7) + d\tau_7 \text{da}} = \frac{1}{(\text{m} + \mu_0 + 1)^7} + \frac{1}{(\text{m} + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ .5. Assume$[f_7](x_7)= K_7 x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)$= $[0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)$=$(1+\omega_7 x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\text{K}][P_0])}{[m_7]_7 + (\mu_0)_7 + [\text{K}][P_0])\frac{3}{7}\ [\text{K}]_7 [P_0]\frac{3}{7}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_7$ i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^*\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^{7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma)^7} + \frac{1\omega_7}{([m_7]_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma_7)^3} < 1$$

And

$$\frac{1}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^7} + \frac{1\omega_7}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the following equation:

$(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7}+(m_7+[\mu_0]_7+\frac{K_7 b_7}{\delta_7-b_7}+\sigma_7)\ \mathrm{Q}_7^*)^3$

$=m_7+[\mu_0]_7+3+\frac{K_7 b_7}{\delta_7-b_7}(1+\mathrm{Q}^*)(1+\omega_7 \mathrm{Q}_7^*$

Now

$$\left[\mathrm{a}\begin{matrix}1\\1\end{matrix}\right]-4\left[\mathrm{a}_{7}\right]\mathrm{e}=\left(\frac{1K_7b_7}{\delta_7-b_7}\right)^{7}+\frac{4K_7b_7\sigma_7\,\mathrm{Q}_{7}^{*}}{(\delta_7-b_7)(7+\mathrm{Q}_{7}^{*})}+\frac{\sigma_{7}^{7}\,\mathrm{Q}_{7}^{*7}}{\left(1+\mathrm{Q}_{7}^{*}\right)^{7}}$$

$$\frac{4\,b_7\sigma_7\,(\delta_{7}-\mathrm{b})\mathrm{Q}^{*}}{\delta_{7}\,(1+\mathrm{Q}_{7}^{*})^{7}}-\frac{4\,(\delta_{7}-b_7)^{7}\,(1-\omega_7)\mathrm{Q}_{7}^{*}}{\delta_{7}\left(1+\omega_7\mathrm{Q}_{7}^{*}\right)(1+\mathrm{Q}_{7}^{*})}\mathrm{P}^{*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{7}\,ae_{7}^{mu}\;\;[\mu_0]_{7}-\frac{a_7}{a_7+7}$$

Hence if $[\mathrm{a}_{7}^{7}>4(a_7)_{7}\mathrm{e}$ and

$$[a_7]=\frac{1K_7b_7}{\delta_7-b_7}+\frac{\sigma_7\,\mathrm{Q}_{7}^{*}}{1+\mathrm{Q}_{7}^{*}}>1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(531)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(532) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(533)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(534) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(535)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(536) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

(537)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States.

(538)Humans are not very good at understanding interacting many-body systems.

(539)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them.

(540)But what happens when we couple (e&eb) the particles strongly?

(541)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments.

(542)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water.

(543)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h.

(544) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1]

(545) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic

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| origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (546)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (547) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (548)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (549) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (550) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (551)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (552)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (553) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (554) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (555)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (556)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (557)When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (558)The energy levels of these quantized orbitals take on (e) discrete values |
| (559):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (560) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (561) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (562)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |

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| (563) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (564)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (565) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (566)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (567)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (568)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (569) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (570)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (571)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (572)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (573) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an "elementary particle." |
| (574)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (575)The remarkable thing is that solitons can become (=) "elementary," in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (576)An example of a duality, which we will use below, arises in (eb) Maxwell's theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (577)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (578)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (579)Another classic example of duality involves (e&eb) vortices. |
| (580) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (581)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (582) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (583)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (584)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |

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| (585) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (586)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (587)TIs are distinguished only by (e) their edge states. |
| (588) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (589)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (590) But the Dirac cone is not (e) the only possible signature. |
| (591) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (592)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (593)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (594)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (595) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (596). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (597)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (598)The bulk duality implies (eb) a duality between edge theories. |
| (599)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (600)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (601) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (602) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (603)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |

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| (604)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (605)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (606)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (607)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (608)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

| |
|---|
| (609) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (610)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (611)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (612) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (613)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (614)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

| |
|---|
| IF we define |
| $$[a_7]_{\,7} = [f_{7*}]_{\,7} + [f_{7*}]_{\,7} + [f_{7*}]\,P^{*}_{7}$$ |
| $$[a_7]_{\,7} = \left[f^{\,7}_{7\,*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$ |
| $$e=[\,g^{7}_{*}]_{\,7}\,Q^{*}_{7}$$ |
| $$\eta_7(t) = \left[h_{*}\right]_{\,7}\beta_{\,7}\left(\theta_{\,7}(t)\right) - \mu_{\,7}\left(\theta(t)\right),$$ |

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{7}\right]M_7 - [a_7]_{7}W_7 - [\mathrm{n(t)}]M_7$$

$$w_7(\mathrm{t}) = \mathrm{e}M_7$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_7]_{7} & -[a_7]_{7} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{7} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{7}(0) + \int_0 \mathrm{e}^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality

function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 52. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(3010) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(3011) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(3012) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(3013) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(3014) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

(3015) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems.

(3016) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe.

(3017) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short.

(3018) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another).

(3019) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited.

(3020) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan.

(3021) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room

(3022) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them.

(3023) "The underlying physics is (=) chaos, essentially," he said.

(3024) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system.

(3025) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior.

(3026) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion.

(3027) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer.

(3028) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state.

(3029) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate.

(3030) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system.

(3031) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map.

(3032) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map.

(3033) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill.

(3034) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement.

(3035) It turns out that they are actually the driving forces behind (eb) thermalization.

(3036) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle."

(3037) The study's findings have fundamental implications for (e) quantum computing.

(3038) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool.

(3039) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

(3040) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box.

(3041) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction.

(3042) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed.

(3043) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer.

(3044) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box.

(3045) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance.

(3046) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction.

(3047) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device.

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(3048) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(3049) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(3050) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(3051) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(3052) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(3053) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(3054) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(3055) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(3056) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(3057) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(3058) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(3059) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(3060) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(3061) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(3062) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(3063) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(3064) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(3065) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(3066) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(3067) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(3068) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(3069) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(3070) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(3071) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(3072) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(3073) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(3074) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(3075) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(3076) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(3077) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until

genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(3078) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(3079) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(3080) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(3081) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(3082) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(3083) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(3084) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(3085) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(3086) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

### Black holes

(3087) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(3088) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(3089) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(3090) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(3091) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(3092) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(3093) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(3094) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(3095) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(3096) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(3097) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(3098) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(3099) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(3100) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(3101) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(3102) However, when summed the entire series appears to be (=) divergent.

(3103) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(3104) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(3105) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(3106) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(3107) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(3108) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(3109) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(3110) The granular properties of space can be (=) explicitly computed.

(3111) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(3112) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(3113) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(3114) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(3115) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(3116) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(3117) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(3118) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(3119) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(3120) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(3121) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(3122) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(3123) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(3124) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(3125) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(3126) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \quad P_8 \text{ } 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \text{ ?}\infty,$$

$$\rho_8(t,0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 53. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^{\infty} u_8(t, a_8) da_8 = \int_0^{\infty} \rho_8(t, a_8) da_8 - \int_0^{\infty} (\rho_0)_8(a_8) da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t, 0) = \rho_8(t, 0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8\,(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8\,(0) \qquad (18)$$

Expanding

$$h_8(Q_8) = 1 + \ (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^{\infty} \beta_8\,(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\ o\ )_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^{8}]\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}da_8\left(\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^{\infty}E_8(a_8)da_8-\int_0^{\infty}E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^{\infty}E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)da_8=1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\neq 0,$$

Since $\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)\int_0^{a}\alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8>0$. Therefore, the characteristic solution is

$$\gamma_8=-(f_{80}^{'})_8(\rho_0)_8\int_0^{\infty}E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8\left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[\dot{g}\right]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - \left[f^{'}_{8*}\right]_8[p_8^*](a_8)M_8 - \left[\dot{f_{8*}}\right]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}_{8*}\right]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[\dot{f_{8*}}\right]_8 P_8^*\right)M_8 - \left[\dot{f_{8*}}\right]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ 2Δ∀t 0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ 4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])_8^3\ [\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8-8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, a e_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (376)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (377)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (378) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (379) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (380)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (381) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (382)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (383)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(384)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(385)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(386) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(387)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(388)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(389)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(390) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(391)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(392)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(393)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(394)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(395)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(396)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(397)Bringing them to thermal equilibrium requires (e) an external bath.

(398) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(399)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(400)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(401)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(402)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(403)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(404) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(405) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(406) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(407)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(408)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(409) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(410)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(411)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(412) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(413)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(414)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (415)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (416) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (417)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (418) This is a realistic scenario for (e) all known systems. |
| (419)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (420)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (421)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (422) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (423)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (424)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (425)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (426)So in those systems, the quantum phase transition does indeed exist. |
| (427)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (428)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 54. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9\,(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^{a}\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^{a}\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{9}^*$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2<4 [a_9] 2<4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} \left| h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9) \right|$. Then $|n_9(t)|$ ?Δ∀t ?. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 <4 a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ?4[$a_9$]e and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e}) > 1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ ?0.5. Assume$[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}[\ \mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_9\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_9 - b) Q^*}{\delta_9 (1 + Q_9^*)^9} - \frac{4\,(\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\,mu}_9 \quad [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO E

## New Class Of Quantum Error-Correcting Codes For A Bosonic Mode

**INTRODUCTION—VARIABLES USED**

**New class of quantum error-correcting codes for a bosonic mode Marios H. Michael, Matti Silveri, R. T. Brierley, Victor V. Albert, Juha Salmilehto, Liang Jiang, S. M. Girvin (Submitted on 29 Jan 2016 (v1), last revised 18 Jul 2016 (this version, v3))**

(1) Authors present an explicit approximate quantum error recovery operation based on (e) projective measurements and unitary operations.
(2) The binomial codes are tailored for (e) detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity.
(3) Authors discuss optimization of the binomial codes and demonstrate that by (e) relaxing the parity structure, codes with (e&eb) even lower unrecoverable error rates can be achieved.
(4) The binomial codes are related to (e&eb) existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors.
**(5)** codes are similar in spirit to (e) 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode.
**(6)** The binomial quantum codes are realizable with (e&eb) current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**. Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version

**NOTATION**

**Module One**

For realistic continuous-time dissipative evolution, the codes can perform approximate quantum error correction to (e) any given order in the timestep between error detection measurements

$G_{13}$ : Category one of any given order in the timestep between error detection measurements

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of realistic continuous-time dissipative evolution, the codes can perform approximate

| | |
|---|---|
| quantum error correction<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Authors present an explicit approximate quantum error recovery operation based on (e) projective measurements and unitary operations | |
| $G_{16}$ : Category one of projective measurements and unitary operations<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of explicit approximate quantum error recovery operation<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>The binomial codes are tailored for (e) detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity | |
| $G_{20}$ : Category one of detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of binomial codes are tailored<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>The binomial codes are tailored for detecting boson loss and gain errors by means of (e) measurements of (e) the generalized number parity | |
| $G_{24}$ : Category one of measurements of the generalized number parity<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of binomial codes are tailored for detecting boson loss and gain errors<br><br>$T_{25}$ : Category two of SAS | |

| | |
|---|---|
| $T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors discuss optimization of the binomial codes and demonstrate that by relaxing the parity structure, codes with (e&eb) even lower unrecoverable error rates can be achieved | |
| $G_{28}$ : Category one of **optimization of the binomial codes and demonstrate that by relaxing the parity structure, codes;** lower unrecoverable error rates can be achieved<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of lower unrecoverable error rates can be achieved; **optimization of the binomial codes and demonstrate that by relaxing the parity structure, codes**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>The binomial codes are related to (e&eb) existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors | |
| $G_{32}$ : Category one of **binomial codes;** existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors **;binomial codes**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>The binomial codes are related to existing two-mode bosonic codes but offer (eb) the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors | |
| $G_{36}$ : Category one of binomial codes are related to existing two-mode bosonic codes | |

| | |
|---|---|
| $G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The binomial codes are related to existing two-mode bosonic codes but offer the advantage of requiring only a single bosonic mode to (e&eb) correct amplitude damping as well as (=) the ability to correct other errors | |
| $G_{40}$ : Category one of **binomial codes are related to existing two-mode bosonic codes but offer the advantage of requiring only a single bosonic mode;** correct amplitude damping as well as (=) the ability to correct other errors<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of correct amplitude damping as well as (=) the ability to correct other errors **;binomial codes are related to existing two-mode bosonic codes but offer the advantage of requiring only a single bosonic mode**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>The binomial codes are related to existing two-mode bosonic codes but offer the advantage of requiring only a single bosonic mode to correct amplitude damping as well as (=) the ability to correct other errors | |
| $G_{44}$ : Category one of binomial codes are related to existing two-mode bosonic codes but offer the advantage of requiring only a single bosonic mode to correct amplitude damping<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of ability to correct other errors<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

| | |
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| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$: $(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$ $(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$, $(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$ $(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$ $(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$ $(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$ are Accentuation coefficients $(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$ , $(a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$ $(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ , $(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$ $(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$, $(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$, $(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$, are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a''_{13})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b''_{13})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a''_{16})^{(2)}(T_{17}, t) =$ First augmentation factor | |

| | |
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| $-(b_{16}^{''})^{(2)}((G_{19}),t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |

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| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b'_{46})^{(9)} - (b''_{46})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a''_{44})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b''_{44})^{(9)}((G_{47}),t) =$ **First detrition factor** | |

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15} \quad (60)$$

Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) \quad +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[\begin{array}{ccc} (a_{17}^{'})^{(2)} & +(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) \quad +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[\begin{array}{ccc} (a_{18}^{'})^{(2)} & +(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) \quad +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{array}\right] G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation

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| coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}-(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}-(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) \end{bmatrix} T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3 $-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3 $-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3 $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3 $-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3 $-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3 $-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3 $-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |

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| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)}\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)}\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}\boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}\boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}\boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}\boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)}\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}\boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}\boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}\boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}\boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22}$ | 69 |
| $\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)}, \boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)}\boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)}\boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20}$ | 70 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)}\boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)}\boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)}\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)}\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21}$ | 71 |

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b'_{22})^{(3)} & -(b''_{22})^{(3)}(G_{23},t) & -(b''_{18})^{(2,2,2)}(G_{19},t) & -(b''_{15})^{(1,1,1,)}(G,t) \\ -(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t) & -(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t) & -(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{38})^{(7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9)}(G_{47},t) & \end{bmatrix} T_{22} \quad (72)$$

$-(b''_{20})^{(3)}(G_{23},t)$, $-(b''_{21})^{(3)}(G_{23},t)$, $-(b''_{22})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b''_{16})^{(2,2,2)}(G_{19},t)$, $-(b''_{17})^{(2,2,2)}(G_{19},t)$, $-(b''_{18})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b''_{13})^{(1,1,1,)}(G,t)$, $-(b''_{14})^{(1,1,1,)}(G,t)$, $-(b''_{15})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b''_{36})^{(7,7,7,7)}(G_{39},t)$, $-(b''_{37})^{(7,7,7,7)}(G_{39},t)$ $-(b''_{38})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b''_{40})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{41})^{(8,8,8,8)}(G_{43},t)$, $-(b''_{42})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b''_{46})^{(9,9,9)}(G_{47},t)$, $-(b''_{45})^{(9,9,9)}(G_{47},t)$, $-(b''_{44})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a'_{24})^{(4)} & +(a''_{24})^{(4)}(T_{25},t) & +(a''_{28})^{(5,5,)}(T_{29},t) & +(a''_{32})^{(6,6,)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1)}(T_{14},t) & +(a''_{16})^{(2,2,2,2)}(T_{17},t) & +(a''_{20})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{36})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{44})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} & +(a''_{25})^{(4)}(T_{25},t) & +(a''_{29})^{(5,5,)}(T_{29},t) & +(a''_{33})^{(6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{37})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} & +(a''_{26})^{(4)}(T_{25},t) & +(a''_{30})^{(5,5,)}(T_{29},t) & +(a''_{34})^{(6,6,)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3)}(T_{21},t) & \\ +(a''_{38})^{(7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{26} \quad (75)$$

$(a''_{24})^{(4)}(T_{25},t)$, $(a''_{25})^{(4)}(T_{25},t)$, $(a''_{26})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* $1, 2\ 3$

$+(a''_{28})^{(5,5,)}(T_{29},t)$, $+(a''_{29})^{(5,5,)}(T_{29},t)$, $+(a''_{30})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$+(a''_{32})^{(6,6,)}(T_{33},t)$, $+(a''_{33})^{(6,6,)}(T_{33},t)$, $+(a''_{34})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$
*are fourth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$
*are fifth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$
*are sixth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$
*are seventh augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$
*are eighth augmentation coefficients for category* $1,2$ *and* $3$
$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}\boxed{-(b_{25}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}\boxed{-(b_{26}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{26} \quad (78)$$

*Where* $\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$
*are fourth detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$
*are fifth detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$
*are sixth detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$
*are seventh detrition coefficients for category* $1,2$ *and* $3$
$\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$
*are eighth detrition coefficients for category* $1,2$ *and* $3$

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| $-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{28}$ | 79 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{29}$ | 80 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,)}(T_{25},t) & +(a_{34}^{''})^{(6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{30}$ | 81 |
| *Where* $+(a_{28}^{''})^{(5)}(T_{29},t)$, $+(a_{29}^{''})^{(5)}(T_{29},t)$, $+(a_{30}^{''})^{(5)}(T_{29},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $+(a_{24}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,)}(T_{25},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3<br>$+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3<br>$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3<br>$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)}-(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)}-(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{30} \quad (84)$$

where $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{32})^{(6)}\boxed{+(a''_{32})^{(6)}(T_{33},t)} & \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[\begin{array}{ccc} (a'_{33})^{(6)}\boxed{+(a''_{33})^{(6)}(T_{33},t)} & \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[\begin{array}{ccc} (a'_{34})^{(6)}\boxed{+(a''_{34})^{(6)}(T_{33},t)} & \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{34} \quad (87)$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6)}(T_{33},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)}$ are third augmentation

| | |
|---|---|
| *coefficients for category* 1, 2 *and* 3<br>$+(a_{13}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}'')^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}'')^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}'')^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}')^{(6)} & -(b_{32}'')^{(6)}(G_{35},t) & -(b_{28}'')^{(5,5,5)}(G_{31},t) & -(b_{24}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{13}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}')^{(6)} & -(b_{33}'')^{(6)}(G_{35},t) & -(b_{29}'')^{(5,5,5)}(G_{31},t) & -(b_{25}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{14}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}')^{(6)} & -(b_{34}'')^{(6)}(G_{35},t) & -(b_{30}'')^{(5,5,5)}(G_{31},t) & -(b_{26}'')^{(4,4,4,)}(G_{27},t) \\ & -(b_{15}'')^{(1,1,1,1,1,1)}(G,t) & -(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}'')^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}'')^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}'')^{(6)}(G_{35},t)$, $-(b_{33}'')^{(6)}(G_{35},t)$, $-(b_{34}'')^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{28}'')^{(5,5,5)}(G_{31},t)$, $-(b_{29}'')^{(5,5,5)}(G_{31},t)$, $-(b_{30}'')^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{25}'')^{(4,4,4,)}(G_{27},t)$, $-(b_{26}'')^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}'')^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}'')^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3<br>$-(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3<br>$-(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3<br>$-(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3 | |

$-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)$
are eighth detrition coefficients for category 1, 2, and 3
$-(b_{46}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}')^{(7)}+(a_{36}'')^{(7)}(T_{37},t) & +(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}')^{(7)}+(a_{37}'')^{(7)}(T_{37},t) & +(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}')^{(7)}+(a_{38}'')^{(7)}(T_{37},t) & +(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}'')^{(7)}(T_{37},t)$, $(a_{37}'')^{(7)}(T_{37},t)$, $(a_{38}'')^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3
$+(a_{16}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}'')^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3
$+(a_{20}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}'')^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}'')^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}'')^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}'')^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3
$+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}'')^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3
$+(a_{42}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}'')^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}'')^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$
are eighth augmentation coefficient for 1,2,3
$+(a_{46}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}'')^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = \quad (94)$$

$$(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} =$$

$$(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} =$$

$$(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{array}{ccc} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{array}{ccc} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[\begin{array}{ccc} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{array}\right] T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

| | |
|---|---|
| $\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and$(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br><br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},(r_i)^{(2)}}$ are positive constants and $\boxed{i=16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |

| | |
|---|---|
| $\|(a_i^{''})^{(2)}(T_{17}^{'},t)-(a_i^{''})^{(2)}(T_{17},t)\| \leq (\hat{k}_{16})^{(2)}\|T_{17}-T_{17}^{'}\|e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\|(b_i^{''})^{(2)}((G_{19})^{'},t)-(b_i^{''})^{(2)}((G_{19}),t)\| < (\hat{k}_{16})^{(2)}\|\|(G_{19})-(G_{19})^{'}\|\|e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and$(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},$ are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i^{'})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i^{'})^{(2)}+(\hat{A}_{16})^{(2)}+(\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)}+(b_i^{'})^{(2)}+(\hat{B}_{16})^{(2)}+(\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i^{'})^{(3)},(a_i^{''})^{(3)},(b_i)^{(3)},(b_i^{'})^{(3)},(b_i^{''})^{(3)}>0, \quad i,j=20,21,22$<br>The functions $(a_i^{''})^{(3)},(b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21},t)\leq(p_i)^{(3)}\leq(\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23},t)\leq\ (r_i)^{(3)}\leq(b_i^{'})^{(3)}\leq(\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t)=(p_i)^{(3)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t)=\ (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)},(p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i=20,21,22}$ | 113 |

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| They satisfy Lipschitz condition:<br><br>$\|(a_i'')^{(3)}(T_{21}', t) - (a_i'')^{(3)}(T_{21}, t)\| \leq (\hat{k}_{20})^{(3)}\|T_{21} - T_{21}'\|e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\|(b_i'')^{(3)}(G_{23}', t) - (b_i'')^{(3)}(G_{23}, t)\| < (\hat{k}_{20})^{(3)}\|\|G_{23} - G_{23}'\|\|e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}', t)$ and$(a_i'')^{(3)}(T_{21}, t)$ . $(T_{21}', t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}$, $(r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i'')^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i'')^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition: | 119 |

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|---|---|
| $\|(a_i^{''})^{(4)}(T_{25}^{'},t)-(a_i^{''})^{(4)}(T_{25},t)\| \leq (\hat{k}_{24})^{(4)}\|T_{25}-T_{25}^{'}\|e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})^{'},t)-(b_i^{''})^{(4)}\big((G_{27}),t\big)\| < (\hat{k}_{24})^{(4)}\|\|(G_{27})-(G_{27})^{'}\|\|e^{-(\hat{M}_{24})^{(4)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)},(\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)}=1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}<1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)},(\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)},(\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)},(a_i^{'})^{(4)},(b_i)^{(4)},(b_i^{'})^{(4)},(p_i)^{(4)},\ (r_i)^{(4)}, i=24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)}+(a_i^{'})^{(4)}+(\hat{A}_{24})^{(4)}+(\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)}+(b_i^{'})^{(4)}+(\hat{B}_{24})^{(4)}+(\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)},(a_i^{'})^{(5)},(a_i^{''})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(b_i^{''})^{(5)}>0, \quad i,j=28,29,30$<br><br>The functions $(a_i^{''})^{(5)},(b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)},\ (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t)\leq(p_i)^{(5)}\leq(\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big)\leq\ (r_i)^{(5)}\leq(b_i^{'})^{(5)}\leq(\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \leq (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

**Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i')^{(6)} \leq (\hat{B}_{32})^{(6)}$$

$lim_{T_2 \to \infty}(a_i'')^{(6)}(T_{33}, t) = (p_i)^{(6)}$ 128

$\lim_{G \to \infty}(b_i'')^{(6)}((G_{35}), t) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

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|---|---|
| Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32{,}33{,}34}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}\big((G_{35}), t\big)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32{,}33{,}34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (KK) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36{,}37{,}38$<br><br>(LL) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(MM)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(NN)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = \ (r_i)^{(7)}$ | 132 |

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(OO) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(PP) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$$

$$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$$

Where we suppose

$(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i, j = 40,41,42$ 136

The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded

**Definition of** $(p_i)^{(8)}, \ (r_i)^{(8)}$: 137

$(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ 138

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| $(b_i^{''})^{(8)}((G_{43}),t) \leq \ (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'},t) - (a_i^{''})^{(8)}(T_{41},t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\widehat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'},t) - (b_i^{''})^{(8)}\left((G_{43}),t\right)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\widehat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\widehat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\widehat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i^{'})^{(8)} + \ (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\widehat{M}_{40})^{(8)}}[\,(b_i)^{(8)} + (b_i^{'})^{(8)} + \ (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i^{'})^{(9)}, (a_i^{''})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (b_i^{''})^{(9)} > 0, \quad i,j = 44,45,46$ | 146 A |

The functions $(a_i^{''})^{(9)}, (b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i^{'})^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i^{''})^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i^{''})^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)} |T_{45} - T_{45}^{'}| e^{-(\hat{M}_{44})^{(9)} t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}((G_{47}), t)| < (\hat{k}_{44})^{(9)} ||(G_{47}) - (G_{47})^{'}|| e^{-(\hat{M}_{44})^{(9)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

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| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 |

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| $G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| Where $s_{(13)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |

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| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a_{24}')^{(4)} + a_{24}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a_{25}')^{(4)} + (a_{25}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
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| $\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{32})^{(6)}$, $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}')^{(6)} + (a_{34}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}')^{(6)} - (b_{32}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}')^{(6)} - (b_{33}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b'_{37})^{(7)} - (b''_{37})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{38})^{(7)} - (b''_{38})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |

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| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166<br>A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that | 168 |

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| $(G_{13}(t)-G_{13}^0)e^{-(\hat{M}_{13})^{(1)}t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{P}_{13})^{(1)}+G_{14}^0)e^{\left(-\frac{(\hat{P}_{13})^{(1)}+G_{14}^0}{G_{14}^0}\right)}+(\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0+(\hat{P}_{16})^{(6)}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1+(a_{16})^{(2)}t\right)G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)}t}-1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t)-G_{16}^0)e^{-(\hat{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | |

that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 (175)

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (176)

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that

$$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6 (177)

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(g) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that (178)

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)}\left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$$ (180)

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| $\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$ | |
| From which it follows that<br>$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$<br>$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left((G^{(1)},T^{(1)}),(G^{(2)},T^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right|\leq\int_{0}^{t}(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_{0}^{t}\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

(186)

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_{+}$.

If instead of proving the existence of the solution on $\mathbb{R}_{+}$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)},i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

From 19 to 24 it results

$$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\geq 0$$

$$T_i\,(t)\geq T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$:

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13}<(\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt}\leq\left((\widehat{M}_{13})^{(1)}\right)_1-(a_{14}^{'})^{(1)}G_{14}$ and by integrating

$$G_{14}\leq\left((\widehat{M}_{13})^{(1)}\right)_2=G_{14}^0+2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$$

(187)

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| In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\widehat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br><br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br><br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t) = 0$ and $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for $t > 0$ | 200 |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1$, $((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)}((\widehat{M}_{16})^{(2)})_1 / (a_{17}^{'})^{(2)}$ | 201 |

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|---|---|
| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |

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| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} \} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t} \; and \; (\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23}) (and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0 \; and \; T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0 \;$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2 \; and \; \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21} \; and \; G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left((\widehat{M}_{20})^{(3)}\right)_1 / (a_{21}^{'})^{(3)}$ | 217 |

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|---|---|
| In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}} < 1$ and to choose<br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}\}$ | 225 |

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})}, \widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\, ds_{(24)} +$$

$$\int_0^t \{(a_{24}^{'})^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} \left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)}, s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

(226)

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

(227)

**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)} t\right)} > 0 \quad \text{for t} > 0$$

(228)

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}^{'})^{(4)}$$

(229)

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| In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$ | 236 |

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{28})^{(5)}\left|G_{29}^{(1)}-G_{29}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)}-G_{28}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left|G_{28}^{(1)}-G_{28}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on it follows

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$$\left|(G_{31})^{(1)}-(G_{31})^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}t}\le$$
$$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

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**Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

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**Remark 22:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

it results

$$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$$

$$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(5)}t\right)}>0\quad \text{for t} > 0$$

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**Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1,\left((\widehat{M}_{28})^{(5)}\right)_2\ and\ \left((\widehat{M}_{28})^{(5)}\right)_3$ :

**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if

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| $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a'_{29})^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a'_{30})^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b''_{30})^{(5)}\left((G_{31})(t),t\right) = (b'_{30})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |

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| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)}+$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\le$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$<br>it results<br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$<br>$T_i\,(t)\ge T_i^0e^{\left(-(b_i^{'})^{(6)}t\right)}>0$ for $\mathrm{t}>0$ | 250 |

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a'_{33})^{(6)} G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)} \quad (257)$$

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| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\le$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\ge 0$ | 261 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(7)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{36})^{(7)})_1, ((\widehat{M}_{36})^{(7)})_2$ *and* $((\widehat{M}_{36})^{(7)})_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq ((\widehat{M}_{36})^{(7)})_1 - (a_{37}')^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq ((\widehat{M}_{36})^{(7)})_2 = G_{37}^0 + 2(a_{37})^{(7)}((\widehat{M}_{36})^{(7)})_1/(a_{37}')^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq ((\widehat{M}_{36})^{(7)})_3 = G_{38}^0 + 2(a_{38})^{(7)}((\widehat{M}_{36})^{(7)})_2/(a_{38}')^{(7)}$<br><br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}((G_{39})(t),t) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + ((\widehat{P}_{40})^{(8)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
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| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$ $\int_0^t\{(a_{40}')^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $G_{40}^{(2)}\left\|(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}'')^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}')^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}'')^{(8)}$ and $(b_{40}'')^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}\ and\ (\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(8)}$ and $(b_i'')^{(8)}, i=40{,}41{,}42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$ | 275 |

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| it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}')^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(8)}((G_{43})(t), t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty} (b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)}+\left((\hat{P}_{44})^{(9)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{44})^{(9)}\right]\leq(\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),\left((G_{47})^{(2)},(T_{47})^{(2)}\right)\right)=$$
$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\;e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\;(G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}')^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}')^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}')^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(9)} ((G_{47})(t), t)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}((G_{47})(t), t) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty} (b_{46}'')^{(9)} ((G_{47})(t), t) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14}, t) + (a_{14}'')^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

| | |
|---|---|
| $-(\tau_2)^{(1)} \le -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \le -(\tau_1)^{(1)}$ | |
| **Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ :<br><br>By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 281 |
| **Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}^{'})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}^{'})^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}^{'})^{(1)}t}\right)$ | 286 |

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|---|---|
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(\mathrm{T}_{17}, t) + (a_{17}'')^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}\big((G_{19}), t\big) - (b_{17}'')^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |

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| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)} t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)} t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)} t}\right] + G_{18}^0 e^{-(S_2)^{(2)} t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)} t} - e^{-(a_{18}')^{(2)} t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)} t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)} t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)} t} - e^{-(b_{18}')^{(2)} t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)} t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)} t}\right] + T_{18}^0 e^{-(R_2)^{(2)} t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ | 317 |

| | |
|---|---|
| $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ :<br>By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and<br>By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | 320 |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,<br>and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$ | 322 |

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| $(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},$ ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br><br>Then the solution of global equations satisfies the inequalities<br><br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$<br><br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$<br><br>$\left. \frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$<br><br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br><br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$<br><br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br><br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$<br><br>$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br><br>**Theorem:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br><br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br><br>$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br><br>$-(\tau_2)^{(4)} \le -(b_{24}')^{(4)} + (b_{25}')^{(4)} - (b_{24}'')^{(4)}((G_{27}), t) - (b_{25}'')^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0, (v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations | 329 |

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| $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$<br>$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$ | 330 |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}^{'})^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}^{'})^{(4)}t} \le T_{26}(t) \le$ | 336 |

$\frac{(a_{26})^{(4)}T_{24}^{0}}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^{0}e^{-(R_2)^{(4)}t}$

**Definition of** $(S_1)^{(4)},(S_2)^{(4)},(R_1)^{(4)},(R_2)^{(4)}$:- 337

Where $(S_1)^{(4)}=(a_{24})^{(4)}(m_2)^{(4)}-(a_{24}^{'})^{(4)}$

$(S_2)^{(4)}=(a_{26})^{(4)}-(p_{26})^{(4)}$

$(R_1)^{(4)}=(b_{24})^{(4)}(\mu_2)^{(4)}-(b_{24}^{'})^{(4)}$

$(R_2)^{(4)}=(b_{26}^{'})^{(4)}-(r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)}\le-(a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}-(a_{28}^{''})^{(5)}(T_{29},t)+(a_{29}^{''})^{(5)}(T_{29},t)\le-(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)}\le-(b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31}),t)-(b_{29}^{''})^{(5)}((G_{31}),t)\le-(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)},(v_2)^{(5)},(u_1)^{(5)},(u_2)^{(5)},v^{(5)},u^{(5)}$ : 339

By $(v_1)^{(5)}>0$ , $(v_2)^{(5)}<0$ and respectively $(u_1)^{(5)}>0$ , $(u_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and

**Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$

**Definition of** $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-

If we define $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)}$ by

$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)}=\frac{G_{28}^{0}}{G_{29}^{0}}}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$

and analogously 341

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| $(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},\ \boldsymbol{if}\ (u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}, \boldsymbol{if}\ (u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)},$<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le\right.$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying | 349 |

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| $-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br><br>$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(v_1)^{(6)}, (v_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, v^{(6)}, u^{(6)}$ :<br><br>By $(v_1)^{(6)} > 0$ , $(v_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_1)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{v}_1)^{(6)},, (\bar{v}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{v}_1)^{(6)} > 0$ , $(\bar{v}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\left(v^{(6)}\right)^2 + (\sigma_2)^{(6)}v^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}\left(u^{(6)}\right)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (v_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (v_0)^{(6)}, (m_1)^{(6)} = (v_1)^{(6)}$, ***if*** $(v_0)^{(6)} < (v_1)^{(6)}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (\bar{v}_6)^{(6)}$ , ***if*** $(v_1)^{(6)} < (v_0)^{(6)} < (\bar{v}_1)^{(6)}$,<br>and $\boxed{(v_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (v_1)^{(6)}, (m_1)^{(6)} = (v_0)^{(6)}$, ***if*** $(\bar{v}_1)^{(6)} < (v_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39}),t)-(b_{37}^{''})^{(7)}((G_{39}),t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

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|---|---|
| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{36}(t) \leq G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{37}(t) \leq \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \leq G_{38}(t) \leq$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \leq T_{38}(t) \leq$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)} (m_2)^{(7)} - (a_{36}')^{(7)}$ | 370 |

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}((G_{43}), t) - (b''_{41})^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$

**Definition of** $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : 372

By $(\nu_1)^{(8)} > 0, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(8)}, , (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{\nu}_1)^{(8)} > 0, (\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}(u^{(8)})^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$, ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,
and $\boxed{(\nu_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$

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| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,
and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,
and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G^0_{44} e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G^0_{44} e^{(S_1)^{(9)} t}$$

| | |
|---|---|
| ( $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t}\le G_{46}(t)\le$ $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a'_{46})^{(9)}t}]+G_{46}^{0}e^{-(a'_{46})^{(9)}t})$ | |
| $\boxed{T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$ | |
| $\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$ | |
| $\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b'_{46})^{(9)}t}\right]+T_{46}^{0}e^{-(b'_{46})^{(9)}t}\le T_{46}(t)\le$ $\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$ | |
| **Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:- Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a'_{44})^{(9)}$ $(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$ $(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b'_{44})^{(9)}$ $(R_2)^{(9)}=(b'_{46})^{(9)}-(r_{46})^{(9)}$ | |

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$ , we obtain

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

383 384 385 386

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain (387)

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ (388)

It follows (389)

$$-\left((a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains (390)

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

$$\text{For } 0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get (391)

| | |
|---|---|
| $\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t)=\frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain | 398 |

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \le \frac{dv^{(3)}}{dt} \le -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \ge \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \le v^{(3)}(t) \le (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \le v^{(3)}(t) \le (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \le \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \le v^{(3)}(t) \le$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \le (\bar{v}_1)^{(3)} \le (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \le v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$, we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} =$

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$(\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

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$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq v^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $v^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)v^{(7)} - (a_{37})^{(7)} v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)} v^{(7)} - (a_{36})^{(7)}\right) \leq \frac{dv^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$v^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq v^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain 419

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

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**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

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In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

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If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

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If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

| | |
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| $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |

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| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ | 436 A |

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| $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |

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| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}))\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}))\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}))\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |

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| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(g) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) +$ | 486 |

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| $(a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |

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| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{13}=\frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}=\frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | 494 |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations | 499 |

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| $G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 504 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if | 510 |

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| $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |

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| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 523 |

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| $T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br>**Proof:**_Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions | 531 |

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| $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |

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| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |

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| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |

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| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ $and$ $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \quad , \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\Big)\right]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\Big)\right]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\Big)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\Big)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\Big)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al**

**Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal**

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(7) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and

review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 7. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \ i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \cap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \cap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K $\subset$ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta_0 > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B*.

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} \boldsymbol{C}_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$ . If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $\mathrm{A_i}(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A_i}(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A_i}(x)\ and\ \mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N_\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots,\tilde{A}_k)$ by

$$\tilde{A}_i(x) = \{\frac{A(x)}{A(x)-\frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $\dot{x} = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1,\ldots,k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $\dot{x} = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x$,$W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $\dot{x} = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1^j))^m_{j=1}$. Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.19) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.20) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.21) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0,' \quad (1.4) \\ \rho_1(0, t) = \int_0^{\infty} b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^{\infty} b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s), a.0, s\epsilon [-\tau, 0], \quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^{\infty} \varphi_i(a, s)\, da, s\in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 7. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +} \times [-\tau, T] \to R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +} \times \{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +} \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t,0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right)ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t) = B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$

$$+\int_0^{\infty} b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, \; t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \; \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^{\infty} \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \; t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2\ ),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in \mathrm{C}([0,T], R^+), \quad B_i \in \mathrm{C}([0,T], R^+)\ , i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in \mathrm{C}([-\tau, T], R\ ) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $\mathrm{C}^+[-\tau, T\ ]$. Then for any

$(x, y) \in \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$\mathrm{C}_T = \mathrm{C}^+[-\tau, T\ ], \quad \mathrm{C}_T^2 = \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ].$$

Members of $\mathrm{C}_T$ will be denoted by $Z_t$ with the meaning that

$Z\ (t+s) = z_t(s)$ *for* $s \in [-\tau, T\ ]$ *and* $t \in [0\ , T\ ]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in \mathrm{C}_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2\ )$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$\mathrm{B}\ (\mathrm{t}) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then ▯ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , ▯ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|▯(P_t^{(1)}) - ▯(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|▯(P^{(1)}(t)) - ▯(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 \| p_t^{(1)} - p_t^{(2)} \|_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left( \int_o^{t-a} d_1 \| P_t^{(1)} - P_t^{(2)} \|_{t^*} ds \right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 \| P_t^{(1)} - P_t^{(2)} \| e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a - 0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$

$$\leq \int_o^a \left| B_1^{(1)}(\sigma)-B_1^{(2)}(\sigma) \right| B_1^{*d}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*} \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a}\right\}$$

$$\leq \beta_1^{*} \int_o^a \left| B_1^{(1)}(\sigma)- B_1^{(2)}(\sigma) \right| do+\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a) \to \beta_1$ as $a \to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)} \|_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)} \|_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)} \|_{t*} o(t^{*})$$

As $t^{*} \to o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\ \ 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

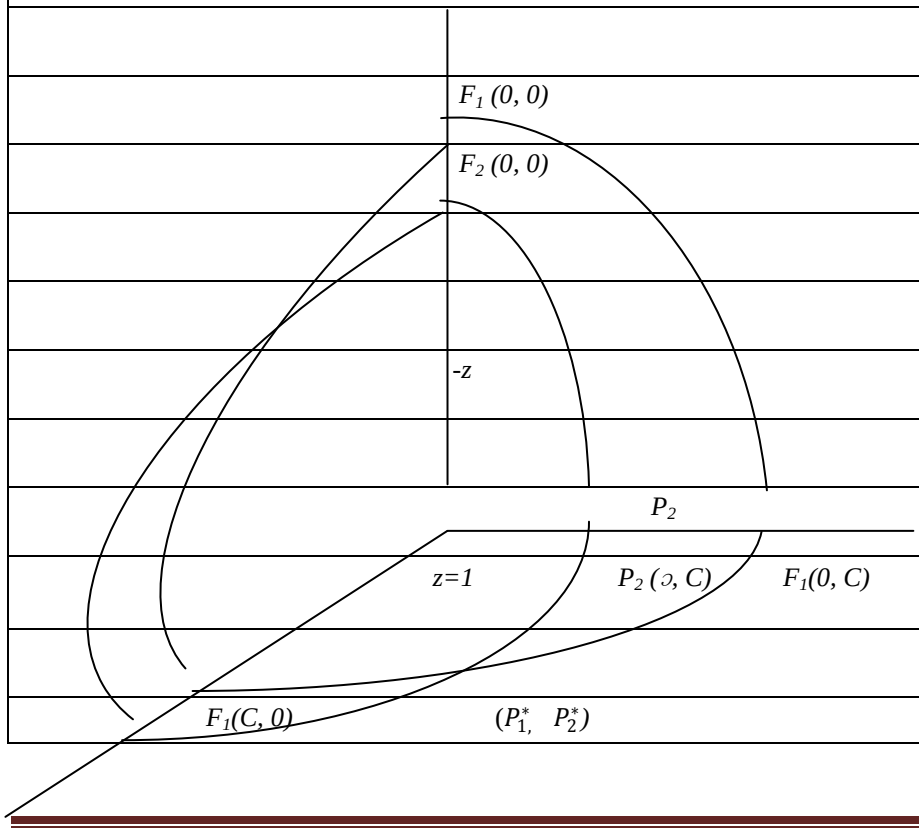
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11)\quad \begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

$$(3.12)\quad \begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)

$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

$$(3.14)\quad \begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{q}_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) q_i(a)da + \int_0^\infty p_i^*(a) \left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_i^*\right\} da ,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \; p_{22}^* = p_2^* .$$

If we integrate the first of (3.15) and use the fact that $q_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right] da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right] q_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right] da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right] q_2(a)da,$$

From the first of (3.15) we get

(3.18) $$q_i(a) = \left\{q_i(0)p_i^*(a)/p_i^*(0) \exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} e^{\lambda s} ds , \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$q_i(a)\left[p_i^*(a)/p_i^*(0)c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a) \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^*\right\} e^{\lambda s} ds , i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 13. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \left\{ \text{an individual's contribution to population growth} \right\} N(t) \qquad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\left\{ \frac{dN(t)}{dt} = N(t) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1,$ 2), (2.2)

(xix) $b_1(0) = 0,$ $m_1(0, x_2) \equiv 0, b_2(0) = 0,$ $m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xx) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2 \quad (\delta_2) - m_2(x_1, \delta_2) < 0,$ for $x_1$ 0, $x_2$ 0,

(2.5)

(xxi) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1$, $X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i,\; j = 1, 2; i \neq j)$.

## 14. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ ,, $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\ (t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\ (t_2)\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\ (t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\ (t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\ (t - \tau_{11}) - m_{12}Y_2\ (t - \tau_{12}),$$

$$\frac{dY_2\ (t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\ (t - \tau_{21}) - m_{22}Y_2\ (t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \qquad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(xxv)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \qquad g_2(0) = 0, \tag{4.3}$$

(xxvi) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xxvii) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(xxviii) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{vmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{vmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 11. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \lambda+m_{11} & 0 \\ 0 & \lambda+22 \end{bmatrix} - \begin{bmatrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{bmatrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \mu(A)$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} \cdot \min \begin{pmatrix} m_{11} \\ m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{11}\left|Y_1(t)\right| + b_{11}\left|Y_1(t-\tau_{11})\right| + m_{12}\left|Y_2(t-\tau_{12})\right|, \\ \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{22}\left|Y_2(t)\right| + m_{21}\left|Y_1(t-\tau_{21})\right| + b_{22}\left|Y_2(t-\tau_{21})\right|, \end{aligned}\right\} \quad (A.2)$$

Where

$$\frac{d}{dt}\left|Y_1(t)\right| = \left\{\begin{aligned} &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0, \\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0, \end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(s) > \left|Y_1(s)\right|, \quad s \in [-\tau, 0], \; i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(t) \text{ for } t \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(t) = Z_i(t) - \left|Y_1(s)\right|$, $i = 1,2$. $t \ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\ge -m_{11}U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\ge -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(s) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \ge U_1(0)\exp[-m_{11}t]$

$$+\int_0^t \left[b_{11}U_1(s-\tau_{11}) + m_{12}U_2(s-\tau_{12})\right]\exp[-m_{11}(t-s)ds > 0,] \qquad (A.9)$$

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11},\tau_{12},\tau_{21},\tau_{22},)$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds + \int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds + \int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0 \leq |\, Y_{1,}(t)\,| \leq Z_1(t) \to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22} m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t) \to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t) \to 0, Z_2(t) \to 0$ as $t \to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s \in [0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty s k_{ij}(s)ds < \infty;\ \int_0^\infty k_{ij}(s)ds = 1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

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| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

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| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

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| **Gopalsamy** |
| **Model For Innocuously Benign Delay In A Periodic System** |
| **Abstract** |
| Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of |

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 19. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 20. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big), \qquad t > 0, \tag{2.1}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ in $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ in $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ Ω and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ Ω then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap\left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ $\in -\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega + t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(xix) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(xx) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t)or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(xxi) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq 2\ max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(m)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(n) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 21. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(s) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(t) N is monotonic nondecreasing on $K^*$.

(u) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \geq \|x_1\| \geq b_1^u/a_1^u,\ b_{2l}/a_{2l} \geq \|x_2\| \geq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \geq -\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \geq -\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_1, X_2)$ where

$$v(t) = V(t, X_1, X_2) = |X_1(t)| + |X_2(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_2(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_1(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21}) x^*_1(t)|X_1(t)| + (|b_{22}| - b_{22}) x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_1, X_2) \geq |X_1(t)| + X_2(t)|$ and $dv/dt \leq 0$, which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_1(t)| + |X_2(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_1(s), X_2(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21}) x^*_1(t_m)|X_1(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} -\ _{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(\mathrm{t})\right)\Big| = \min_{t}, \beta/K_2(\mathrm{t})$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(\mathrm{t})\right)\Big| = \min_{t} \alpha/K_1(\mathrm{t})$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha,\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}$=0 or $T_{12}$=0 or both $T_{12}$=0 and $T_{21}$=0: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that “In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$\frac{dx_1(\mathrm{t})}{dt} x_1(\mathrm{t})[1 - x_1(\mathrm{t}) + 0.4x_2(\mathrm{t} - \mathcal{T})/k_1(\mathrm{t})]$

$\frac{dx_2(\mathrm{t})}{dt} x_2(\mathrm{t})[1 - \{0.8\, x_1(\mathrm{t} - \mathcal{T}) + x_2(\mathrm{t})\}/k_2(\mathrm{t})]$

Where $k_1(\mathrm{t}) = 20 + \sin\ (\pi\mathrm{t})$ and $k_2(\mathrm{t})$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 13. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \ldots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.7) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \quad b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right)\Bigg)(1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \quad x_i^l \quad for\ t \quad t_0,\ i = 1, 2, \ldots, n. \qquad (1.7)$$

14. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* $\boldsymbol{R}$ *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \quad x_l \quad x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1 \quad n} x_l^l; \qquad x_*$$

$$= \min_{1 \quad n} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(133) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(134) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(135) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(136) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(137) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(138) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(139) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(140) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(141) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(142) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(143) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(144) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(145) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (146) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (147) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (148) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (149) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (150) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (151) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (152) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (153) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (154) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like

Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left||f\ (t+\ \tau,x) - \ f(t,x)|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^{*}\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{ij}\;(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^{*})/(2x^{*}). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k),x(t+\tau_m)\big)\ \ v\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_{*}(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_{*})/(2x^{*})]/\mu x_{*})$$

$$[(2x_n^{*})/x_{*}]\exp[-\mu x_{*}(t+t_k-t_0)]\ +(\varepsilon/2x^{*})$$

$$<(\varepsilon/x^{*})\ \text{On using (2.8), (2.10)} \qquad (2.14)$$

For $m\ \ k\ \ n_0(\varepsilon,\beta)$ and for all $t\in\cup\subset[\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n}|logx_i\,(t+\tau_k)-logx_l(t+\tau_m)|<(\varepsilon/x^{*}) \qquad (2.15)$$

And hence using $x_{*}\ \ x_i(t)\ \ x^{*}, i=1,2,\dots,n; t\ \ t_0$,

$$\sum_{i=1}^{n}|x_i\,(t+\tau_k)-x_i(t+\tau_m)|<\varepsilon \text{ for all } t\in\cup\subset[\beta,\infty], m\ \ k\ \ n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t)=\{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t)\ \in[x_{*},x^{*}], i=1,2,\dots,n; t\in[\beta,\infty]$ follows from (2.16) and the fact that $x_i(t)\ \in[x_{*},x^{*}], i=1,2,\dots,n;$ for $t\ \ t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k)\to w_l(t)$ as $k\to\infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i=1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ & $k$ $\geq n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ & $k$ $\geq n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\,(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k))$$

$$-f_l(t+\tau_k,w_1(t),\dots,w_n(t))$$

$$+-f_l(t+\tau_k,w_1(t),\dots,w_n(t))]$$

$$=-f_l(t+\tau_k,w_1(t),\dots,w_n(t)); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=(w_1(t),\dots,w_n(t))$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=(w_1(t),\dots,w_n(t))$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$ e^{-\mu x * T_k} v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon / 2x^*) $$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{l};\quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{l};\quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad \beta_1 \leq y_i(t) \leq \beta_2, \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V(x(t)y(t)) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t)y(t))$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \rho_1(t, a_1) \rho_1^*(t, a_1),$$

$$0 Q_1(t) Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0)$$

$$= \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1(P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1(P_0)a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1} da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1} da_{,1}$$

And assume that

$$[k_3]_1 \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** $$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{(m_1 m_2)(m_1+m_2)}{r^3},$$

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

$$t = \frac{5}{256}\frac{c^5}{G^3}\frac{r^4}{(m_1 m_2)(m_1+m_2)}.$$

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu} \; [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^\infty \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^\infty u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_3 a_3-(f_{30})_3 a_3-\int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3,a_3)da_3\right)u_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0, \qquad (24)$$

$$\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^\infty Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^a(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^l)_3\int_0^\infty Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3} da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{,3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g^{'}{}_{*}]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_{*}]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{,3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2\,[a_3^2]_3<4\,[a_2]_3<4\,a_2 e$,

Re $\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\ ]_3 < 0$.

Let $\Delta=\max_{2<u<\infty}\left|h_{*}[h_{*}]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then$|n_3(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_3^2 <4\,a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2]$ ⊠4$[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^*\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

**correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1 \log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 55. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \ 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \ \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a) e^{-\int_0^a \mu_4(\tau)dt} e^{-ra}\, da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(t)dt}\, da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 56. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40_4}^{l}\left(\int_0^{\infty} Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a\mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\,da\right);$$

$$\int_0^{\infty}Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^{\infty}E_4(a_4)e^{-\gamma_4a_4}\,da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4\ [a_2]_4<1\ a_2 e$,

Re $\lambda_4[\lambda_4]_4 = \text{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0$.

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_4^2 <4\ a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\ x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae_4^{mu} \; [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 57. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ } 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ } \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ 0 $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t,a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 58. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0) = \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(3127) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(3128) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(3129) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(3130) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(3131) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(3132) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(3133) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(3134) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(3135) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(3136) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(3137) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(3138) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(3139) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(3140) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(3141) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(3142) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(3143) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(3144) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(3145) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

## Black holes

(3146) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(3147) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(3148) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(3149) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(3150) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(3151) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(3152) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(3153) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(3154) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have**

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| **experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (3155) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (3156) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

**String theory**

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| (3157) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (3158) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (3159) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (3160) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (3161) **However, when summed the entire series appears to be (=) divergent.** |
| (3162) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (3163) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (3164) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (3165) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (3166) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (3167) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (3168) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |

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| (3169) **The granular properties of space can be (=) explicitly computed.** |
| (3170) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (3171) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (3172) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (3173) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (3174) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (3175) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (3176) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (3177) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (3178) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (3179) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (3180) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (3181) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (3182) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (3183) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (3184) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (3185) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (3186) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (3187) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (3188) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (3189) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (3190) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (3191) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (3192) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (3193) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (3194) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (3195) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (3196) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (3197) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

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| (3198) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (3199) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (3200) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (3201) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (3202) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (3203) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (3204) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (3205) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (3206) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (3207) **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (3208) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (3209) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (3210) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (3211) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (3212) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—** |

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (3213) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (3214) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (3215) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (3216) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (3217) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (3218) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (3219) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (3220) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (3221) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (3222) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (3223) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (3224) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (3225) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (3226) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (3227) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (3228) **This is a realistic scenario for (e) all known systems.** |
| (3229) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (3230) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (3231) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (3232) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (3233) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (3234) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (3235) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (3236) **So in those systems, the quantum phase transition does indeed exist.** |
| (3237) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (3238) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (3239) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (3240) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (3241) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (3242) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (3243) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (3244) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (3245) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (3246) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (3247) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (3248) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (3249) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (3250) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (3251) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (3252) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (3253) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (3254) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (3255) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (3256) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (3257) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (3258) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (3259) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (3260) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (3261) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (3262) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (3263) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (3264) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (3265) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (3266) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (3267) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (3268) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (3269) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (3270) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (3271) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (3272) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (3273) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (3274) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (3275) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (3276) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (3277) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (3278) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (3279) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (3280) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (3281) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (3282) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (3283) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (3284) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (3285) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (3286) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (3287) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (3288) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (3289) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (3290) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (3291) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (3292) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (3293) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (3294) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (3295) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (3296) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (3297) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (3298) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (3299) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (3300) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (3301) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (3302) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (3303) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (3304) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (3305) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (3306) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (3307) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (3308) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (3309) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (3310) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (3311) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (3312) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (3313) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (3314) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (3315) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (3316) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (3317) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (3318) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (3319) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (3320) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (3321) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (3322) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (3323) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (3324) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (3325) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (3326) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

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| (3327) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (3328) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (3329) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (3330) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (3331) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (3332) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (3333) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (3334) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (3335) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (3336) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (3337) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (3338) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (3339) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (3340) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |

(3341) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(3342) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(3343) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(3344) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(3345) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(3346) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(3347) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(3348) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(3349) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(3350) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(3351) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(3352) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(3353) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(3354) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(3355) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (3356) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (3357) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (3358) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (3359) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (3360) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (3361) **We know dark matter exists only from (e) its gravitational effects.** |
| (3362) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (3363) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (3364) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (3365) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (3366) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (3367) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (3368) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (3369) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (3370) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (3371) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (3372) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (3373) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (3374) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (3375) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (3376) **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (3377) **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (3378) **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (3379) **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (3380) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (3381) **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (3382) **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (3383) **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (3384) **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (3385) **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (3386) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (3387) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (3388) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (3389) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (3390) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (3391) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (3392) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (3393) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (3394) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (3395) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (3396) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (3397) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (3398) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (3399) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (3400) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (3401) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (3402) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (3403) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (3404) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (3405) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (3406) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (3407) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (3408) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (3409) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (3410) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (3411) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (3412) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (3413) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (3414) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (3415) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (3416) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (3417) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (3418) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (3419) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (3420) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (3421) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (3422) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (3423) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (3424) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (3425) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (3426) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (3427) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (3428) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (3429) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (3430) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (3431) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (3432) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (3433) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (3434) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (3435) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (3436) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (3437) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (3438) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (3439) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (3440) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(577) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(578) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(579) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(580) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(581) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(582) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(583) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(584) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(585) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(586) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(587) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(588) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(589) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(590) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(591) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(592) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(593) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

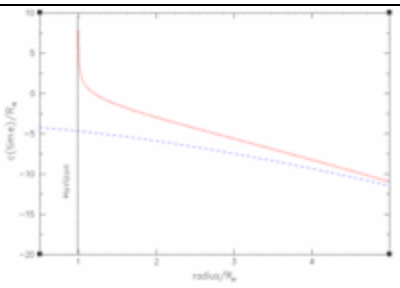
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| **(594) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(595) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(596) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(597) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(598) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(599) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(600) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(601) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(602) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(603) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(604) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(605) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(606) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(607) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(608) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(609) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(610) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(611) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(612) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(613) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(614) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(615) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(616) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(617) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(618) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(619) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(620) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(621) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(622) One of these coordinates is (=) called the "coordinate time."** |
| **(623) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(624) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(625) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(626) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(627) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(628) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(629) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(630) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(631) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(632) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(633) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(634) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(635) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(636) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(637) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(638) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(639) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(640) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(641) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(642) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(643) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(644) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(645) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(646) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(647) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(648) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(649) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(650) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(651) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(652) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(653) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(654) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(655) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(656) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(657) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(658) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(659) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(660) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(661) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(662) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(663) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(664) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(665) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(666) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(667) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(668) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(669) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(670) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(671) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(672) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6\left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*$, $h_6^*$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[f^{'}{}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[f_{2*}^{'}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right)M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$\mathrm{Re}\,\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\, P^*\, [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\, [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\, g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^2}+\frac{2\omega_6}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6}+(m_6+[\mu_0]_6+\frac{K_6b_6}{\delta_6-b_6}+\sigma_6)\,Q_6^*)^3$$

$$=m_6+[\mu_0]_6+3+\frac{K_6b_6}{\delta_6-b_6}(1+Q^*)(1+\omega_6 Q_6^*$$

Now

$$\left[a_6^2\right]-4\left[a_2\right]e=\left(\frac{2K_6b_6}{\delta_6-b_6}\right)^2+\frac{4K_6b_6\sigma_6\,Q_6^*}{(\delta_6-b_6)(1+Q_6^*)}+\frac{\sigma_6^2\,Q_6^{*2}}{(1+Q_6^*)^2}$$

$$\frac{4\,b_6\sigma_6\,(\delta_6-b)Q^*}{\delta_6(1+Q_6^*)^2}-\frac{4\,(\delta_6-b_6)^2(1-\omega_6)Q_6^*}{\delta_6(1+\omega_6Q_6^*)1+Q_6^*)}P*(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_6\,ae_6^{mu}\;[\mu_0]_6-\frac{a_6}{a_6+6}$$

Hence if $[a_6^2>4(a_2)_6e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6} = [f_{6*}]_{\,6} + [f_{2*}]_{\,6} + [f_{6*}] \, P_6^*$$

$$[a_2]_{\,6} = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [\, g_*^6]_{\,6} \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_{\,6} \beta_6 \left(\theta_6(t)\right) - \mu_6\left(\theta(t)\right),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right] M_6 - [a_2]_{\,6} W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 59. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \ P_7 \ 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \ \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(3441) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(3442) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(3443) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(3444) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(3445) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(3446) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(3447) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(3448) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(3449) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(3450) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(3451) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(3452) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(3453) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(3454) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(3455) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(3456) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(3457) The electronic and (e&eb) structural variability among them is extraordinary.

(3458) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(3459) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(3460) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(3461) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(3462) Nudge your bath tub just right and (eb) you might end up reversing time.

(3463) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(3464) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(3465) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(3466) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(3467) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(3468) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(3469) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(3470) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(3471) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(3472) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(3473) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(3474) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(3475) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(3476) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(3477) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(3478) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(3479) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(3480) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(3481) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(3482) Quantum mechanics rules the dynamics of (e) light and matter.

(3483) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(3484) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(3485) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(3486) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (3487) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (3488) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (3489) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (3490) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (3491) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (3492) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (3493) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (3494) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (3495) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (3496) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (3497) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (3498) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (3499) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (3500) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (3501) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (3502) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (3503) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (3504) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (3505) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (3506) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (3507) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (3508) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (3509) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (3510) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (3511) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (3512) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (3513) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (3514) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (3515) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (3516) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (3517) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (3518) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (3519) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (3520) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (3521) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (3522) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (3523) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (3524) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (3525) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (3526) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (3527) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (3528) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (3529) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |
| (3530) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big(f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\,\rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\,\rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 60. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7)e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds}d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_J) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7)da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7)m_7(t,a_7)\,da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0)W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7\left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}^{'}$ denoting $h_{7}^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7[a_7^7]2<4\,[a_7]2<4\,a_7e$,

Re $\lambda_7\left[\lambda_7\right]_7 = \mathrm{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\,]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^{*}(a), \boldsymbol{Q}^{*}$) is asymptotically stable if either $a_7^7 <4\,a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⊠4$[a_7]e$ and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e}) > 1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\ x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])_{\,7}^{\,3}\ [\mathrm{K}]_{\,7}[P_0]_{\,7}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)\ [P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^{*}\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\,7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^{\,7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\,7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,7}\right]e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7\, Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^{\,7} Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7\, (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\, (\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae^{mu}_{\,7}\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7\, Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (615)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (616) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (617)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (618) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

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| **Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).** |

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| (619)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (620) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (621)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (622)Humans are not very good at understanding interacting many-body systems. |
| (623)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (624)But what happens when we couple (e&eb) the particles strongly? |
| (625)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (626)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (627)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (628) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (629) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (630)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (631) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (632)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (633) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (634) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (635)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (636)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (637) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (638) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (639)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (640)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (641)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (642)The energy levels of these quantized orbitals take on (e) discrete values |
| (643):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (644) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (645) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (646)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (647) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (648)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (649) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (650)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (651)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (652)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (653) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (654)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (655)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (656)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (657) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (658)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (659)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (660)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (661)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (662)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (663)Another classic example of duality involves (e&eb) vortices. |
| (664) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (665)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (666) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (667)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (668)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (669) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (670)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (671)TIs are distinguished only by (e) their edge states. |
| (672) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (673)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (674) But the Dirac cone is not (e) the only possible signature. |
| (675) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (676)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (677)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (678)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (679) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (680). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (681)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (682)The bulk duality implies (eb) a duality between edge theories. |
| (683)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |

(684)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids.

(685) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles.

(686) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold.

(687)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables.

(688)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI.

(689)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors.

(690)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field.

(691)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions.

(692)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3.

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

(693) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $TH=\hbar\kappa/2\pi$.

(694)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration.

(695)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP).

(696) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole.

(697)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process.

(698)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version)

IF we define

$$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,\mathrm{P}^{*}_{7}$$

$$[a_7]_7 = \left[f^{7}_{7*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$

$$e=[g^{7}_{*}]_7 Q^{*}_{7}$$

$$\eta_7(t) = \left[h_{*}\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 61. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(3531) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(3532) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(3533) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(3534) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(3535) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

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| (3536) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (3537) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (3538) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (3539) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (3540) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (3541) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (3542) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (3543) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (3544) "The underlying physics is (=) chaos, essentially," he said. |
| (3545) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (3546) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (3547) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (3548) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (3549) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (3550) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (3551) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (3552) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (3553) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (3554) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |

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| (3555) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (3556) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (3557) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (3558) The study's findings have fundamental implications for (e) quantum computing. |
| (3559) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (3560) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (3561) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (3562) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (3563) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (3564) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer. |
| (3565) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (3566) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (3567) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (3568) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (3569) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (3570) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |

(3571) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(3572) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(3573) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(3574) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(3575) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(3576) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(3577) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(3578) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(3579) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(3580) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(3581) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(3582) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(3583) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as

contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(3584) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(3585) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(3586) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(3587) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(3588) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(3589) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(3590) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(3591) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(3592) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(3593) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(3594) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(3595) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(3596) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(3597) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(3598) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(3599) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(3600) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(3601) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(3602) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(3603) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(3604) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(3605) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(3606) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(3607) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background

radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(3608) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(3609) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(3610) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(3611) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(3612) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(3613) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(3614) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(3615) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(3616) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(3617) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(3618) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(3619) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(3620) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

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| (3621) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime. |
| (3622) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite. |
| (3623) However, when summed the entire series appears to be (=) divergent. |
| (3624) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime. |
| (3625) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects. |
| (3626) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing. |
| (3627) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory. |

**Loop quantum gravity and spin foams**

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| (3628) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics. |
| (3629) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity. |
| (3630) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime. |
| (3631) The granular properties of space can be (=) explicitly computed. |
| (3632) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory. |
| (3633) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks. |
| (3634) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure. |
| (3635) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit. |
| (3636) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism. |
| (3637) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism. |

**Noncommutative geometry**

(3638) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(3639) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(3640) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(3641) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(3642) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(3643) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(3644) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(3645) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(3646) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(3647) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \; P_8 \; \text{0}, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \; \text{∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \; \text{∞},$$

$$\rho_8(t, 0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly

eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8 \; h_8(Q_8) = 1, \qquad and \;\; g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 62. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))u_8 - (\rho_0)_8(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8^{'}(P_0) = f_{80}^{'}$, and $f_8^{'}(0) = f_{80}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \quad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a \mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\,(\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(a_8)da_8-\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^\infty E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)([f_8^{'}]_8(z_8))\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$-h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[\dot{g}\right]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - \left[f^{'}_{8*}\right]_8[p_8^*](a_8)M_8 - \left[\dot{f_{8*}}\right]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}_{8*}\right]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[\dot{f_{8*}}\right]_8 P_8^*\right)M_8 - \left[\dot{f_{8*}}\right]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if a $^8_8$ [a $^8_8$]2<4 [a $_8$]2<4 a $_8$e,

Re $\lambda_8\left[\lambda_8\right]_8=\mathrm{Re}\lambda_8\left[\lambda_8\right]_8=-\frac{8}{8}a_8[\mathrm{a2}\ ]_8<0.$

Let ⊿=max$_{8<u<\infty}\left|\mathrm{h}_*\,[\mathrm{h}_*]_8\beta_8(a_8)-\mu_8(a_8)\right|$. Then|$n_8$(t)| ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $^8_8$ <4 a $_8$ e and [a $_8$] > 1⊿,or [a $^8_8$] [a $_8$]e and (a $_8-\sqrt{(\mathrm{a}^8_8)_8-4(\mathrm{a}_8)_8\,\mathrm{e})}>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{\mathrm{a}}{\mathrm{a}+8}\right)$ and β $_8(a_8)$ = ae-$^{\mathrm{ma}}$ Then ⊿=[$\mu_0$] + 2 Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])_8^3\ [\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma Q_8^*}{8+Q_8^*}$ $[h_*] = \frac{8+\omega_8 Q_8^*}{8+Q_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+Q_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+P_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8-8}{(8+Q_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, ae_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\qquad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (429)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (430)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (431) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (432) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (433)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (434) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (435)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (436)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(437)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(438)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(439) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(440)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(441)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(442)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(443) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(444)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(445)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(446)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(447)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(448)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(449)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(450)Bringing them to thermal equilibrium requires (e) an external bath.

(451) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(452)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(453)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(454)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(455)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(456)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(457) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(458) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(459) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(460)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(461)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(462) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(463)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(464)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(465) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(466)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(467)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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|---|
| (468)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (469) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (470)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (471) This is a realistic scenario for (e) all known systems. |
| (472)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (473)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (474)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (475) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (476)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (477)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (478)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (479)So in those systems, the quantum phase transition does indeed exist. |
| (480)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (481)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 63. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \quad P_9 \; \square 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \square \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \square \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ∞; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9\,(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\ a_9)da_9 - \int_0^\infty E_9(\ a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\ a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2<4 [a_9] 2<4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} \left| h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9) \right|$. Then $|n_9(t)| \ 2\Delta \forall t \ 0$. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 <4 a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9] \ 4 [a_9] e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0] \ 0.5$. Assume $[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$ and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$ (See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}[\ \mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a_{\ 9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_{\ 9} - b)Q^*}{\delta_{\ 9}(1 + Q_9^*)^9} - \frac{4\,(\delta_{\ 9} - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_{\ 9}(1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\ mu}_{\ 9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a^{\ 9}_{\ 9} > 4(a_9)_{\ 9}e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION TWO F

## Bosonic Quantum Memories, Photonic Quantum Communication, And Optical-To-Microwave Up- And Down-Conversion

### INTRODUCTION—VARIABLES USED

**New class of quantum error-correcting codes for a bosonic mode Marios H. Michael, Matti Silveri, R. T. Brierley, Victor V. Albert, Juha Salmilehto, Liang Jiang, S. M. Girvin (Submitted on 29 Jan 2016 (v1), last revised 18 Jul 2016 (this version, v3))**

**(1)** codes are similar in spirit to (e) 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode.

**(2)** The binomial quantum codes are realizable with (e&eb) current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**. Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version

**Quasi-Spin-Wave Quantum Memories with a Dynamical Symmetry C. P. Sun, Y. Li, and X. F. Liu Phys. Rev. Lett 91, 147903 – Published 2 October 2003**

(3) For the two-mode exciton system formed by (e) the quasi-spin-wave collective excitation of many Λ atoms fixed at (eb) the lattice sites of a crystal, authors discover a dynamical symmetry depicted by (e) the semi direct product algebra SU (2)⊗‾‾h2 in (eb) the large N limit with low excitations.

(4) With the help of the spectral generating algebra method, authors obtain (eb) a larger class of exact zero-eigenvalue states adiabatically interpolating between the initial state of photon-type and (er&eb) the final state of quasi-spin-wave exciton-type.

(5) The conditions for the adiabatic passage of dark states are shown to be valid, even with (e&eb) the presence of the level degeneracy.

**(6)** These theoretical results can lead to (eb) the proposals of a new protocol for implementing quantum memory robust against (e&eb) quantum decoherence. Received 27 October 2002 DOI:http://dx.doi.org/10.1103/PhysRevLett.91.147903

### NOTATION

**Module One**

codes are similar in spirit to (e) 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code

| | |
|---|---|
| words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode | |
| $G_{13}$ : Category one of'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of codes are similar in spirit<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>codes are similar in spirit to 'cat codes' based on (e) superpositions of the coherent states, but offer (eb) several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode | |
| $G_{16}$ : Category one of codes are similar in spirit to 'cat codes' based on (e) superpositions of the coherent states<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>codes are similar in spirit to 'cat codes' based on superpositions of the coherent states, but offer several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for (e) repumping energy into (e&eb) the bosonic mode | |
| $G_{20}$ : Category one of repumping energy into (e&eb) the bosonic mode<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of codes are similar in spirit to 'cat codes' based on superpositions of the coherent states, but offer several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation<br><br>$T_{21}$ : Category two of SAS | |

| | |
|---|---|
| $T_{22}$ : Category three of SAS | |
| **Module four**<br><br>codes are similar in spirit to 'cat codes' based on superpositions of the coherent states, but offer several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for repumping energy into (e&eb) the bosonic mode | |
| $G_{24}$ : Category one of **codes are similar in spirit to 'cat codes' based on superpositions of the coherent states, but offer several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for repumping energy;** bosonic mode<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of bosonic mode; **codes are similar in spirit to 'cat codes' based on superpositions of the coherent states, but offer several advantages such as smaller mean number, exact rather than approximate orthonormality of the code words, and an explicit unitary operation for repumping energy**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>The binomial quantum codes are realizable with (e&eb) current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**.<br><br>Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version | |
| $G_{28}$ : Category one of **binomial quantum codes are realizable;** current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**.<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**. ;**binomial quantum codes are realizable**<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |

| | |
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| **Module six**<br><br>The binomial quantum codes are realizable with current superconducting circuit technology and they should prove useful in other quantum technologies, including (e) **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**.<br><br>Comments: Published version Subjects:Quantum Physics (quant-ph) Journal reference:Phys. Rev. X 6, 031006 (2016) DOI:10.1103/PhysRevX.6.031006 Cite as: arXiv:1602.00008 [quant-ph] (or arXiv:1602.00008v3 [quant-ph] for this version | |
| $G_{32}$ : Category one of **bosonic quantum memories, photonic quantum communication, and optical-to-microwave up- and down-conversion**<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of binomial quantum codes are realizable with current superconducting circuit technology and they should prove useful in other quantum technologies<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>For the two-mode exciton system formed by (e) the quasi-spin-wave collective excitation of many Λ atoms fixed at (eb) the lattice sites of a crystal, authors discover a dynamical symmetry depicted by (e) the semi direct product algebra SU (2)⊗¯¯h2 in (eb) the large N limit with low excitations | |
| $G_{36}$ : Category one ofquasi-spin-wave collective excitation of many Λ atoms fixed at (eb) the lattice sites of a crystal, authors discover a dynamical symmetry depicted by (e) the semi direct product algebra SU (2)⊗¯¯h2 in (eb) the large N limit with low excitations<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of two-mode exciton system formed<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>For the two-mode exciton system formed by the quasi-spin-wave collective excitation of many Λ atoms fixed at the lattice sites of a crystal, authors discover a dynamical symmetry depicted by (e) the semi direct product algebra SU (2)⊗¯¯h2 in (eb) the large N limit with low excitations | |

$G_{40}$ : Category one of two-mode exciton system formed by the quasi-spin-wave collective excitation of many Λ atoms fixed

$G_{41}$ : Category two of SAS

$G_{42}$ : Category three of SAS

$T_{40}$ : Category one of lattice sites of a crystal, authors discover a dynamical symmetry depicted by (e) the semi direct product algebra SU (2)⊗¯¯h2 in (eb) the large N limit with low excitations

$T_{41}$ : Category two of SAS

$T_{42}$ : Category three of SAS

**Module Nine**

For the two-mode exciton system formed by the quasi-spin-wave collective excitation of many Λ atoms fixed at the lattice sites of a crystal, authors discover a dynamical symmetry depicted by the semi direct product algebra SU (2)⊗¯¯h2 in (eb) the large N limit with low excitations

$G_{44}$ : Category one of two-mode exciton system formed by the quasi-spin-wave collective excitation of many Λ atoms fixed at the lattice sites of a crystal, authors discover a dynamical symmetry depicted by the semi direct product algebra SU (2)⊗¯¯h2

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of large N limit with low excitations

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$

| | |
|---|---|
| $(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)},$ $(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)},$ $(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)},$ are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |

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| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}),t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\left((G_{31}),t\right)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\left((G_{31}),t\right) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}\left((G_{35}),t\right)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |

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| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |

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| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3 | |

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| $+(a_{16}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{38}^{''})^{(7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7)}(T_{37},t)$ are seventh augmentation coefficient for 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\left[-(b_{13}^{''})^{(1)}(G,t)\right] & \left[-(b_{16}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\left[-(b_{14}^{''})^{(1)}(G,t)\right] & \left[-(b_{17}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\left[-(b_{15}^{''})^{(1)}(G,t)\right] & \left[-(b_{18}^{''})^{(2,2,)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 | |

$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt}=(a_{16})^{(2)}G_{17}-\left[\begin{array}{ccc}(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{16} \quad (61)$$

$$\frac{dG_{17}}{dt}=(a_{17})^{(2)}G_{16}-\left[\begin{array}{ccc}(a_{17}^{'})^{(2)}+(a_{17}^{''})^{(2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,)}(T_{14},t) & +(a_{21}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt}=(a_{18})^{(2)}G_{17}-\left[\begin{array}{ccc}(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,)}(T_{14},t) & +(a_{22}^{''})^{(3,3,3)}(T_{21},t)\\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9)}(T_{45},t)\end{array}\right]G_{18} \quad (63)$$

Where $+(a_{16}^{''})^{(2)}(T_{17},t)$, $+(a_{17}^{''})^{(2)}(T_{17},t)$, $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3

$+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt}=(b_{16})^{(2)}T_{17}-\left[\begin{array}{ccc}(b_{16}^{'})^{(2)}-(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t)\\ -(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)\\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t)\end{array}\right]T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a'_{20})^{(3)}\boxed{+(a''_{20})^{(3)}(T_{21},t)} & \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{20} \quad (67)$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a'_{21})^{(3)}\boxed{+(a''_{21})^{(3)}(T_{21},t)} & \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{21} \quad (68)$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a'_{22})^{(3)}\boxed{+(a''_{22})^{(3)}(T_{21},t)} & \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

| | |
|---|---|
| $+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}-(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}-(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3* | |

$\boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt}=(a_{24})^{(4)}G_{25}-\begin{bmatrix}(a_{24}^{'})^{(4)}\boxed{+(a_{24}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt}=(a_{25})^{(4)}G_{24}-\begin{bmatrix}(a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)}\\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt}=(a_{26})^{(4)}G_{25}-\begin{bmatrix}(a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}\\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}\\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{25}^{''})^{(4)}(T_{25},t)}$, $\boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}\\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}\\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)}\boxed{-(b''_{25})^{(4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)}\boxed{-(b''_{26})^{(4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

*Where* $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$
*are fourth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$
*are fifth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$
*are sixth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$
*are seventh detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$
*are eighth detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)}\boxed{+(a''_{28})^{(5)}(T_{29},t)} & \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)}\boxed{+(a''_{29})^{(5)}(T_{29},t)} & \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

*Where* $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* $1, 2$ *and* $3$

*And* $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ *are second augmentation*

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| $coefficient\ for\ category\ 1,2\ and\ 3$ $+(a_{32}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6)}(T_{33},t)$ $are\ third\ \ augmentation$ $coefficient\ for\ category\ 1,2\ and\ 3$ $+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)$ are fourth augmentation coefficients for category 1,2, and 3 $+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)$ are fifth augmentation coefficients for category 1,2,and 3 $+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)$ are sixth augmentation coefficients for category 1,2, 3 $+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1,2, 3 $+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3 $+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28}$ | 82 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29}$ | 83 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30}$ | 84 |
| $where$ $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ $are\ first\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{24}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ $are\ second\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ $are\ \ third\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3 $-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3 $-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3 $-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition | |

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| coefficients for category 1,2, and 3 $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3 | |
| $\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}$ $-\left[\begin{matrix}(a_{32}^{'})^{(6)}\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{28}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{32}$ | *85* |
| $\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\left[\begin{matrix}(a_{33}^{'})^{(6)}\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{29}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{33}$ | *86* |
| $\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\left[\begin{matrix}(a_{34}^{'})^{(6)}\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{30}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}\end{matrix}\right]G_{34}$ | *87* |
| $\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{28}^{''})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5)}(T_{29},t)}$ *are second augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)}$ *are third augmentation coefficients for category* 1, 2 *and* 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)}$ - are fourth augmentation coefficients $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)}$ - fifth augmentation coefficients $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)}$ sixth augmentation coefficients $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)}$ seventh augmentation coefficients $\boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$ Eighth augmentation coefficients $\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)}$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt}=(b_{32})^{(6)}T_{33}-\left[\begin{matrix}(b_{32}^{'})^{(6)}\boxed{-(b_{32}^{''})^{(6)}(G_{35},t)} & \boxed{-(b_{28}^{''})^{(5,5,5)}(G_{31},t)} & \boxed{-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)}\end{matrix}\right]T_{32}$ | *88* |

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)}\boxed{-(b''_{33})^{(6)}(G_{35},t)} & \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)}\boxed{-(b''_{34})^{(6)}(G_{35},t)} & \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}, \boxed{-(b''_{33})^{(6)}(G_{35},t)}, \boxed{-(b''_{34})^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}, \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}, \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a'_{36})^{(7)}\boxed{+(a''_{36})^{(7)}(T_{37},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

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Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

| | |
|---|---|
| Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[\begin{matrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{matrix} (a_{41}^{'})^{(8)}\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{matrix} (a_{42}^{'})^{(8)}\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{15}$ | |
| Where $\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second | |

augmentation coefficient for category 1, 2 and 3
$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3
$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3
$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3
$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3
$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3
$+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3
$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}-(b_{40}^{''})^{(8)}(G_{43},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}-(b_{41}^{''})^{(8)}(G_{43},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}-(b_{42}^{''})^{(8)}(G_{43},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3
$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3
$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3
$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

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| $-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{44}}{dt}$<br>$= (a_{44})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{44}^{'})^{(9)}+(a_{44}^{''})^{(9)}(T_{45},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{13}$ | *96* |
| $\frac{dG_{45}}{dt}$<br>$= (a_{45})^{(9)}G_{44}$<br>$-\begin{bmatrix} (a_{45}^{'})^{(9)}+(a_{45}^{''})^{(9)}(T_{45},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{14}$ | |
| $\frac{dG_{46}}{dt}$<br>$= (a_{46})^{(9)}G_{45}$<br>$-\begin{bmatrix} (a_{46}^{'})^{(9)}+(a_{46}^{''})^{(9)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{bmatrix} G_{15}$ | |
| Where $+(a_{44}^{''})^{(9)}(T_{45},t)$, $+(a_{45}^{''})^{(9)}(T_{45},t)$, $+(a_{46}^{''})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 | |

$+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3

$+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b_{44}^{'})^{(9)}\left[-(b_{44}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\left[-(b_{45}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\left[-(b_{46}^{''})^{(9)}(G_{47},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{44}^{''})^{(9)}(G_{47},t)$, $-(b_{45}^{''})^{(9)}(G_{47},t)$, $-(b_{46}^{''})^{(9)}(G_{47},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)$ are seventh detrition coefficients for category 1, 2 and 3

| | |
|---|---|
| $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| Where we suppose | |
| $(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$<br><br>The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:<br><br>$(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | *97* |
| $lim_{T_2 \to \infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lVert G - G^{'}\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}$, $(r_i)^{(1)}, i = 13,14,15,$ | 101 |

satisfy the inequalities

$$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$$

$$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$$

Where we suppose

$(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i,j = 16,17,18$

The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$:

$$(a_i'')^{(2)}(T_{17},t) \le (p_i)^{(2)} \le (\hat{A}_{16})^{(2)} \quad (102)$$

$$(b_i'')^{(2)}(G_{19},t) \le (r_i)^{(2)} \le (b_i')^{(2)} \le (\hat{B}_{16})^{(2)} \quad (103)$$

$$\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)} \quad (104)$$

$$\lim_{G\to\infty}(b_i'')^{(2)}((G_{19}),t) = (r_i)^{(2)} \quad (105)$$

**Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : (106)

Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)| \le (\hat{k}_{16})^{(2)}|T_{17} - T_{17}'|e^{-(\hat{M}_{16})^{(2)}t} \quad (107)$$

$$|(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}((G_{19}),t)| < (\hat{k}_{16})^{(2)}||(G_{19}) - (G_{19})'||e^{-(\hat{M}_{16})^{(2)}t} \quad (108)$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and $(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}]$. It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ :

$(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants (109)

$$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$$

**Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :

There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants

| | |
|---|---|
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$ satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)}$, $(r_i)^{(3)}$:<br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$<br>$\lim_{G \to \infty}(b_i^{''})^{(3)}(G_{23}, t) = (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'}, t) - (a_i^{''})^{(3)}(T_{21}, t)\rvert \leq (\hat{k}_{20})^{(3)} \lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br>$\lvert(b_i^{''})^{(3)}(G_{23}^{'}, t) - (b_i^{''})^{(3)}(G_{23}, t)\rvert < (\hat{k}_{20})^{(3)} \lVert G_{23} - G_{23}^{'}\rVert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'}, t)$ and $(a_i^{''})^{(3)}(T_{21}, t)$ . $(T_{21}^{'}, t)$ And $(T_{21}, t)$ are points belonging to the interval $[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |

There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$$

$$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$$ 116

Where we suppose

$$(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$$ 117

The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:

$$(a_i^{''})^{(4)}(T_{25}, t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$$

$$(b_i^{''})^{(4)}((G_{27}), t) \le (r_i)^{(4)} \le (b_i^{'})^{(4)} \le (\hat{B}_{24})^{(4)}$$

$$lim_{T_2 \to \infty}(a_i^{''})^{(4)}(T_{25}, t) = (p_i)^{(4)}$$ 118

$$\lim_{G \to \infty}(b_i^{''})^{(4)}((G_{27}), t) = (r_i)^{(4)}$$

**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :

Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$

They satisfy Lipschitz condition: 119

$$|(a_i^{''})^{(4)}(T_{25}^{'}, t) - (a_i^{''})^{(4)}(T_{25}, t)| \le (\hat{k}_{24})^{(4)}|T_{25} - T_{25}^{'}|e^{-(\hat{M}_{24})^{(4)}t}$$

$$|(b_i^{''})^{(4)}((G_{27})^{'}, t) - (b_i^{''})^{(4)}((G_{27}), t)| < (\hat{k}_{24})^{(4)}||(G_{27}) - (G_{27})^{'}||e^{-(\hat{M}_{24})^{(4)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'}, t)$ and $(a_i^{''})^{(4)}(T_{25}, t)$ . $(T_{25}^{'}, t)$ and $(T_{25}, t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ : 120

$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants

$$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$$

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| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)} (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}((G_{31}), t) \leq (r_i)^{(5)} \leq (b_i^{'})^{(5)} \leq (\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(5)}(T_{29}, t) = (p_i)^{(5)}$<br><br>$\lim_{G \to \infty}(b_i^{''})^{(5)}(G_{31}, t) = (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(5)}(T_{29}^{'}, t) - (a_i^{''})^{(5)}(T_{29}, t)\| \leq (\hat{k}_{28})^{(5)}\|T_{29} - T_{29}^{'}\|e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\|(b_i^{''})^{(5)}((G_{31})^{'}, t) - (b_i^{''})^{(5)}((G_{31}), t)\| < (\hat{k}_{28})^{(5)}\|\|(G_{31}) - (G_{31})^{'}\|\|e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'}, t)$ and $(a_i^{''})^{(5)}(T_{29}, t)$ . $(T_{29}^{'}, t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants | 125 |

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|---|---|
| $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)} (\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$<br><br>The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$<br><br>$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$ | 127 |
| $lim_{T_2 \to \infty} (a_i^{''})^{(6)} (T_{33}, t) = (p_i)^{(6)}$<br><br>$\lim_{G \to \infty} (b_i^{''})^{(6)} ((G_{35}), t) = (r_i)^{(6)}$<br><br>**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :<br><br>Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | 128 |
| They satisfy Lipschitz condition:<br><br>$\|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)\| \leq (\hat{k}_{32})^{(6)} \|T_{33} - T_{33}^{'}\| e^{-(\hat{M}_{32})^{(6)} t}$<br><br>$\|(b_i^{''})^{(6)}((G_{35})', t) - (b_i^{''})^{(6)}((G_{35}), t)\| < (\hat{k}_{32})^{(6)} \|(G_{35}) - (G_{35})'\| e^{-(\hat{M}_{32})^{(6)} t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and $(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants | 129 |

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

(QQ) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

(RR) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$$

$$(b_i'')^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$$

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*(SS)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(TT)*

$\lim_{G \to \infty} (b_i'')^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}'|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})'||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the

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| system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :<br><br>(UU) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants<br><br>$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ | 134 |
| **Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :<br><br>(VV) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i')^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$<br><br>$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i')^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | 135 |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i'')^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i')^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i'')^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i'')^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i'')^{(8)}(T_{41}', t) - (a_i'')^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}'\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |

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| $\|(b_i^{''})^{(8)}((G_{43})',t)-(b_i^{''})^{(8)}\big((G_{43}),t\big)\|<(\hat{k}_{40})^{(8)}\|(G_{43})-(G_{43})'\|e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}',t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}',t)$ and $(T_{41},t)$ are points belonging to the interval $\big[(\hat{k}_{40})^{(8)},(\hat{M}_{40})^{(8)}\big]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)}=1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\ ,\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}<1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)},(\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)},(\hat{k}_{40})^{(8)},(\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)},(a_i')^{(8)},(b_i)^{(8)},(b_i')^{(8)},(p_i)^{(8)},\ (r_i)^{(8)},i=40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)}+(a_i')^{(8)}+\ (\hat{A}_{40})^{(8)}+(\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}]<1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\ (b_i)^{(8)}+(b_i')^{(8)}+\ (\hat{B}_{40})^{(8)}+(\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}]<1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)},(a_i')^{(9)},(a_i^{''})^{(9)},(b_i)^{(9)},(b_i')^{(9)},(b_i^{''})^{(9)}>0,\quad i,j=44,45,46$<br><br>The functions $(a_i^{''})^{(9)},(b_i^{''})^{(9)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(9)},\ (r_i)^{(9)}$:<br><br>$(a_i^{''})^{(9)}(T_{45},t)\le(p_i)^{(9)}\le(\hat{A}_{44})^{(9)}$<br><br>$(b_i^{''})^{(9)}(G_{47},t)\le\ (r_i)^{(9)}\le(b_i')^{(9)}\le(\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i^{''})^{(9)}(T_{45},t)=(p_i)^{(9)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(9)}(G_{47},t)=\ (r_i)^{(9)}$<br><br>**Definition of** $(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)}$ :<br><br>Where $\boxed{(\hat{A}_{44})^{(9)},(\hat{B}_{44})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)}}$ are positive constants and $\boxed{i=44,45,46}$ | |
| They satisfy Lipschitz condition: | |

$$|(a_i^{''})^{(9)}(T_{45}^{'},t)-(a_i^{''})^{(9)}(T_{45},t)|\le(\hat{k}_{44})^{(9)}|T_{45}-T_{45}^{'}|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'},t)-(b_i^{''})^{(9)}\big((G_{47}),t\big)|<(\hat{k}_{44})^{(9)}||(G_{47})-(G_{47})^{'}||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'},t)$ and$(a_i^{''})^{(9)}(T_{45},t)$ . $(T_{45}^{'},t)$ and $(T_{45},t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)},(\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)}=1$ then the function $(a_i^{''})^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},$ are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}<1$$

**Definition of** $(\hat{P}_{44})^{(9)},(\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)},(\hat{k}_{44})^{(9)},(\hat{A}_{44})^{(9)}and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)},(a_i^{'})^{(9)},(b_i)^{(9)},(b_i^{'})^{(9)},(p_i)^{(9)},\ (r_i)^{(9)},i=44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)}+(a_i^{'})^{(9)}+(\hat{A}_{44})^{(9)}+(\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)}+(b_i^{'})^{(9)}+(\hat{B}_{44})^{(9)}+(\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}]<1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0),T_i(0)$ :

$$G_i(t)\le(\hat{P}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{G_i(0)=G_i^0>0}$$

$$T_i(t)\le(\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}\ ,\ \boxed{T_i(0)=T_i^0>0}$$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0),T_i(0)$

$$G_i(t)\le(\hat{P}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ G_i(0)=G_i^0>0$$

$$T_i(t)\le(\hat{Q}_{16})^{(2)}e^{(\hat{M}_{16})^{(2)}t}\ ,\ T_i(0)=T_i^0>0$$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$$G_i(t)\le(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}t}\ ,\ G_i(0)=G_i^0>0$$

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| $T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ : | 153 B |

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| $G_i(t) \leq \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{13})^{(1)}$ , $T_i^0 \leq (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br><br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br><br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |

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| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a_{20}^{'})^{(3)} + a_{20}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{24})^{(4)}$ , $T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |

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| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{28})^{(5)}$ , $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |

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| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}')^{(7)} - (b_{37}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}')^{(7)} - (b_{38}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a_{40}')^{(8)} + a_{40}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a_{41}')^{(8)} + (a_{41}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a_{42}')^{(8)} + (a_{42}'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b_{40}')^{(8)} - (b_{40}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b_{41}')^{(8)} - (b_{41}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b_{42}')^{(8)} - (b_{42}'')^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ | |

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''^{(9)}_{44}\right)\left(T_{45}(s_{(44)}), s_{(44)}\right)\right) G_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}\left(T_{45}(s_{(44)}), s_{(44)}\right)\right) G_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}\left(T_{45}(s_{(44)}), s_{(44)}\right)\right) G_{46}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{\mathrm{T}}_{46}(\mathrm{t}) = \mathrm{T}_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{46}(s_{(44)})\right] ds_{(44)}$$

Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$

The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)}\left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$$

$$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)}(\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$$

(167)

From which it follows that

$$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 1

(168)

Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$$

(169)

From which it follows that

$$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$$

(170)

Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

(171)

$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that 172

$$(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$$

From which it follows that 174

$$(G_{24}(t) - G_{24}^0) e^{-(\hat{M}_{24})^{(4)} t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right) e^{\left(-\frac{(\hat{P}_{24})^{(4)} + G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$$

From which it follows that 175

$$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

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(h) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

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$$G_{36}(t) \le G_{36}^0 + \int_0^t\left[(a_{36})^{(7)}\left(G_{37}^0+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$
$$\left(1+(a_{36})^{(7)}t\right)G_{37}^0+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^0)e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^0)e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^0}{G_{37}^0}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0 + \int_0^t\left[(a_{40})^{(8)}\left(G_{41}^0+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$
$$\left(1+(a_{40})^{(8)}t\right)G_{41}^0+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$$

180

From which it follows that

$$(G_{40}(t)-G_{40}^0)e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^0)e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^0}{G_{41}^0}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

181

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0 + \int_0^t\left[(a_{44})^{(9)}\left(G_{45}^0+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)} =$$
$$\left(1+(a_{44})^{(9)}t\right)G_{45}^0+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^0)e^{-(\hat{M}_{44})^{(9)}t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^0)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^0}{G_{45}^0}\right)}+(\hat{P}_{44})^{(9)}\right]$$

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| $(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}} < 1$ and to choose<br>$(\widehat{\mathrm{P}}_{13})^{(1)}$ and $(\widehat{\mathrm{Q}}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[(\widehat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\widehat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \leq (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)}, T^{(1)}\right), \left(G^{(2)}, T^{(2)}\right)\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G}, \tilde{T}$ : $(\tilde{G}, \tilde{T}) = \mathcal{A}^{(1)}(G, T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{13})^{(1)}\left\|G_{14}^{(1)} - G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)} +$<br>$\int_0^t\{(a_{13}')^{(1)}\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right)\left\|G_{13}^{(1)} - G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}} +$<br>$G_{13}^{(2)}\left\|(a_{13}'')^{(1)}\left(T_{14}^{(1)}, s_{(13)}\right) - (a_{13}'')^{(1)}\left(T_{14}^{(2)}, s_{(13)}\right)\right\| \; e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)} - G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t} \leq$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)} + (a_{13}')^{(1)} + (\widehat{A}_{13})^{(1)} + (\widehat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)}, T^{(1)}; G^{(2)}, T^{(2)}\right)\right)$ | 186 |

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| And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i = 13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>From 19 to 24 it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(1)} - (a_i^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(1)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}^{'})^{(1)}G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}^{'})^{(1)}$<br><br>If $G_{14}\ or\ G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}\ and\ G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t), t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t), t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results | |

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| $T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded. The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ $and$ $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $(\widetilde{G_{19}}, \widetilde{T_{19}}) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)}\left\|G_{17}^{(1)} - G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)} +$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}} +$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t} \leq$ | |

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| $\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+(a'_{16})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)\mathrm{d}\left(((G_{19})^{(1)},(T_{19})^{(1)};(G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a''_{16})^{(2)}$ and $(b''_{16})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(2)}$ and $(b''_i)^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a'_i)^{(2)}-(a''_i)^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}\mathrm{d}s_{(16)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{(-(b'_i)^{(2)}t)}>0$ for $t>0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1,\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16}<(\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt}\leq\left((\widehat{M}_{16})^{(2)}\right)_1-(a'_{17})^{(2)}G_{17}$ and by integrating<br><br>$G_{17}\leq\left((\widehat{M}_{16})^{(2)}\right)_2=G_{17}^0+2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1/(a'_{17})^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18}\leq\left((\widehat{M}_{16})^{(2)}\right)_3=G_{18}^0+2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a'_{18})^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(2)}((G_{19})(t),t))=(b'_{17})^{(2)}$ then $T_{17}\to\infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t>t_2$<br><br>$(b_{17})^{(2)}-(b''_i)^{(2)}((G_{19})(t),t)<\varepsilon_2, T_{16}(t)>(m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt}\geq(a_{17})^{(2)}(m)^{(2)}-\varepsilon_2 T_{17}$ which leads to | 204 |

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| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1-e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t), t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)}, (T_{23})^{(1)}), ((G_{23})^{(2)}, (T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left(\widetilde{(G_{23})}, \widetilde{(T_{23})}\right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)}\left\|G_{21}^{(1)} - G_{21}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)} +$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{-(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right)\left\|G_{20}^{(1)} - G_{20}^{(2)}\right\|e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}} +$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}'')^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\hat{M}_{20})^{(3)}s_{(20)}}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br>From the hypotheses it follows | 213 |

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| $\left\|{G_{23}}^{(1)} - {G_{23}}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)}t} \leq$ $\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a'_{20})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right) d\left(((G_{23})^{(1)}, (T_{23})^{(1)};\ (G_{23})^{(2)}, (T_{23})^{(2)})\right)$ And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a''_{20})^{(3)}$ and $(b''_{20})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ and $(\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(3)}$ and $(b''_i)^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})$ (and not on $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t) = 0$ and $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(3)} - (a''_i)^{(3)}(T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(3)}t\right)} > 0$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ and $\left((\widehat{M}_{20})^{(3)}\right)_3$ : **Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ and $G_{22}$ . indeed if $G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a'_{21})^{(3)} G_{21}$ and by integrating $G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1 / (a'_{21})^{(3)}$ In the same way , one can obtain $G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2 / (a'_{22})^{(3)}$ If $G_{21}$ or $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ is bounded, from below, the same property holds for $G_{21}$ and $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(3)}((G_{23})(t), t)) = (b'_{21})^{(3)}$ then $T_{21} \to \infty$. **Definition of** $(m)^{(3)}$ and $\varepsilon_3$ : Indeed let $t_3$ be so that for $t > t_3$ $(b_{21})^{(3)} - (b''_i)^{(3)}((G_{23})(t), t) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to | 220 |

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1-e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results

$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right), \quad t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.
The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose

$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have

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$$\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)} \qquad 222$$

$$\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)} \qquad 223$$

In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself 224

The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric 225

$$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{24})^{(4)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right) = \mathcal{A}^{(4)}((G_{27}),(T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{24})^{(4)}\left|G_{25}^{(1)} - G_{25}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)} +$$

$$\int_0^t\{(a_{24}^{'})^{(4)}\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}} +$$

$$G_{24}^{(2)}\left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses on Equations it follows

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| $\left\|(G_{27})^{(1)} - (G_{27})^{(2)}\right\| e^{-(\widehat{M}_{24})^{(4)}t} \leq$ $\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a'_{24})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$ And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a''_{24})^{(4)}$ and $(b''_{24})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$. If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(4)}$ and $(b''_i)^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$ it results $G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(4)} - (a''_i)^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$ $T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(4)}t\right)} > 0$ for t > 0 | 228 |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ : **Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if $G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a'_{25})^{(4)} G_{25}$ and by integrating $G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a'_{25})^{(4)}$ In the same way , one can obtain $G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2 / (a'_{26})^{(4)}$ If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(4)}((G_{27})(t), t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$. **Definition of** $(m)^{(4)}$ and $\varepsilon_4$ : Indeed let $t_4$ be so that for $t > t_4$ $(b_{25})^{(4)} - (b''_i)^{(4)}((G_{27})(t), t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to | 232 |

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1-e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

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$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad (234)$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad (235)$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{31})}, \widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})}, \widetilde{(T_{31})}\right) = \mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$

It results

$$\left|\tilde{G}_{28}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{28})^{(5)}\left|G_{29}^{(1)} - G_{29}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)} +$$

$$\int_0^t\{(a_{28}^{'})^{(5)}\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{-(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right)\left|G_{28}^{(1)} - G_{28}^{(2)}\right|e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}} +$$

$$G_{28}^{(2)}\left|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)}, s_{(28)}\right) - (a_{28}^{''})^{(5)}\left(T_{29}^{(2)}, s_{(28)}\right)\right|\ e^{-(\hat{M}_{28})^{(5)}s_{(28)}}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$$

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| Where $s_{(28)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)} - (G_{31})^{(2)}\right\| e^{-(\widehat{M}_{28})^{(5)}t} \leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)} + (a'_{28})^{(5)} + (\widehat{A}_{28})^{(5)} + (\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right) d\left(((G_{31})^{(1)}, (T_{31})^{(1)};\ (G_{31})^{(2)}, (T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)} e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i = 28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(5)} - (a''_i)^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b'_i)^{(5)}t\right)} > 0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}\ and\ G_{30}$ . indeed if<br><br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br><br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1 / (a'_{29})^{(5)}$<br><br>In the same way , one can obtain<br><br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2 / (a'_{30})^{(5)}$<br><br>If $G_{29}\ or\ G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}\ and\ G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(5)}((G_{31})(t), t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br><br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br><br>Indeed let $t_5$ be so that for $t > t_5$ | 242 |

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| $(b_{29})^{(5)} - (b_i^{''})^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}^{''})^{(5)}\left((G_{31})(t),t\right) = (b_{30}^{'})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}} < 1$ and to choose<br>$(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[(\hat{P}_{32})^{(6)} + \left((\hat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\bar{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{32})^{(6)}\right] \leq (\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right) = \mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{32})^{(6)}\left|G_{33}^{(1)} - G_{33}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)} +$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{-(\bar{M}_{32})^{(6)}s_{(32)}} +$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left|G_{32}^{(1)} - G_{32}^{(2)}\right|e^{-(\bar{M}_{32})^{(6)}s_{(32)}}e^{(\bar{M}_{32})^{(6)}s_{(32)}} +$ | 247 |

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| $G_{32}^{(2)}\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br><br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}\ and\ (\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32{,}33{,}34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0 e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0 e^{(-(b_i^{'})^{(6)}t)}>0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1,\left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32}<(\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt}\leq\left((\widehat{M}_{32})^{(6)}\right)_1-(a_{33}^{'})^{(6)}G_{33}$ and by integrating<br><br>$G_{33}\leq\left((\widehat{M}_{32})^{(6)}\right)_2=G_{33}^0+2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a_{33}^{'})^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34}\leq\left((\widehat{M}_{32})^{(6)}\right)_3=G_{34}^0+2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a_{34}^{'})^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}\ is$ bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t))=(b_{33}^{'})^{(6)}$ then $T_{33}\to\infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ : | 253 |

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| Indeed let $t_6$ be so that for $t > t_6$<br>$(b_{33})^{(6)} - (b_i^{''})^{(6)}\big((G_{35})(t), t\big) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$ | |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br>$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}^{'})^{(6)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$<br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ large to have | 255 |
| $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\hat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br>$d\left(((G_{39})^{(1)}, (T_{39})^{(1)}), ((G_{39})^{(2)}, (T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{36})^{(7)}t}\}$<br>Indeed if we denote<br>**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left(\widetilde{(G_{39})}, \widetilde{(T_{39})}\right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$<br>It results<br>$\left\|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{36})^{(7)}\left|G_{37}^{(1)} - G_{37}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)} +$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{-(\hat{M}_{36})^{(7)}s_{(36)}} +$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right|e^{-(\hat{M}_{36})^{(7)}s_{(36)}}e^{(\hat{M}_{36})^{(7)}s_{(36)}} +$ | 258 |

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| $G_{36}^{(2)}\|(a_{36}^{''})^{(7)}(T_{37}^{(1)},s_{(36)})-(a_{36}^{''})^{(7)}(T_{37}^{(2)},s_{(36)})\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}\ and\ (\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br><br>it results<br><br>$G_i(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\geq 0$<br><br>$T_i(t)\geq T_i^0e^{\left(-(b_i^{'})^{(7)}t\right)}>0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1,\left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36}<(\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt}\leq\left((\widehat{M}_{36})^{(7)}\right)_1-(a_{37}^{'})^{(7)}G_{37}$ and by integrating<br><br>$G_{37}\leq\left((\widehat{M}_{36})^{(7)}\right)_2=G_{37}^0+2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1/(a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38}\leq\left((\widehat{M}_{36})^{(7)}\right)_3=G_{38}^0+2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2/(a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |

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| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t),t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$. | 264 |
| **Definition of** $(m)^{(7)}$ and $\varepsilon_7$ : | |
| Indeed let $t_7$ be so that for $t > t_7$ | |
| $(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to | 265 |
| $T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results | |
| $T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right), \quad t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded. The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}((G_{39})(t),t) = (b_{38}^{'})^{(7)}$ We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ and to choose $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[(\hat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right) =$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\hat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote | 270 |
| **Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | |
| It results | 271 |

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| $\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)}-G_{41}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$ $\int_0^t \{(a_{40}^{'})^{(8)} \left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$ $(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$ $G_{40}^{(2)} \left\|(a_{40}^{''})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)-(a_{40}^{''})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$ | |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\| e^{-(\widehat{M}_{40})^{(8)} t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}^{'})^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right) d\left(\left((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}^{''})^{(8)}$ and $(b_{40}^{''})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)} t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$ | 276 |

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| If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t),t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\big((G_{43})(t),t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1-e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\big((G_{43})(t),t(t),t\big) = (b_{42}^{'})^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\bar{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\bar{M}_{44})^{(9)}} < 1$ and to choose $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ large to have | 279 A |
| $\frac{(a_i)^{(9)}}{(\bar{M}_{44})^{(9)}}\left[(\hat{P}_{44})^{(9)} + \big((\hat{P}_{44})^{(9)} + G_j^0\big)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$ | |
| $\frac{(b_i)^{(9)}}{(\bar{M}_{44})^{(9)}}\left[\big((\hat{Q}_{44})^{(9)} + T_j^0\big)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$ | |
| In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself | |
| The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric<br><br>$d\Big(((G_{47})^{(1)},(T_{47})^{(1)}),\big((G_{47})^{(2)},(T_{47})^{(2)}\big)\Big) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\bar{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\bar{M}_{44})^{(9)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}$ : $\big(\widetilde{(G_{47})},\widetilde{(T_{47})}\big) = \mathcal{A}^{(9)}\big((G_{47}),(T_{47})\big)$<br>It results | |

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{44})^{(9)} \left|G_{45}^{(1)}-G_{45}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\, ds_{(44)} +$$
$$\int_0^t \{(a_{44}^{'})^{(9)} \left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)} s_{(44)}} e^{(\widehat{M}_{44})^{(9)} s_{(44)}}\} ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right| e^{-(\widehat{M}_{44})^{(9)} t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)} (\widehat{k}_{44})^{(9)}\right) d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ *and* $(\widehat{Q}_{44})^{(9)} e^{(\widehat{M}_{44})^{(9)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$

From 99 to 44 it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)} (T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{45}^{'})^{(9)}$ then $T_{45}\to\infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t>t_9$

$$(b_{45})^{(9)}-(b_i^{''})^{(9)}\left((G_{47})(t),t\right)<\varepsilon_9, T_{44}(t)>(m)^{(9)}$$

Then $\frac{dT_{45}}{dt}\geq(a_{45})^{(9)}(m)^{(9)}-\varepsilon_9 T_{45}$ which leads to

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1-e^{-\varepsilon_9 t})+T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t}=\frac{1}{2}$ it results

$T_{45}\geq\left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right),\quad t=log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t),t\right)=(b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)},(\sigma_2)^{(1)},(\tau_1)^{(1)},(\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)}\leq-(a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}-(a_{13}^{''})^{(1)}(T_{14},t)+(a_{14}^{''})^{(1)}(T_{14},t)\leq-(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)}\leq-(b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(b_{13}^{''})^{(1)}(G,t)-(b_{14}^{''})^{(1)}(G,t)\leq-(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)},(\nu_2)^{(1)},(u_1)^{(1)},(u_2)^{(1)},\nu^{(1)},u^{(1)}$ : 281

By $(\nu_1)^{(1)}>0,(\nu_2)^{(1)}<0$ and respectively $(u_1)^{(1)}>0,(u_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_1)^{(1)}\nu^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_1)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(\bar{\nu}_1)^{(1)},,(\bar{\nu}_2)^{(1)},(\bar{u}_1)^{(1)},(\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)}>0,(\bar{\nu}_2)^{(1)}<0$ and respectively $(\bar{u}_1)^{(1)}>0,(\bar{u}_2)^{(1)}<0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_2)^{(1)}\nu^{(1)}-(a_{13})^{(1)}=0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2+(\tau_2)^{(1)}u^{(1)}-(b_{13})^{(1)}=0$

**Definition of** $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)},(\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)},(m_2)^{(1)},(\mu_1)^{(1)},(\mu_2)^{(1)}$ by
$(m_2)^{(1)}=(\nu_0)^{(1)},(m_1)^{(1)}=(\nu_1)^{(1)},\ if\ (\nu_0)^{(1)}<(\nu_1)^{(1)}$

$(m_2)^{(1)}=(\nu_1)^{(1)},(m_1)^{(1)}=(\bar{\nu}_1)^{(1)},if\ (\nu_1)^{(1)}<(\nu_0)^{(1)}<(\bar{\nu}_1)^{(1)},$

and $\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$

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| $(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | |
| and analogously<br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$<br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$ are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)} t}$<br>where $(p_i)^{(1)}$ is defined by equation<br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t}\right] + G_{15}^0 e^{-(S_2)^{(1)} t} \le G_{15}(t) \le \right.$<br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}')^{(1)})}\left[e^{(S_1)^{(1)} t} - e^{-(a_{15}')^{(1)} t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)} t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b_{15}')^{(1)})}\left[e^{(R_1)^{(1)} t} - e^{-(b_{15}')^{(1)} t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)} t} \le T_{15}(t) \le$<br>$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})}\left[e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t}\right] + T_{15}^0 e^{-(R_2)^{(1)} t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:-<br>Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$<br>$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$<br>$(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$<br>$(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |

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| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}^{'})^{(2)} + (a_{17}^{'})^{(2)} - (a_{16}^{''})^{(2)}(T_{17}, t) + (a_{17}^{''})^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}^{'})^{(2)} + (b_{17}^{'})^{(2)} - (b_{16}^{''})^{(2)}\big((G_{19}), t\big) - (b_{17}^{''})^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously $(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$ $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities | 310 |

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| $G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^{0}e^{(S_1)^{(2)}t}$ | |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}}G_{16}^{0}e^{((S_1)^{(2)}-(p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}}G_{16}^{0}e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_1)^{(2)}((S_1)^{(2)}-(p_{16})^{(2)}-(S_2)^{(2)})}\left[e^{((S_1)^{(2)}-(p_{16})^{(2)})t}-e^{-(S_2)^{(2)}t}\right]+G_{18}^{0}e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$ $\left.\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a_{18}')^{(2)})}[e^{(S_1)^{(2)}t}-e^{-(a_{18}')^{(2)}t}]+G_{18}^{0}e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t}-e^{-(b_{18}')^{(2)}t}\right]+T_{18}^{0}e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}')^{(2)}$ $(R_2)^{(2)} = (b_{18}')^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions** **Theorem 3:** If we denote and define **Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$: $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying $-(\sigma_2)^{(3)} \le -(a_{20}')^{(3)} + (a_{21}')^{(3)} - (a_{20}'')^{(3)}(T_{21}, t) + (a_{21}'')^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$ $-(\tau_2)^{(3)} \le -(b_{20}')^{(3)} + (b_{21}')^{(3)} - (b_{20}'')^{(3)}(G_{23}, t) - (b_{21}'')^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)}, (v_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$: By $(v_1)^{(3)} > 0$, $(v_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$ and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and | 320 |

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| By $(\bar{v}_1)^{(3)} > 0$ , $(\bar{v}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$ , $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}\left(u^{(3)}\right)^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | |
| **Definition of** $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}$ , $(m_2)^{(3)}$ , $(\mu_1)^{(3)}$, $(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (v_0)^{(3)}, (m_1)^{(3)} = (v_1)^{(3)}$, ***if*** $(v_0)^{(3)} < (v_1)^{(3)}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)} < (v_0)^{(3)} < (\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (v_1)^{(3)}, (m_1)^{(3)} = (v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)} < (v_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$ , ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}$, ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br>Then the solution of global equations satisfies the inequalities<br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)}t}$<br>$(p_i)^{(3)}$ is defined by equation | 322 |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \le G_{22}(t) \le\right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)}t} - e^{-(a_{22}')^{(3)}t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)}t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)}t} - e^{-(b_{22}')^{(3)}t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)}t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$ | 327 |

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

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**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$

**Definition of** $(\nu_1)^{(4)}, (\nu_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, \nu^{(4)}, u^{(4)}$ :

By $(\nu_1)^{(4)} > 0$, $(\nu_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$, $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_1)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

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**Definition of** $(\bar{\nu}_1)^{(4)},, (\bar{\nu}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ :

By $(\bar{\nu}_1)^{(4)} > 0$, $(\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$, $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$

$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$, ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,

and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$

$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$

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and analogously

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$, ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

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| and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}, \boldsymbol{if}\ (\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)} t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)} t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)} t}\right] + G_{26}^0 e^{-(S_2)^{(4)} t} \le G_{26}(t) \le\right.$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)} t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}')^{(4)})}\left[e^{(R_1)^{(4)} t} - e^{-(b_{26}')^{(4)} t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)} t} \le T_{26}(t) \le$<br>$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)} + (r_{24})^{(4)} + (R_2)^{(4)})}\left[e^{((R_1)^{(4)} + (r_{24})^{(4)})t} - e^{-(R_2)^{(4)} t}\right] + T_{26}^0 e^{-(R_2)^{(4)} t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:-<br>Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$<br>$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$<br>$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$<br>$(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :<br>$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying<br>$-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$<br>$-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 338 |
| **Definition of** $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ : | 339 |

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| By $(v_1)^{(5)} > 0$ , $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$ , $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ :<br>By $(\bar{v}_1)^{(5)} > 0$ , $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$<br>**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$, $(v_0)^{(5)}$ :-<br>If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}$, $(\mu_2)^{(5)}$ by<br>$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,<br>and $\boxed{(v_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)} < (v_0)^{(5)}$ | 340 |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}$, ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)}t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |

$$\frac{(b_{30})^{(5)}T_{28}^{0}}{(\mu_1)^{(5)}((R_1)^{(5)}-(b'_{30})^{(5)})}\left[e^{(R_1)^{(5)}t}-e^{-(b'_{30})^{(5)}t}\right]+T_{30}^{0}e^{-(b'_{30})^{(5)}t}\le T_{30}(t)\le$$

$$\frac{(a_{30})^{(5)}T_{28}^{0}}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t}-e^{-(R_2)^{(5)}t}\right]+T_{30}^{0}e^{-(R_2)^{(5)}t}$$ (347)

**Definition of** $(S_1)^{(5)},(S_2)^{(5)},(R_1)^{(5)},(R_2)^{(5)}$:- (348)

Where $(S_1)^{(5)}=(a_{28})^{(5)}(m_2)^{(5)}-(a'_{28})^{(5)}$

$(S_2)^{(5)}=(a_{30})^{(5)}-(p_{30})^{(5)}$

$(R_1)^{(5)}=(b_{28})^{(5)}(\mu_2)^{(5)}-(b'_{28})^{(5)}$

$(R_2)^{(5)}=(b'_{30})^{(5)}-(r_{30})^{(5)}$

**Behavior of the solutions of equation** (349)

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)},(\sigma_2)^{(6)},(\tau_1)^{(6)},(\tau_2)^{(6)}$ four constants satisfying

$-(\sigma_2)^{(6)}\le-(a'_{32})^{(6)}+(a'_{33})^{(6)}-(a''_{32})^{(6)}(T_{33},t)+(a''_{33})^{(6)}(T_{33},t)\le-(\sigma_1)^{(6)}$

$-(\tau_2)^{(6)}\le-(b'_{32})^{(6)}+(b'_{33})^{(6)}-(b''_{32})^{(6)}((G_{35}),t)-(b''_{33})^{(6)}((G_{35}),t)\le-(\tau_1)^{(6)}$

**Definition of** $(\nu_1)^{(6)},(\nu_2)^{(6)},(u_1)^{(6)},(u_2)^{(6)},\nu^{(6)},u^{(6)}$ : (350)

By $(\nu_1)^{(6)}>0$ , $(\nu_2)^{(6)}<0$ and respectively $(u_1)^{(6)}>0$ , $(u_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2+(\sigma_1)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}(u^{(6)})^2+(\tau_1)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$ and

**Definition of** $(\bar{\nu}_1)^{(6)},,(\bar{\nu}_2)^{(6)},(\bar{u}_1)^{(6)},(\bar{u}_2)^{(6)}$ : (351)

By $(\bar{\nu}_1)^{(6)}>0$ , $(\bar{\nu}_2)^{(6)}<0$ and respectively $(\bar{u}_1)^{(6)}>0$ , $(\bar{u}_2)^{(6)}<0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2+(\sigma_2)^{(6)}\nu^{(6)}-(a_{32})^{(6)}=0$ and $(b_{33})^{(6)}(u^{(6)})^2+(\tau_2)^{(6)}u^{(6)}-(b_{32})^{(6)}=0$

**Definition of** $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)},(\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)},(m_2)^{(6)},(\mu_1)^{(6)},(\mu_2)^{(6)}$ by

$(m_2)^{(6)}=(\nu_0)^{(6)},(m_1)^{(6)}=(\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)}<(\nu_1)^{(6)}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)}<(\nu_0)^{(6)}<(\bar{\nu}_1)^{(6)}$,

and $\boxed{(\nu_0)^{(6)}=\frac{G_{32}^{0}}{G_{33}^{0}}}$

$(m_2)^{(6)}=(\nu_1)^{(6)},(m_1)^{(6)}=(\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)}<(\nu_0)^{(6)}$

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| and analogously<br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}, \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$<br>and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$<br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br>$G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G_{32}^0 e^{(S_1)^{(6)} t}$<br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)} t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)} t}\right] + G_{34}^0 e^{-(S_2)^{(6)} t} \le G_{34}(t) \le\right.$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)} t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)} + (r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)} - (b'_{34})^{(6)})}\left[e^{(R_1)^{(6)} t} - e^{-(b'_{34})^{(6)} t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)} t} \le T_{34}(t) \le$<br>$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)} + (r_{32})^{(6)} + (R_2)^{(6)})}\left[e^{((R_1)^{(6)} + (r_{32})^{(6)})t} - e^{-(R_2)^{(6)} t}\right] + T_{34}^0 e^{-(R_2)^{(6)} t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:-<br>Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$<br>$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$<br>$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$<br>$(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ : | |

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$$

$$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}\big((G_{39}), t\big) - (b''_{37})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0$, $(\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0$, $(u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0$, $(\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0$, $(\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

Then the solution of global equations satisfies the inequalities 364

$$G^0_{36} e^{\left((S_1)^{(7)} - (p_{36})^{(7)}\right)t} \le G_{36}(t) \le G^0_{36} e^{(S_1)^{(7)}t}$$

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| where $(p_i)^{(7)}$ is defined by equation | |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $(\frac{(a_{38})^{(7)}G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le$ $\frac{(a_{38})^{(7)}G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a_{38}')^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}] + G_{38}^0 e^{-(a_{38}')^{(7)}t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)}T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$ $\frac{(a_{38})^{(7)}T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- <br> Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ <br> $(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$ <br> $(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b_{36}')^{(7)}$ <br> $(R_2)^{(7)} = (b_{38}')^{(7)} - (r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation** <br> **Theorem 2:** If we denote and define <br> **Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ : <br> $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying <br> $-(\sigma_2)^{(8)} \le -(a_{40}')^{(8)} + (a_{41}')^{(8)} - (a_{40}'')^{(8)}(T_{41}, t) + (a_{41}'')^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$ <br> $-(\tau_2)^{(8)} \le -(b_{40}')^{(8)} + (b_{41}')^{(8)} - (b_{40}'')^{(8)}((G_{43}), t) - (b_{41}'')^{(8)}((G_{43}), t) \le -(\tau_1)^{(8)}$ | 371 |
| **Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : <br> By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}(v^{(8)})^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$ | 372 |

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| and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(8)},,(\bar{v}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{v}_1)^{(8)} > 0$ , $(\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the<br>roots of the equations $(a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$<br>and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$<br>**Definition of** $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,<br>and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}$, ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)}t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$ | 377 |

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|---|---|
| | |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \le T_{42}(t) \le$<br><br>$\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br><br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br><br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br><br>**Theorem 2:** If we denote and define<br><br>**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :<br><br>$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying<br><br>$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$<br><br>$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}((G_{47}), t) - (b''_{45})^{(9)}((G_{47}), t) \le -(\tau_1)^{(9)}$ | 382 |
| **Definition of** $(v_1)^{(9)}, (v_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, v^{(9)}, u^{(9)}$ :<br><br>By $(v_1)^{(9)} > 0$, $(v_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$, $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(9)}, , (\bar{v}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :<br><br>By $(\bar{v}_1)^{(9)} > 0$, $(\bar{v}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$, $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}(v^{(9)})^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)} = 0$<br>and $(b_{45})^{(9)}(u^{(9)})^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$<br>**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (v_0)^{(9)}$ :-<br><br>If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by<br><br>$(m_2)^{(9)} = (v_0)^{(9)}, (m_1)^{(9)} = (v_1)^{(9)}$, ***if*** $(v_0)^{(9)} < (v_1)^{(9)}$ | |

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{44}(t) \leq G_{44}^0 e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \leq G_{45}(t) \leq \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)} t}$$

$$\frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)} - (p_{44})^{(9)} - (S_2)^{(9)})} \left[ e^{((S_1)^{(9)} - (p_{44})^{(9)})t} - e^{-(S_2)^{(9)} t} \right] + G_{46}^0 e^{-(S_2)^{(9)} t} \leq G_{46}(t) \leq$$

$$\frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)} - (a'_{46})^{(9)})} [e^{(S_1)^{(9)} t} - e^{-(a'_{46})^{(9)} t}] + G_{46}^0 e^{-(a'_{46})^{(9)} t})$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)} t} \leq T_{44}(t) \leq \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)} + (r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)} - (b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)} t} - e^{-(b'_{46})^{(9)} t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)} t} \leq T_{46}(t) \leq$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)} + (r_{44})^{(9)} + (R_2)^{(9)})} \left[ e^{((R_1)^{(9)} + (r_{44})^{(9)})t} - e^{-(R_2)^{(9)} t} \right] + T_{46}^0 e^{-(R_2)^{(9)} t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get 384

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{v}_1)^{(1)}$ we find like in the previous case, 385

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq v^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{v^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $v^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

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| In the same manner , we get<br>$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | 391 |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br>**Definition of** $u^{(2)}(t)$ :-<br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |

**Proof** : From global equations we obtain 398

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \leq \frac{dv^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \geq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \leq v^{(3)}(t) \leq (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \leq v^{(3)}(t) \leq (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \leq \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \leq v^{(3)}(t) \leq$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \leq (\bar{v}_1)^{(3)} \leq (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \leq v^{(3)}(t) \leq \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \leq (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq v^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{v^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $v^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{dv^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)v^{(4)} - (a_{25})^{(4)}v^{(4)}$$

**Definition of** $v^{(4)}$ :- $\boxed{v^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)}\right) \leq \frac{dv^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_4)^{(4)}v^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$v^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq v^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

$$v^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

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From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)\le\frac{dv^{(5)}}{dt}\le-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(5)},(v_0)^{(5)}$ :-

For $0<\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}<(v_1)^{(5)}<(\bar{v}_1)^{(5)}$

$$v^{(5)}(t)\ge\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}{5+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_0)^{(5)}\right)t\right]}}\quad,\quad\boxed{(C)^{(5)}=\frac{(v_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(v_2)^{(5)}}}$$

it follows $(v_0)^{(5)}\le v^{(5)}(t)\le(v_1)^{(5)}$

In the same manner , we get

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$$v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(5)}=\frac{(\bar{v}_1)^{(5)}-(v_0)^{(5)}}{(v_0)^{(5)}-(\bar{v}_2)^{(5)}}}$$

From which we deduce $(v_0)^{(5)}\le v^{(5)}(t)\le(\bar{v}_5)^{(5)}$

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If $0<(v_1)^{(5)}<(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}<(\bar{v}_1)^{(5)}$ we find like in the previous case,

$$(v_1)^{(5)}\le\frac{(v_1)^{(5)}+(C)^{(5)}(v_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((v_1)^{(5)}-(v_2)^{(5)}\right)t\right]}}\le v^{(5)}(t)\le$$

$$\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(\bar{v}_1)^{(5)}$$

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If $0<(v_1)^{(5)}\le(\bar{v}_1)^{(5)}\le\boxed{(v_0)^{(5)}=\frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(v_1)^{(5)}\le v^{(5)}(t)\le\frac{(\bar{v}_1)^{(5)}+(\bar{C})^{(5)}(\bar{v}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{v}_1)^{(5)}-(\bar{v}_2)^{(5)}\right)t\right]}}\le(v_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(5)}(t)$ :-

$(m_2)^{(5)}\le v^{(5)}(t)\le(m_1)^{(5)}$, $\boxed{v^{(5)}(t)=\frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)}\le u^{(5)}(t)\le(\mu_1)^{(5)}$, $\boxed{u^{(5)}(t)=\frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case, 414

$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain 415

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}, \quad \boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}, \quad \boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right) - (a_{37}^{''})^{(7)}(T_{37},t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\quad \boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \ge \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \le \nu^{(8)}(t) \le (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \le \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \le \nu^{(8)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \le (\bar{\nu}_1)^{(8)} \le \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \le \nu^{(8)}(t) \le \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \le (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq v^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{v^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $v^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{dv^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)v^{(9)} - (a_{45})^{(9)}v^{(9)}$$

**Definition of** $v^{(9)}$ :- $\boxed{v^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_2)^{(9)}v^{(9)} - (a_{44})^{(9)}\right) \leq \frac{dv^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(v^{(9)}\right)^2 + (\sigma_1)^{(9)}v^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$v^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq v^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \le (\bar{\nu}_1)^{(9)} \le \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \le \nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \le \nu^{(9)}(t) \le (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \le u^{(9)}(t) \le (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

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**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

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| $(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,<br>$(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}')^{(1)}(r_{14})^{(1)} - (b_{14}')^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$<br>$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i'')^{(2)} and\ (b_i'')^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}')^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}')^{(2)}(r_{17})^{(2)} - (b_{17}')^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i'')^{(3)} and\ (b_i'')^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}')^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}')^{(3)}(r_{21})^{(3)} - (b_{21}')^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i'')^{(4)} and\ (b_i'')^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}')^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}')^{(4)}(r_{25})^{(4)} - (b_{25}')^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i'')^{(5)} and\ (b_i'')^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}')^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}')^{(5)}(b_{29}')^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ , | 433 |

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| $(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$ | 436 A |

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| $(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |

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| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27})\right)]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27})\right)]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27})\right)]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |

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| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(h) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if | 486 |

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| $F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) +$ | 492 A |

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| $(a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations | 494 |
| $G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first | 499 |

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| equations<br><br>$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |

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| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |

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| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $\mathrm{G}_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $\mathrm{T}_{17}^{*}$ given by $f(\mathrm{T}_{17}^{*}) = 0$ and | 512 |
| $\mathrm{G}_{16}^{*} = \frac{(\mathrm{a}_{16})^{(2)}\mathrm{G}_{17}^{*}}{[(\mathrm{a}_{16}^{'})^{(2)} + (\mathrm{a}_{16}^{''})^{(2)}(\mathrm{T}_{17}^{*})]}$ , $\mathrm{G}_{18}^{*} = \frac{(\mathrm{a}_{18})^{(2)}\mathrm{G}_{17}^{*}}{[(\mathrm{a}_{18}^{'})^{(2)} + (\mathrm{a}_{18}^{''})^{(2)}(\mathrm{T}_{17}^{*})]}$ | 513 |
| $\mathrm{T}_{16}^{*} = \frac{(\mathrm{b}_{16})^{(2)}\mathrm{T}_{17}^{*}}{[(\mathrm{b}_{16}^{'})^{(2)} - (\mathrm{b}_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $\mathrm{T}_{18}^{*} = \frac{(\mathrm{b}_{18})^{(2)}\mathrm{T}_{17}^{*}}{[(\mathrm{b}_{18}^{'})^{(2)} - (\mathrm{b}_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$ | 515 |

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| Obviously, these values represent an equilibrium solution of global equations | |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)} (T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)} (T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)} ((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)} ((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br>$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and<br>$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)} (T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)} (T_{29}^*)]}$ | 518 |
| $T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)} ((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)} ((G_{31})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and<br>$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)} (T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)} (T_{33}^*)]}$ | 520 |
| $T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)} ((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)} ((G_{35})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and<br>$G_{36}^* = \frac{(a_{36})^{(7)} G_{37}^*}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)} (T_{37}^*)]}$ , $G_{38}^* = \frac{(a_{38})^{(7)} G_{37}^*}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)} (T_{37}^*)]}$<br>$T_{36}^* = \frac{(b_{36})^{(7)} T_{37}^*}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)} ((G_{39})^*)]}$ , $T_{38}^* = \frac{(b_{38})^{(7)} T_{37}^*}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)} ((G_{39})^*)]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^*$ given by $\varphi((G_{43})^*) = 0$ , $T_{41}^*$ given by $f(T_{41}^*) = 0$ and | 523 |

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| $G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)} (T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)} (T_{41}^{*})]}$<br><br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)} ((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)} ((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br><br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br><br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)} (T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)} (T_{45}^{*})]}$<br><br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)} ((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)} ((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS** | 531 |

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| **Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)}$ , $\frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |

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| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}$ $and$ $(b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}$ , $\frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ $and$ $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}$ , $\frac{\partial(b_i^{''})^{(5)}}{\partial G_j}((G_{31})^{*}) = s_{ij}$ | 556 |

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| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{33}^{''})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 571 |

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 572

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \quad , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain from

$$\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37} \quad 573$$

$$\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37} \quad 574$$

$$\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37} \quad 575$$

$$\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right) \quad 576$$

$$\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right) \quad 578$$

$$\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right) \quad 579$$

Obviously, these values represent an equilibrium solution

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**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 580

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \quad , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41} \quad 581$$

$$\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41} \quad 582$$

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| | |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \ , \ \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$<br><br>$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$ | |

$$+\left(\left((\lambda)^{(1)}+(a_{14}^{'})^{(1)}+(p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^{*}+(a_{13})^{(1)}(q_{14})^{(1)}G_{14}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(13)}T_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^{2}+\left((b_{13}^{'})^{(1)}+(b_{14}^{'})^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$+\left(\left((\lambda)^{(1)}\right)^{2}+\left((a_{13}^{'})^{(1)}+(a_{14}^{'})^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}^{'})^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left(\left((\lambda)^{(1)}+(b_{13}^{'})^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}^{'})^{(2)}-(r_{18})^{(2)})\{\left((\lambda)^{(2)}+(a_{18}^{'})^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left(\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$$

$$+\left(\left((\lambda)^{(2)}+(a_{17}^{'})^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$\left(\left((\lambda)^{(2)}\right)^{2}+\left((b_{16}^{'})^{(2)}+(b_{17}^{'})^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$$

$$+\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}^{'})^{(2)}+(a_{17}^{'})^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}^{'})^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left(\left((\lambda)^{(2)}+(b_{16}^{'})^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}^{'})^{(3)}-(r_{22})^{(3)})\{\left((\lambda)^{(3)}+(a_{22}^{'})^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left(\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$$

$$+\left(\left((\lambda)^{(3)}+(a'_{21})^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20}+(a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\right)$$

$$\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$\left(\left((\lambda)^{(3)}\right)^{2}+\left((b'_{20})^{(3)}+(b'_{21})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$$

$$+\left(\left((\lambda)^{(3)}\right)^{2}+\left((a'_{20})^{(3)}+(a'_{21})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a'_{20})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left(\left((\lambda)^{(3)}+(b'_{20})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21}+(b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\right)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b'_{26})^{(4)}-(r_{26})^{(4)}\}\{\left((\lambda)^{(4)}+(a'_{26})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left(\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(25)}T^{*}_{25}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a'_{25})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G^{*}_{24}+(a_{24})^{(4)}(q_{25})^{(4)}G^{*}_{25}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(24)}T^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b'_{24})^{(4)}+(b'_{25})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a'_{24})^{(4)}+(a'_{25})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a'_{24})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G^{*}_{25}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G^{*}_{24}\right)$$

$$\left(\left((\lambda)^{(4)}+(b'_{24})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T^{*}_{25}+(b_{25})^{(4)}s_{(24),(26)}T^{*}_{24}\right)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b'_{30})^{(5)}-(r_{30})^{(5)}\}\{\left((\lambda)^{(5)}+(a'_{30})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(29)}T^{*}_{29}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a'_{29})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G^{*}_{28}+(a_{28})^{(5)}(q_{29})^{(5)}G^{*}_{29}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(28)}T^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b'_{28})^{(5)}+(b'_{29})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a'_{28})^{(5)}+(a'_{29})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a'_{28})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G^{*}_{29}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G^{*}_{28}\right)$$

$$\left(\left((\lambda)^{(5)}+(b'_{28})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T^{*}_{29}+(b_{29})^{(5)}s_{(28),(30)}T^{*}_{28}\right)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b'_{34})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a'_{34})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(33)}T^{*}_{33}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^{*}_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^{*}_{33}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(32)}T^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(34)}T^{*}_{32}\right)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(37)}T^{*}_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^{*}_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^{*}_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(36)}T^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(38)}T^{*}_{36}\right)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\right)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45} + (b_{45})^{(9)}s_{(44),(45)}T^*_{45}\Big)$$

$$+\Big(\left((\lambda)^{(9)} + (a'_{45})^{(9)} + (p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44} + (a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\Big(\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\Big(\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future**. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_{i,}$ $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i Xi(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k \tag{8}$$

Where $x_i$ is the abundance of population $i$, $x= (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to

replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 8. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For $K \subset C$ and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty}(x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. *If* $x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on ***C***, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} C_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$ . If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$ , $\mathrm{A}_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A}_i(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A}_i(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i\ \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N_\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^\infty$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp\,(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $\dot{x} = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $\dot{x} = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $\dot{x} = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\mathrm{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.22) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.23) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.24) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a,t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a,t) = 0, & \text{'} \\ \rho_1(0,t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a,t)da, & t>0, \\ \rho_2(0,t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a,t)da, \end{cases}$$ (1.4)

(1.5) $\rho_i(a,s) = \varphi_i(a,s),\ a.0,\ s\epsilon[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a,s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 8. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +} \times [-\tau, T] \to R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +} \times \{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +} \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t,0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right)ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t) = B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0,t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t,p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a,t,p_i(t)da + \int_o^{\infty} \varphi_i(a,0)\ L_i(a,t,p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a,s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$

$$+\int_0^\infty b_i\left(a+t,P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i\,(t)\right)da$$

$$i = 1,\ 2,\ t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*\,(t-a)] \ \le\ 1, if\ \ t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^*\,\beta_i(a)\exp[-d_i^*(t-a)]\,da + \beta_i^*\Phi_i(0)\exp[-d_i^* t],\ \ i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^*\Phi_i(0)\exp[\delta_i t],\ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i\,(t) - 0_i\,(0)| \le \beta(0)\,exp[\delta_i\,a - d_i^*(t-a)]\,da$$

$$+\int_0^\infty \varphi_i\ (a,0)|1-exp(-d_i^* t)|da$$

$$\le \beta_i^*\Phi_i\ (0)\,exp(\beta_i^* t\,)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0)d_i^* t$$

$$\le [\beta_i^*\phi_i(0)\,exp(\beta_i^* t)\,\phi_i(0)d_i^*]t\ \ for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0,T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0,T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then □ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , □ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|□(P_t^{(1)}) - □(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|□(P^{(1)}(t)) - □(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1(a-0, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} \| P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2 / 2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta}\, 1^{a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\exp\left[-\int_o^\infty f_i(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty b_1(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty b_2(a, P_1^*, P_2^*)\exp\left[-\int_o^a f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) = \int_o^\infty b_i(a, P_1, P_2)\exp\left[-\int_o^\infty f_i(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $P_1 \geq 0$, $P_2 \geq 0$,

**And suppose the following hold:**

(3.7) $A_4$. $F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_5$. *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 = F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i$ =1, 2, $a \geq 0$,

Proof Consider the elements of the surfaces defined by

Z= $F_1(P_1, P_2), z = F_2(P_1, P_2), P_1 \geq 0, P_2 \geq 0$,

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

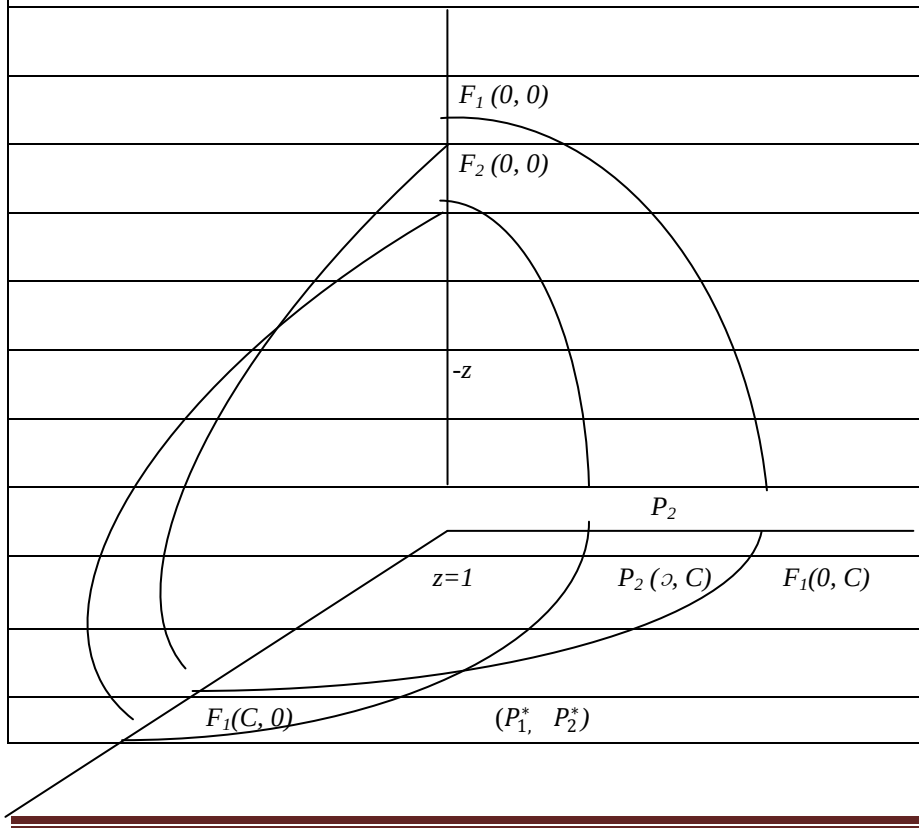
$P_1 \geq 0, P_2 \geq 0$, $i$ =1, 2,

$j$ =1, 2, Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, F_1(P_1, P_2 \geq 0$,

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11)\quad \begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

$$(3.12)\quad \begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)

$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

$$(3.14)\quad \begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{q}(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) q(a)da + \int_0^\infty p_i^*(a) \left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*} p_i^* q\right\} da ,$$

Where $i = 1, 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*, \exp\{-\lambda\tau\}, \; p_{22}^* = p_2^* .$$

If we integrate the first of (3.15) and use the fact that $q(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right) p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right) p_2^* e^{-\lambda\tau}\right] da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right] q(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right) p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right) p_2^*\right] da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right] q(a)da,$$

From the first of (3.15) we get

(3.18) $$q(a) = \left\{q(0) p_i^*(a)/p_i^*(0) \exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a] \int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* q\right\} e^{\lambda s} ds , \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$q(a)\left[p_i^*(a)/p_i^*(0) c_i \exp(\lambda a)\right]\left[\int_0^a p_i^*(a) \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* q\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a) e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* q\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a) e^{-\lambda a} \int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*} p_i^* q\right\} e^{\lambda s} ds , i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{i\varphi} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_\varphi^*} - \frac{\partial f_i}{\partial p_\varphi^*}\right] da,$$

(3.23) $$g_{i\varphi}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \varphi$$
$$\left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_\varphi^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_\varphi^*} e^{\lambda s} ds\right) da, \quad i, \varphi = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 15. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1,$ 2), (2.2)

(xxii) $b_1(0) = 0,\quad m_1(0, x_2) \equiv 0, b_2(0) = 0,\quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xxiii) There exist positive constants $\delta_1$, $\delta_2$ such that

$$b_1(\delta_1) - m_1(\delta_1, x_2) < 0,$$

$$b_2(\delta_2) - m_2(x_1, \delta_2) < 0,$$

for $x_1$ 0, $x_2$ 0, (2.5)

(xxiv) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i, \; j = 1, 2; i \neq j \;)$.

## 16. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \geq 0, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\ (t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\ (t_2)\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\ (t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\ (t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\ (t - \tau_{11}) - m_{12}Y_2\ (t - \tau_{12}),$$

$$\frac{dY_2\ (t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\ (t - \tau_{21}) - m_{22}Y_2\ (t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \qquad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(xxix)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \qquad g_2(0) = 0, \tag{4.3}$$

(xxx) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xxxi) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(xxxii) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 12. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{matrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{matrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} \cdot \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{11}\left|Y_1(t)\right| + b_{11}\left|Y_1(t-\tau_{11})\right| + m_{12}\left|Y_2(t-\tau_{12})\right|, \\ \frac{d\left|Y_1(t)\right|}{dt} &\le -m_{22}\left|Y_2(t)\right| + m_{21}\left|Y_1(t-\tau_{21})\right| + b_{22}\left|Y_2(t-\tau_{21})\right|, \end{aligned}\right\} \quad (A.2)$$

Where

$$\frac{d}{dt}\left|Y_1(t)\right| = \left\{\begin{aligned} &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0, \\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0, \end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(s) > \left|Y_1(s)\right|, \quad s \in [-\tau, 0], \; i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(t) \text{ for } t \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(t) = Z_i(t) - \left|Y_1(s)\right|$, $i = 1,2$. $t \ge -\tau$. Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\ge -m_{11}U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\ge -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(s) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[\tau, 0]$ : It will follow from (A.7) that $U_1(t) \ge U_1(0) \exp[-m_{11}t]$

$$+\int_0^t \left[b_{11}U_1(s-\tau_{11}) + m_{12}U_2(s-\tau_{12})\right] \exp[-m_{11}(t-s)ds > 0,] \qquad (A.9)$$

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11},\tau_{12},\tau_{21},\tau_{22},)$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds+\int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds+\int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0 \leq |\, Y_{1,}(t)\,| \leq Z_1(t) \to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}+\frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22} m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t) \to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t) \to 0, Z_2(t) \to 0$ as $t \to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s \in [0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty s k_{ij}(s)ds < \infty;\ \int_0^\infty k_{ij}(s)ds=1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 22. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 23. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big), \qquad t > 0, \tag{2.1}$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \tag{2.2}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ *in* $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap\left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ ∈ $-\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega+t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(xxii) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(xxiii) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t)or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(xxiv) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$*and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(o)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(p) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 24. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(v) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(w) N is monotonic nondecreasing on $K^*$.

(x) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \|x_1\| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \|x_2\| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \geq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \geq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \geq -\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \geq -\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x^*_1(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21})\, x^*_1(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} - {}_{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(t)\right)\Big| = \min_{t}, \beta/K_2(t)$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(t)\right)\Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality $0< \alpha,\beta<1$, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter Ƭ:

$\frac{dx_1(t)}{dt} x_1(t)[1 - x_1(t) + 0.4x_2(t - \mathcal{T})/k_1(t)]$

$\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$

Where $k_1(t) = 20 + \sin\ (\pi t)$ and $k_2(t)$=2.5+sin (πt)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 15. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \ldots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.8) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \; b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \; (b_j^u / a_{jj}^l)) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \; x_i^l \; for \; t \; t_0, \; i = 1, 2, \ldots, n. \qquad (1.7)$$

16. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \; x_l \; x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1} x_l^l ; \qquad x_*$$

$$= \min_{1} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(155) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(156) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(157) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(158) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(159) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(160) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(161) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(162) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(163) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(164) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(165) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(166) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(167) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (168) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (169) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (170) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (171) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (172) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (173) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (174) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (175) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (176) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like

Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left||f\ (t+\ \tau,x) - \ f(t,x)|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

*$S, \{g(t+,x)\}$ is also uniformly convergent on $\boldsymbol{R} \times S$ then the module of $g(t+,x)$ is contained in the module of $f(t+,x)$.*

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number $\mu$ such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1, 2, \dots, n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1, x_2, \dots, x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t + \tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t + \tau_k, t_0, x_1(t_0)), x_2(t + \tau_k, t_0, x_2(t_0)), \dots, x_n(t + \tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))\ by\ x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{ij}\ (\sigma+\tau_k) - a_{ij}\,(\sigma+\tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k), x(t+\tau_m)\big) \le v\big(x(t_0), x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$$\le [(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] \ + (\varepsilon/2x^*)$$

$$< (\varepsilon/x^*) \text{ On using (2.8), (2.10)} \qquad (2.14)$$

For $m \ge k \ge n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \le x_i(t) \le x^*, i = 1,2,\dots,n; t \ge t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \ge k \ge n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*, x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*, x^*], i = 1,2,\dots,n;$ for $t \ge t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ [illegible] $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ [illegible] $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} |\frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m)| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\left(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k)\right)$$

$$-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big)$$

$$+-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big)]$$

$$=-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=\big(w_1(t),\dots,w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=\big(w_1(t),\dots,w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^*)$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\beta_1 \leq x_i(t) \leq \beta_2, \quad \beta_1 \leq y_i(t) \leq \beta_2, \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V(x(t)y(t)) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t)y(t))$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\,\beta_1 V\left(x(t),y(t)\right) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t),y(t)) \leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \leq \rho_1(t, a_1) \leq \rho_1^*(t, a_1),$$

$$0 \leq Q_1(t) \leq Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})_\tau}{}_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1}da_{}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1}da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1[P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** **d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \; P_3 \; 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \; \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_3 a_3-(f_{30})_3 a_3-\int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3,a_3)da_3\right)u_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0, \qquad (24)$$

$$\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^\infty Y_3(\gamma_3,a_3)\int_0^a (a_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^a(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l\left(\int_0^\infty Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^\infty \beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^l)_3\int_0^\infty Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3} da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{.3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3, \qquad (33)$$

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2\,[a_3^2]_3$<4 $[a_2]_3$<4 $a_2 e$,

Re $\lambda_3[\lambda_3]_3 = \mathrm{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\ ]_3 < 0$.

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then$|n_3(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_3^2$ <4 $a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2]$ ⊠4$[a_2]$e and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

**correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. 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Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1 \log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 64. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \ 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \ \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^{\infty} [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 65. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40_4}^{l}\left(\int_0^{\infty} Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a\mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\,da\right);$$

$$\int_0^{\infty}Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^{\infty}E_4(a_4)e^{-\gamma_4a_4}\,da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^{-}(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4\ [a_2]_4<1\ a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\ a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\ x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4+ Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b)Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\,(\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 a e_4^{mu} [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α~|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 66. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \text{ } 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \text{ } \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 67. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0)$$

$$= \int_0^\infty \beta_6 (a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(3648) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(3649) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(3650) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(3651) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(3652) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(3653) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(3654) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(3655) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(3656) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(3657) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(3658) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(3659) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(3660) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(3661) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(3662) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(3663) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(3664) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(3665) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(3666) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

## Black holes

(3667) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(3668) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(3669) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(3670) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(3671) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(3672) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(3673) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(3674) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(3675) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have**

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| **experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.** |
| (3676) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.** |
| (3677) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.** |

**String theory**

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| (3678) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.** |
| (3679) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.** |
| (3680) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.** |
| (3681) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (3682) **However, when summed the entire series appears to be (=) divergent.** |
| (3683) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (3684) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (3685) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (3686) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (3687) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (3688) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (3689) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |

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| (3690) **The granular properties of space can be (=) explicitly computed.** |
| (3691) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (3692) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (3693) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (3694) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (3695) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (3696) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (3697) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (3698) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (3699) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (3700) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (3701) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (3702) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (3703) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (3704) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (3705) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (3706) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (3707) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (3708) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (3709) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (3710) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (3711) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (3712) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (3713) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (3714) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (3715) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (3716) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (3717) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (3718) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

(3719) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(3720) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

(3721) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.**

(3722) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016**
**DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society**

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(3723) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.**

(3724) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.**

(3725) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.**

(3726) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.**

(3727) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).**

(3728) **Bringing them to thermal equilibrium requires (e) an external bath.**

(3729) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.**

(3730) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).**

(3731) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.**

(3732) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.**

(3733) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—**

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (3734) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (3735) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (3736) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (3737) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (3738) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (3739) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (3740) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (3741) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (3742) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (3743) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (3744) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (3745) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (3746) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (3747) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (3748) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (3749) **This is a realistic scenario for (e) all known systems.** |
| (3750) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (3751) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (3752) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (3753) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (3754) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (3755) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (3756) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (3757) **So in those systems, the quantum phase transition does indeed exist.** |
| (3758) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (3759) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (3760) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (3761) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (3762) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (3763) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (3764) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (3765) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (3766) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (3767) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (3768) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (3769) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (3770) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (3771) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (3772) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (3773) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (3774) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (3775) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (3776) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (3777) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (3778) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (3779) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (3780) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (3781) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (3782) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (3783) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (3784) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (3785) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (3786) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (3787) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (3788) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (3789) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (3790) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (3791) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (3792) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (3793) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (3794) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (3795) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (3796) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (3797) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (3798) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (3799) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (3800) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (3801) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (3802) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (3803) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (3804) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (3805) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (3806) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (3807) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (3808) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (3809) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (3810) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (3811) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (3812) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (3813) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (3814) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (3815) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (3816) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (3817) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (3818) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (3819) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (3820) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (3821) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (3822) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (3823) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (3824) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (3825) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (3826) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (3827) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (3828) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (3829) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (3830) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (3831) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (3832) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (3833) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (3834) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (3835) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (3836) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (3837) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (3838) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (3839) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (3840) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (3841) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (3842) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (3843) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (3844) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (3845) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (3846) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (3847) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

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| (3848) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (3849) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (3850) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (3851) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (3852) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (3853) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (3854) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (3855) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (3856) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (3857) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (3858) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (3859) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (3860) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (3861) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |

(3862) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(3863) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(3864) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(3865) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(3866) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(3867) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(3868) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(3869) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(3870) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(3871) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(3872) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(3873) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(3874) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(3875) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(3876) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (3877) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (3878) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (3879) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (3880) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (3881) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (3882) **We know dark matter exists only from (e) its gravitational effects.** |
| (3883) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (3884) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (3885) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (3886) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (3887) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (3888) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (3889) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (3890) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (3891) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (3892) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (3893) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (3894) | **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (3895) | **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (3896) | **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (3897) | **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (3898) | **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (3899) | **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (3900) | **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (3901) | **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (3902) | **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (3903) | **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (3904) | **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (3905) | **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (3906) | **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (3907) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (3908) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (3909) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (3910) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (3911) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (3912) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (3913) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (3914) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (3915) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (3916) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (3917) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (3918) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (3919) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (3920) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (3921) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (3922) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (3923) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (3924) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (3925) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (3926) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson’s problem**

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| (3927) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking “palm tree” organization.** |
| (3928) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (3929) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (3930) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (3931) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (3932) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (3933) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (3934) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (3935) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (3936) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (3937) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (3938) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (3939) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (3940) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (3941) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (3942) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (3943) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (3944) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (3945) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (3946) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (3947) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (3948) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (3949) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (3950) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (3951) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (3952) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (3953) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (3954) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (3955) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (3956) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (3957) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (3958) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (3959) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (3960) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (3961) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(673) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(674) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(675) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(676) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(677) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(678) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(679) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(680) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(681) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(682) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(683) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(684) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(685) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(686) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(687) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(688) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(689) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

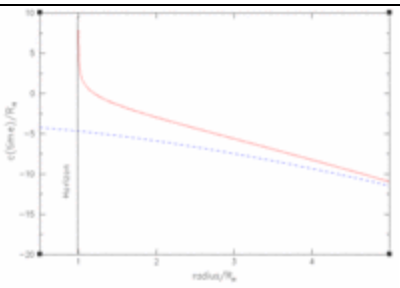
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| **(690) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(691) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(692) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(693) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(694) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(695) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(696) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(697) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(698) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(699) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(700) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(701) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(702) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(703) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(704) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(705) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(706) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(707) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(708) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(709) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(710) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(711) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(712) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(713) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(714) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(715) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(716) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(717) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(718) One of these coordinates is (=) called the "coordinate time."** |
| **(719) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(720) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(721) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(722) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(723) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(724) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(725) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(726) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(727) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(728) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(729) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(730) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(731) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(732) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(733) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(734) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(735) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(736) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(737) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(738) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(739) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(740) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(741) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(742) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(743) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(744) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(745) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(746) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(747) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(748) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(749) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(750) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(751) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(752) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(753) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(754) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(755) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(756) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(757) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(758) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(759) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(760) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(761) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(762) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(763) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(764) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(765) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(766) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(767) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(768) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)([f_2^{'}]_6(z_6))\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{6}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6\left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,\quad h_{6}^{*}$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[f^{'}{}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[f_{2*}^{'}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right)M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$\mathrm{Re}\,\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\, P^*\, [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*} \quad [h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\, [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2} [\, g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2} \quad [h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^2}+\frac{2\omega_6}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6}+(m_6+[\mu_0]_6+\frac{K_6b_6}{\delta_6-b_6}+\sigma_6)\,Q_6^*)^3$$

$$=m_6+[\mu_0]_6+3+\frac{K_6b_6}{\delta_6-b_6}(1+Q^*)(1+\omega_6Q_6^*$$

Now

$$\left[a_6^2\right]-4\left[a_2\right]e=\left(\frac{2K_6b_6}{\delta_6-b_6}\right)^2+\frac{4K_6b_6\sigma_6\,Q_6^*}{(\delta_6-b_6)(1+Q_6^*)}+\frac{\sigma_6^2Q_6^{*2}}{(1+Q_6^*)^2}$$

$$\frac{4\,b_6\sigma_6\,(\delta_6-b)Q^*}{\delta_6(1+Q_6^*)^2}-\frac{4\,(\delta_6-b_6)^2(1-\omega_6)Q_6^*}{\delta_6(1+\omega_6Q_6^*)1+Q_6^*)}P^*(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_6\,ae_6^{mu}\;[\mu_0]_6-\frac{a_6}{a_6+6}$$

Hence if $[a_6^2>4(a_2)_6e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6} = [f_{6*}]_{\,6} + [f_{2*}]_{\,6} + [f_{6*}] \, P_6^*$$

$$[a_2]_{\,6} = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [\, g_*^6]_{\,6} \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_{\,6} \beta_{\,6} \left(\theta_{\,6}(t)\right) - \mu_{\,6}\left(\theta(t)\right),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_{\,6}\right] M_6 - [a_2]_{\,6} W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\qquad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 68. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \ P_7 \ 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \ \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(3962) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(3963) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(3964) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(3965) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(3966) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(3967) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(3968) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(3969) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(3970) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(3971) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(3972) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(3973) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(3974) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(3975) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(3976) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(3977) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(3978) The electronic and (e&eb) structural variability among them is extraordinary.

(3979) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(3980) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(3981) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(3982) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(3983) Nudge your bath tub just right and (eb) you might end up reversing time.

(3984) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(3985) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(3986) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(3987) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(3988) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(3989) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(3990) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(3991) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(3992) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(3993) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(3994) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(3995) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(3996) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(3997) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(3998) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(3999) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(4000) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(4001) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(4002) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(4003) Quantum mechanics rules the dynamics of (e) light and matter.

(4004) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(4005) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(4006) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(4007) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (4008) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (4009) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (4010) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (4011) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (4012) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (4013) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (4014) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (4015) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (4016) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (4017) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (4018) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (4019) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (4020) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (4021) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (4022) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (4023) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (4024) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (4025) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (4026) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (4027) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (4028) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (4029) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (4030) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (4031) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (4032) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (4033) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (4034) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (4035) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (4036) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (4037) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (4038) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (4039) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (4040) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (4041) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (4042) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (4043) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (4044) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (4045) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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|---|
| (4046) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (4047) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (4048) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (4049) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (4050) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |
| (4051) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\,\rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\,\rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 69. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7) e^{-\gamma_7\tau_7} e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds} d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\, da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\,\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_J) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7 \left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}^{'}$ denoting $h_{7}^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7 [a_7^7]2<4 [a_7]2<4 a_7 e$,

$\text{Re}\,\lambda_7\left[\lambda_7\right]_7 = \text{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\ ]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⍰⊿∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{*}(a), \boldsymbol{Q}^{*})$ is asymptotically stable if either $a_7^7 <4 a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⍰$[a_7]e$ and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e}) > 1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7}da = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)$=(1+$\omega_7 x_7$)/(1 + $x_7$)(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])\,{}_7^3\ [\mathrm{K}]_{\,7}[P_0]\,{}_7^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^{*}\,[f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$ $[h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}\,{}_{7*}^{7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}\,{}_*^7] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7}$ $[\mathrm{h}\,{}_*^7] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a_{1}^{1}\right] - 4\left[a_{7}\right] e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7 Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^7 Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7 (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\, (\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae^{mu}_7\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7 Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (699)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (700) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (701)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (702) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

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| (703)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (704) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (705)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (706)Humans are not very good at understanding interacting many-body systems. |
| (707)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (708)But what happens when we couple (e&eb) the particles strongly? |
| (709)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (710)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (711)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (712) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (713) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (714)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (715) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (716)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (717) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (718) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (719)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (720)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (721) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (722) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (723)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (724)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (725)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (726)The energy levels of these quantized orbitals take on (e) discrete values |
| (727):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (728) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (729) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (730)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (731) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (732)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (733) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (734)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (735)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (736)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (737) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (738)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (739)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (740)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (741) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (742)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (743)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (744)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (745)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (746)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (747)Another classic example of duality involves (e&eb) vortices. |
| (748) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (749)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (750) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (751)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (752)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (753) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (754)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (755)TIs are distinguished only by (e) their edge states. |
| (756) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (757)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (758) But the Dirac cone is not (e) the only possible signature. |
| (759) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (760)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (761)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (762)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (763) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (764). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (765)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (766)The bulk duality implies (eb) a duality between edge theories. |
| (767)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |

(768)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids.

(769) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles.

(770) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold.

(771)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables.

(772)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI.

(773)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors.

(774)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field.

(775)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions.

(776)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3.

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

(777) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$.

(778)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration.

(779)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP).

(780) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole.

(781)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process.

(782)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version)

IF we define

$$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,\mathrm{P}^{*}_{7}$$

$$[a_7]_7 = \left[f^{7}_{7*}\right] P^{*}_{7} - \frac{h^{7}_{*}}{h_{*}} P^{*}(0)$$

$$e=[g^{7}_{*}]_7 Q^{*}_{7}$$

$$\eta_7(t) = \left[h_{*}\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 70. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(4052) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(4053) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(4054) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(4055) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(4056) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

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| (4057) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (4058) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (4059) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (4060) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (4061) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (4062) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (4063) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (4064) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (4065) "The underlying physics is (=) chaos, essentially," he said. |
| (4066) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (4067) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (4068) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (4069) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (4070) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (4071) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (4072) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (4073) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (4074) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (4075) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |

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| (4076) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (4077) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (4078) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (4079) The study's findings have fundamental implications for (e) quantum computing. |
| (4080) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (4081) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (4082) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (4083) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (4084) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (4085) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer. |
| (4086) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (4087) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (4088) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (4089) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (4090) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (4091) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |

(4092) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(4093) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(4094) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(4095) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(4096) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(4097) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(4098) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(4099) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(4100) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(4101) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(4102) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(4103) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(4104) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as

contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(4105) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(4106) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(4107) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(4108) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(4109) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(4110) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(4111) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(4112) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(4113) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(4114) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(4115) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(4116) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(4117) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(4118) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(4119) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(4120) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(4121) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(4122) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(4123) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(4124) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(4125) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(4126) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(4127) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(4128) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background

radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(4129) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(4130) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(4131) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(4132) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(4133) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(4134) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(4135) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(4136) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(4137) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(4138) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(4139) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(4140) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(4141) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

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| (4142) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime. |
| (4143) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite. |
| (4144) However, when summed the entire series appears to be (=) divergent. |
| (4145) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime. |
| (4146) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects. |
| (4147) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing. |
| (4148) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory. |

**Loop quantum gravity and spin foams**

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| (4149) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics. |
| (4150) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity. |
| (4151) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime. |
| (4152) The granular properties of space can be (=) explicitly computed. |
| (4153) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory. |
| (4154) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks. |
| (4155) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure. |
| (4156) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit. |
| (4157) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism. |
| (4158) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism. |

**Noncommutative geometry**

(4159) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(4160) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(4161) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(4162) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(4163) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(4164) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(4165) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(4166) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(4167) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(4168) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \ P_8 \ \square 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \square \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \square \infty,$$

$$\rho_8(t, 0) = \int_0^\infty n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly

eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ∞; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big((f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8$$

$$= -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0)$$

$$= \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt}$$

$$= -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8 \; h_8(Q_8) = 1, \qquad and \;\; g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 71. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))u_8 - (\rho_0)_8(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8^{'}(P_0) = f_{80}^{'}$, and $f_8^{'}(0) = f_{80}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \quad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^\infty e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^\infty e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a (\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^\infty Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^\infty Y_8(\gamma_8,a_8)\int_0^a (\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a \mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\,(\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(a_8)da_8-\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1-\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8\,(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8\int_0^\infty E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $[g^{'}]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - [f^{'}_{8*}]_8[p_8^*](a_8)M_8 - [f^{'}_{8*}]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [f^{'}_{8*}]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [f^{'}_{8*}]_8 P_8^*\right)M_8 - [f^{'}_{8*}]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])_8^3\ [\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8-8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, ae_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}]\, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8}\, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8}\, \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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|---|
| (482)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (483)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (484) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (485) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (486)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (487) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (488)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (489)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(490)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(491)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(492) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(493)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(494)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(495)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(496) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(497)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(498)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(499)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(500)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(501)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(502)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(503)Bringing them to thermal equilibrium requires (e) an external bath.

(504) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(505)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(506)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(507)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(508)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(509)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(510) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(511) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(512) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(513)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(514)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(515) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(516)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(517)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(518) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(519)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(520)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (521)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (522) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (523)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (524) This is a realistic scenario for (e) all known systems. |
| (525)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (526)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (527)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (528) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (529)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (530)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (531)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (532)So in those systems, the quantum phase transition does indeed exist. |
| (533)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (534)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant $\Lambda$, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive $\Lambda$ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the $\Lambda>0$ generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins $>+0.6$ and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 72. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$



PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\ a_9)da_9 - \int_0^\infty E_9(\ a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\ a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2 < 4 [a_9] 2 < 4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} \left| h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9) \right|$. Then $|n_9(t)| \ 2 \Delta \forall t \ 0$. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 < 4\, a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]\ 4 [a_9] e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]\ 0.5$. Assume $[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$ and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$ (See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}[\ \mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_9\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_9 - b) Q^*}{\delta_9 (1 + Q_9^*)^9} - \frac{4\,(\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{mu}_9 \quad [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION THREE

## Mixed State Entanglement And Quantum Error Correction

### INTRODUCTION—VARIABLES USED

**Mixed State Entanglement and Quantum Error Correction Charles H. Bennett, David P. DiVincenzo, John A. Smolin, William K. Wootters (Submitted on 23 Apr 1996 (v1), last revised 8 Aug 1996 (this version, v2))**

(1) Entanglement purification protocols (EPP) and quantum error-correcting codes (QECC) provide (eb) two ways of protecting quantum states from (e) interaction with the environment.
(2) In an EPP, perfectly entangled pure states are extracted, with some yield D, from (e) a mixed state M shared by (e) two parties; with (e&*eb) a QECC, an arbi- trary quantum state $|\xi\rangle$ can be transmitted at some rate Q through (e&eb) a noisy channel $\chi$ without degradation.
(3) Authors prove that an EPP involving one- way classical communication and acting on (e&eb) mixed state M^ ($\chi$) (obtained by (e) sharing halves of EPR pairs through (e&eb) a channel $\chi$) yields (eb) a QECC on $\chi$ with rate Q=D, and vice versa.
(4) Authors compare the amount of entanglement E (M) required to (e) prepare a mixed state M by (e) local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal.
(5) While EPPs require (e) classical communica- tion, QECCs do not, and authors prove Q is not increased by (e) adding one-way classical communication.
(6) However, both D and Q can be increased by (e) adding two-way communication.
(7) They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for (e) reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available.
(8) Authors exhibit a family of codes based on (e) universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy.
(9) They also obtain a specific, simple 5-bit single- error-correcting quantum block code. They prove that {\em iff} a QECC results in (eb) high fidelity for the case of no error the QECC can be (=) recast into a form where (e) the encoder is the matrix inverse of the decoder.
(10) Comments: Subjects:Quantum Physics (quant-ph) Journal reference:Phys.Rev.A54:3824-3851,1996 DOI:10.1103/PhysRevA.54.3824 Cite as:arXiv:quant-ph/9604024 (or arXiv:quant-ph/9604024v2 for this version)

### NOTATION

### Module One

Entanglement purification protocols (EPP) and quantum error-correcting codes (QECC) provide (eb) two

| | |
|---|---|
| ways of protecting quantum states from (e) interaction with the environment | |
| $G_{13}$ : Category one of Entanglement purification protocols (EPP) and quantum error-correcting codes (QECC)<br><br>$G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of two ways of protecting quantum states from (e) interaction with the environment<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Entanglement purification protocols (EPP) and quantum error-correcting codes (QECC) provide two ways of protecting quantum states from (e) interaction with the environment | |
| $G_{16}$ : Category one of interaction with the environment<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of Entanglement purification protocols (EPP) and quantum error-correcting codes (QECC) provide two ways of protecting quantum states<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>perfectly entangled pure states in an EPP are extracted, with some yield D, from (e) a mixed state M shared by (e) two parties; with (e&*eb) a QECC, an arbi- trary quantum state $\|\xi\rangle$ can be transmitted at some rate Q through (e&eb) a noisy channel $\chi$ without degradation | |
| $G_{20}$ : Category one of mixed state M shared by (e) two parties; with (e&*eb) a QECC, an arbi- trary quantum state $\|\xi\rangle$ can be transmitted at some rate Q through (e&eb) a noisy channel $\chi$ without degradation<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of perfectly entangled pure states in an EPP are extracted, with some yield D<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by (e) two parties; with (e&*eb) a QECC, an arbi- trary quantum state $\|\xi\rangle$ can be transmitted at some rate Q | |

| | |
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| through (e&eb) a noisy channel χ without degradation | |
| $G_{24}$ : Category one of **perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M;** two parties; with (e&*eb) a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of two parties; with (e&*eb) a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation **;perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation | |
| $G_{28}$ : Category one of QECC<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation | |
| $G_{32}$ : Category one of **entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩;** some rate Q through (e&eb) a noisy channel χ without degradation<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |

| | |
|---|---|
| $T_{32}$ : Category one of some rate Q through (e&eb) a noisy channel χ without degradation **;entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>perfectly entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q through (e&eb) a noisy channel χ without degradation | |
| $G_{36}$ : Category one of **entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q;** noisy channel χ without degradation<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of noisy channel χ without degradation **;entangled pure states in an EPP are extracted, with some yield D, from a mixed state M shared by two parties; with a QECC, an arbi- trary quantum state \|ξ⟩ can be transmitted at some rate Q**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>Authors prove that an EPP involving one- way classical communication and acting on (e&eb) mixed state M^ (χ) (obtained by (e) sharing halves of EPR pairs through (e&eb) a channel χ) yields (eb) a QECC on χ with rate Q=D, and vice versa | |
| $G_{40}$ : Category one of **EPP involving one- way classical communication;** mixed state M^ (χ) (obtained by (e) sharing halves of EPR pairs through (e&eb) a channel χ) yields (eb) a QECC on χ with rate Q=D, and vice versa<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of mixed state M^ (χ) (obtained by (e) sharing halves of EPR pairs through (e&eb) a channel χ) yields (eb) a QECC on χ with rate Q=D, and vice versa; **EPP involving one- way classical communication**<br><br>$T_{41}$ : Category two of SAS | |

$T_{42}$ : Category three of SAS

**Module Nine**

Authors prove that an EPP involving one- way classical communication and acting on mixed state M^ (χ) (obtained by (e) sharing halves of EPR pairs through (e&eb) a channel χ) yields (eb) a QECC on χ with rate Q=D, and vice versa

$G_{44}$ : Category one of EPP involving one- way classical communication and acting on mixed state M^ (χ)

$G_{45}$ : Category two of SAS

$G_{46}$ : Category three of SAS

$T_{44}$ : Category one of QECC on χ with rate Q=D, and vice versa

$T_{45}$ : Category two of SAS

$T_{46}$ : Category three of SAS

**The Coefficients:**

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
$, (a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}\ (a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,
are Dissipation coefficients

**Module Numbered One**

**The differential system of this model is now (Module Numbered one)**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}, t)\right]G_{13} \quad (1)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}, t)\right]G_{14} \quad (2)$$

| | |
|---|---|
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14},t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G,t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G,t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14},t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G,t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17},t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17},t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17},t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17},t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}),t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21},t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21},t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21},t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}),t\right)\right]T_{26}$ | 24 |

| | |
|---|---|
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\big((G_{27}),t\big) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \big[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\big]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \big[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\big]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \big[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\big]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \big[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\big((G_{31}),t\big)\big]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \big[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\big((G_{31}),t\big)\big]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \big[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}\big((G_{31}),t\big)\big]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}\big((G_{31}),t\big) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \big[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\big]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \big[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\big]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \big[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\big]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \big[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}\big((G_{35}),t\big)\big]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \big[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}\big((G_{35}),t\big)\big]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \big[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}\big((G_{35}),t\big)\big]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \big[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\big]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \big[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\big]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \big[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\big]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \big[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}\big((G_{39}),t\big)\big]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \big[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}\big((G_{39}),t\big)\big]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \big[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}\big((G_{39}),t\big)\big]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \big[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\big]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \big[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\big]G_{41}$ | *44* |

$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ *45*

$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ *46*

$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ *47*

$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ *48*

**Module Numbered Nine**

The differential system of this model is now

$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ 49

$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ 50

$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ 51

$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ 52

$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ 53

$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ 54

$+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor**

$-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$ *55*

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$ *56*

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$ *57*

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for

| | |
|---|---|
| 1,2,3<br>$+(a_{40}^{''})^{(8,8)}(T_{41},t)$ $+(a_{41}^{''})^{(8,8)}(T_{41},t)$ $+(a_{42}^{''})^{(8,8)}(T_{41},t)$ are eight augmentation coefficient for 1,2,3<br>$+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)} & -(b_{13}^{''})^{(1)}(G,t) & -(b_{16}^{''})^{(2,2,)}(G_{19},t) & -(b_{20}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,)}(G_{39},t) & -(b_{40}^{''})^{(8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)} & -(b_{14}^{''})^{(1)}(G,t) & -(b_{17}^{''})^{(2,2,)}(G_{19},t) & -(b_{21}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,)}(G_{39},t) & -(b_{41}^{''})^{(8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)} & -(b_{15}^{''})^{(1)}(G,t) & -(b_{18}^{''})^{(2,2,)}(G_{19},t) & -(b_{22}^{''})^{(3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,)}(G_{39},t) & -(b_{42}^{''})^{(8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$ | *60* |
| Where $-(b_{13}^{''})^{(1)}(G,t)$, $-(b_{14}^{''})^{(1)}(G,t)$, $-(b_{15}^{''})^{(1)}(G,t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{16}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3<br>$-(b_{20}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3<br>$-(b_{37}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)} & +(a_{16}^{''})^{(2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,)}(T_{14},t) & +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{bmatrix} G_{16}$ | *61* |

$$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}\ \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17} \quad (62)$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}\ \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}\ \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}\ \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}\ \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}\ \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}\ \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)}\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{13}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{40}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{44}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)}\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{14}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{41}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{45}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)}\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)}\ \boxed{-(b_{15}^{''})^{(1,1,)}(G,t)}\ \boxed{-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)}\ \boxed{-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)}\ \boxed{-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7)}(G_{39},t)}\ \boxed{-(b_{42}^{''})^{(8,8,8)}(G_{43},t)}\ \boxed{-(b_{46}^{''})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b_{16}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{13}^{''})^{(1,1,)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,)}(G,t)}$ are second detrition coefficients for category

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| 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7,)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{ccc} (a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right] G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{ccc} (a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right] G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{ccc} (a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t) & +(a_{18}^{''})^{(2,2,2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{array}\right] G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation | |

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| coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\left[-(b_{20}^{''})^{(3)}(G_{23},t)\right] & \left[-(b_{16}^{''})^{(2,2,2)}(G_{19},t)\right] & \left[-(b_{13}^{''})^{(1,1,1,)}(G,t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{20}$ | *70* |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\left[-(b_{21}^{''})^{(3)}(G_{23},t)\right] & \left[-(b_{17}^{''})^{(2,2,2)}(G_{19},t)\right] & \left[-(b_{14}^{''})^{(1,1,1,)}(G,t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{21}$ | *71* |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}\left[-(b_{22}^{''})^{(3)}(G_{23},t)\right] & \left[-(b_{18}^{''})^{(2,2,2)}(G_{19},t)\right] & \left[-(b_{15}^{''})^{(1,1,1,)}(G,t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{22}$ | *72* |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)}\left[+(a_{24}^{''})^{(4)}(T_{25},t)\right] & \left[+(a_{28}^{''})^{(5,5,)}(T_{29},t)\right] & \left[+(a_{32}^{''})^{(6,6,)}(T_{33},t)\right] \\ \left[+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)\right] & \left[+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)\right] & \left[+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)\right] \\ \left[+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)\right] & \left[+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)\right] & \left[+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)\right] \end{bmatrix} G_{24}$ | *73* |

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)}\boxed{+(a_{25}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{25}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* $1, 2\ 3$

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}\boxed{-(b_{25}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}\boxed{-(b_{26}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

*Where* $\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients*

| | |
|---|---|
| $for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}\ are\ third\ detrition\ coefficients$<br>$for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$<br>$are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$<br>$are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$<br>$are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$<br>$are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$<br>$are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{30}$ | *81* |
| $Where\ \boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}\ are\ first\ augmentation$<br>$coefficients\ for\ category\ 1,2\ and\ 3$<br>$And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation$<br>$coefficient\ for\ category\ 1,2\ and\ 3$<br>$\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ \ augmentation$<br>$coefficient\ for\ category\ 1,2\ and\ 3$<br>$\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation | |

coefficients for category 1,2, 3
$+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)$ are eighth augmentation coefficients for category 1,2, 3
$+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)$ , $+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) & -(b_{32}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) & -(b_{33}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) & -(b_{34}^{''})^{(6,6,6)}(G_{35},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3
$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3
$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3
$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3
$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3
$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3
$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3
$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3
$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a'_{33})^{(6)} & +(a''_{33})^{(6)}(T_{33},t) & +(a''_{29})^{(5,5,5)}(T_{29},t) & +(a''_{25})^{(4,4,4,)}(T_{25},t) \\ +(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a'_{34})^{(6)} & +(a''_{34})^{(6)}(T_{33},t) & +(a''_{30})^{(5,5,5)}(T_{29},t) & +(a''_{26})^{(4,4,4,)}(T_{25},t) \\ +(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t) & +(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{34} \quad (87)$$

$+(a''_{32})^{(6)}(T_{33},t)$, $+(a''_{33})^{(6)}(T_{33},t)$, $+(a''_{34})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{28})^{(5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{24})^{(4,4,4,)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients

$+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b'_{32})^{(6)} & -(b''_{32})^{(6)}(G_{35},t) & -(b''_{28})^{(5,5,5)}(G_{31},t) & -(b''_{24})^{(4,4,4,)}(G_{27},t) \\ -(b''_{13})^{(1,1,1,1,1,1)}(G,t) & -(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b'_{33})^{(6)} & -(b''_{33})^{(6)}(G_{35},t) & -(b''_{29})^{(5,5,5)}(G_{31},t) & -(b''_{25})^{(4,4,4,)}(G_{27},t) \\ -(b''_{14})^{(1,1,1,1,1,1)}(G,t) & -(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b'_{34})^{(6)} & -(b''_{34})^{(6)}(G_{35},t) & -(b''_{30})^{(5,5,5)}(G_{31},t) & -(b''_{26})^{(4,4,4,)}(G_{27},t) \\ -(b''_{15})^{(1,1,1,1,1,1)}(G,t) & -(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t) & -(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34} \quad (90)$$

$-(b''_{32})^{(6)}(G_{35},t)$, $-(b''_{33})^{(6)}(G_{35},t)$, $-(b''_{34})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)} & +(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)} & +(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)} & +(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ & +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3

$+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

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$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}-(b_{36}^{''})^{(7)}(G_{39},t) & -(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{13}$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}-(b_{37}^{''})^{(7)}(G_{39},t) & -(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}-(b_{38}^{''})^{(7)}(G_{39},t) & -(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)$, $-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)$

| | |
|---|---|
| are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *95* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}^{'})^{(8)}\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}^{'})^{(8)}\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | |
| Where $\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3 $\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3 | |

$+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\left[-(b_{40}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\left[-(b_{41}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\left[-(b_{42}^{''})^{(8)}(G_{43},t)\right] & \left[-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)\right] & \left[-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)\right] & \left[-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)\right] & \left[-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)\right] \end{bmatrix} T_{15}$$

Where $-(b_{36}^{''})^{(7)}(G_{39},t)$, $-(b_{37}^{''})^{(7)}(G_{39},t)$, $-(b_{38}^{''})^{(7)}(G_{39},t)$ are first detrition coefficients for category 1, 2 and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)$ are second detrition coefficients for category 1, 2 and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)$ are third detrition coefficients for category 1, 2 and 3

$-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1, 2 and 3

$-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3

$-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$ are sixth detrition coefficients for category 1, 2 and 3

$-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)$, $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)$ are eighth detrition coefficients for category 1, 2 and 3

$-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} -$$

$$\begin{bmatrix} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

| | |
|---|---|
| $(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lvert\lvert G - G^{'}\rvert\rvert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |

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|---|---|
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t) - (a_i^{''})^{(2)}(T_{17},t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t) - (b_i^{''})^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and $(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\widehat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$ | 112 |

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|---|---|
| The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded. | 117 |

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| **Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \le (r_i)^{(4)} \le (b_i^{'})^{(4)} \le (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

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| **Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br>$(a_i^{''})^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br>$(b_i^{''})^{(5)}\big((G_{31}),t\big) \le (r_i)^{(5)} \le (b_i^{'})^{(5)} \le (\hat{B}_{28})^{(5)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$<br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)\rvert \le (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and $(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$ | 127 |

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$

$\lim_{G \to \infty}(b_i^{''})^{(6)}((G_{35}), t) = (r_i)^{(6)}$ 128

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}((G_{35}), t)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

Where we suppose

(WW) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(XX) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(YY)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(ZZ)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}$ are positive constants and $i = 36,37,38$

They satisfy Lipschitz condition: 133

$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$

$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(AAA) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(BBB) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}} [(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$

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|---|---|
| $\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$ | |

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| Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with | |

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0), T_i(0)$ :

$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0), T_i(0)$

$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$

$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$

$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

**Definition of** $G_i(0), T_i(0)$ :

$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

**Definition of** $G_i(0), T_i(0)$ :

$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 152

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| **Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[ (a_{13})^{(1)} G_{14}(s_{(13)}) - \left( (a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{13}(s_{(13)}) \right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[ (a_{14})^{(1)} G_{13}(s_{(13)}) - \left( (a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)}) \right) G_{14}(s_{(13)}) \right] ds_{(13)}$ | |

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| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{16})^{(2)}$ , $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}^{'})^{(2)} + a_{16}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{20})^{(3)}$ , $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to$ | |

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| $\mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |

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| $\bar{T}_{33}(t) = T_{33}^{0} + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^{0} + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{\mathrm{T}}_{38}(\mathrm{t}) = \mathrm{T}_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[ (b_{44})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[ (b_{45})^{(9)} T_{44}(s_{(44)}) - \left( (b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[ (b_{46})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{46}(s_{(44)}) \right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)}\left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$ $\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that $(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |

From which it follows that

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t} \leq \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right] \quad 174$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \leq G_{28}^{0}+\int_0^t\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)} =$$

$$\left(1+(a_{28})^{(5)}t\right)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right] \quad 175$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right] \quad 177$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(i) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 178

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^0)e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^0)e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^0}{G_{37}^0}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \le G_{40}^0+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^0+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)} =$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^0+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right) \quad (180)$$

From which it follows that (181)

$$(G_{40}(t)-G_{40}^0)e^{-(\hat{M}_{40})^{(8)}t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^0)e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^0}{G_{41}^0}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \le G_{44}^0+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^0+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)} =$$

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^0+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^0)e^{-(\hat{M}_{44})^{(9)}t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^0)e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^0}{G_{45}^0}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose (182)

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric

$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{13})^{(1)}t}\}$$

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Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_0^t\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\le$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

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And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i=13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$

From 19 to 24 it results

$$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\ge 0$$

$$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

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| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | 193 |

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| Equations into itself | |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}},\widetilde{T_{19}}$ : $\left(\widetilde{G_{19}},\widetilde{T_{19}}\right)=\mathcal{A}^{(2)}(G_{19},T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{16})^{(2)}\left\|G_{17}^{(1)}-G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)}+$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)-(a_{16}^{''})^{(2)}\left(T_{17}^{(2)},s_{(16)}\right)\right\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)}-(G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t}\le$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+(a_{16}^{'})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)},(T_{19})^{(1)};(G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(2)}-(a_i^{''})^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}ds_{(16)}\right]}\ge 0$ | 200 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)}t)} > 0$ for $t > 0$ | |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1, ((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}')^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)} ((\widehat{M}_{16})^{(2)})_1 / (a_{17}')^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)} ((\widehat{M}_{16})^{(2)})_2 / (a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t), t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t), t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}((G_{19})(t), t) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + ((\widehat{P}_{20})^{(3)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{20})^{(3)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[((\widehat{Q}_{20})^{(3)} + T_j^0)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)} + T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |

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| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})-(a_{20}'')^{(3)}(T_{21}^{(2)},s_{(20)})\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}'')^{(3)}$ and $(b_{20}'')^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(3)}$ and $(b_i'')^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i')^{(3)}-(a_i'')^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$ | 216 |

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| $T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}\,(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$ | 228 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(4)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{24})^{(4)})_1, ((\widehat{M}_{24})^{(4)})_2$ *and* $((\widehat{M}_{24})^{(4)})_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq ((\widehat{M}_{24})^{(4)})_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq ((\widehat{M}_{24})^{(4)})_2 = G_{25}^0 + 2(a_{25})^{(4)} ((\widehat{M}_{24})^{(4)})_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq ((\widehat{M}_{24})^{(4)})_3 = G_{26}^0 + 2(a_{26})^{(4)} ((\widehat{M}_{24})^{(4)})_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}((G_{27})(t),t) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + ((\widehat{P}_{28})^{(5)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |

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| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(5)}$ and $(b_i'')^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$ | 239 |

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| it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(5)} - (a_i'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(5)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1, \left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a_{29}')^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)} \left((\widehat{M}_{28})^{(5)}\right)_1 / (a_{29}')^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)} \left((\widehat{M}_{28})^{(5)}\right)_2 / (a_{30}')^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t), t)) = (b_{29}')^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}'')^{(5)}\left((G_{31})(t), t\right) = (b_{30}')^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |

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| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)}+\left((\hat{P}_{32})^{(6)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on | 249 |

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| $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i')^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}\ ((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33}\ \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t\ > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\widehat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\widehat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\widehat{Q}_{36})^{(7)}\right]\leq(\widehat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval [0, t]<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 260 |

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| suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |

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| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose $(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$ $\sup_i\{\max_{t\in\mathbb{R}_+} \left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+} \left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote <br> **Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results <br> $\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\, ds_{(40)} +$ $\int_0^t\{(a'_{40})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $G_{40}^{(2)}\left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)}, (T_{43})^{(1)}; (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ <br> And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 274 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(8)}t)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results | 279 |

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{47}}), (\widetilde{T_{47}}) : \left((\widetilde{G_{47}}), (\widetilde{T_{47}})\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by

$(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \le -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \le -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \le -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G, t) - (b''_{14})^{(1)}(G, t) \le -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

are defined

Then the solution of global equations satisfies the inequalities 285

$G^0_{13} e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G^0_{13} e^{(S_1)^{(1)} t}$

where $(p_i)^{(1)}$ is defined by equation

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| $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$ $\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}')^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}')^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(T_{17}, t) + (a_{17}'')^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}((G_{19}), t) - (b_{17}'')^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |

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| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$,<br>and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ | 315 |

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| $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}^{'})^{(2)}$<br>$(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)}=(b_{18}^{'})^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)}\le-(a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}-(a_{20}^{''})^{(3)}(T_{21},t)+(a_{21}^{''})^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)}\le-(b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{21}^{''})^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$ :<br>By $(v_1)^{(3)}>0$ , $(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0$ , $(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and<br>By $(\bar{v}_1)^{(3)}>0$ , $(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0$ , $(\bar{u}_2)^{(3)}<0$ the<br>roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)}=(v_0)^{(3)},(m_1)^{(3)}=(v_1)^{(3)}$, ***if*** $(v_0)^{(3)}<(v_1)^{(3)}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)}<(v_0)^{(3)}<(\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)}=\frac{G_{20}^{0}}{G_{21}^{0}}}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)}<(v_0)^{(3)}$ | 321 |

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)},\ \boldsymbol{if}\ (u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}, \boldsymbol{if}\ (u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)},$ and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},\ \boldsymbol{if}\ (\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$

$(p_i)^{(3)}$ is defined by equation

$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ 323

$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$

$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ 324

$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ 325

$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ 326

$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$

$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ 327

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$

$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

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| $-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)} + (b_{25}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27}),t\big) - (b_{25}^{''})^{(4)}\big((G_{27}),t\big) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)},(v_2)^{(4)},(u_1)^{(4)},(u_2)^{(4)},v^{(4)},u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(\nu_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br><br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |

$$T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$$

$$\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t} \quad (335)$$

$$\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b'_{26})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b'_{26})^{(4)}t}\right] + T_{26}^0 e^{-(b'_{26})^{(4)}t} \le T_{26}(t) \le$$

$$\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t} \quad (336)$$

**Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- (337)

Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$

$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$

$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$

$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$

**Behavior of the solutions of equation** (338)

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying

$$-(\sigma_2)^{(5)} \le -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$$

$$-(\tau_2)^{(5)} \le -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$$

**Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : (339)

By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and

**Definition of** $(\bar{v}_1)^{(5)}, , (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : (340)

By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$

**Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :-

If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by

$(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$

$(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$,

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| and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)},$ ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},$ ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)},$ ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{28}(t) \leq G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{29}(t) \leq \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \leq G_{30}(t) \leq$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \leq T_{30}(t) \leq$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** | 349 |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying

$$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$$

$$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}), t\big) - (b''_{33})^{(6)}\big((G_{35}), t\big) \le -(\tau_1)^{(6)}$$

**Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ : 350

By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(6)}, , (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$

**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by

$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,
and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$

and analogously 352

$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,
and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$

Then the solution of global equations satisfies the inequalities 353

$$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$$

where $(p_i)^{(6)}$ is defined by equation

| | |
|---|---|
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G_{34}^0 e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$ *(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}^{'})^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}^{'})^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}^{'})^{(6)}$ $(R_2)^{(6)} = (b_{34}^{'})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$: $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}), t) - (b_{37}^{''})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$: By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)} u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the | 362 |

roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$

and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (v_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (v_0)^{(7)}, (m_1)^{(7)} = (v_1)^{(7)}$, ***if*** $(v_0)^{(7)} < (v_1)^{(7)}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (\bar{v}_1)^{(7)}$, ***if*** $(v_1)^{(7)} < (v_0)^{(7)} < (\bar{v}_1)^{(7)}$,

and $\boxed{(v_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (v_1)^{(7)}, (m_1)^{(7)} = (v_0)^{(7)}$, ***if*** $(\bar{v}_1)^{(7)} < (v_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$$

where $(p_i)^{(7)}$ is defined by equation

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$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$$

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$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le \right.$$

$$\left.\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)}t}\right)$$

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$$\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$

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$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$

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$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$$

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| | |
|---|---|
| $\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | |
| **Definition of** $(S_1)^{(7)},(S_2)^{(7)},(R_1)^{(7)},(R_2)^{(7)}$:-<br>Where $(S_1)^{(7)}=(a_{36})^{(7)}(m_2)^{(7)}-(a_{36}')^{(7)}$<br>$(S_2)^{(7)}=(a_{38})^{(7)}-(p_{38})^{(7)}$<br>$(R_1)^{(7)}=(b_{36})^{(7)}(\mu_2)^{(7)}-(b_{36}')^{(7)}$<br>$(R_2)^{(7)}=(b_{38}')^{(7)}-(r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 371 |
| **Definition of** $(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)}\le-(a_{40}')^{(8)}+(a_{41}')^{(8)}-(a_{40}'')^{(8)}(T_{41},t)+(a_{41}'')^{(8)}(T_{41},t)\le-(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)}\le-(b_{40}')^{(8)}+(b_{41}')^{(8)}-(b_{40}'')^{(8)}((G_{43}),t)-(b_{41}'')^{(8)}((G_{43}),t)\le-(\tau_1)^{(8)}$ | 372 |
| **Definition of** $(\nu_1)^{(8)},(\nu_2)^{(8)},(u_1)^{(8)},(u_2)^{(8)},\nu^{(8)},u^{(8)}$ :<br>By $(\nu_1)^{(8)}>0,(\nu_2)^{(8)}<0$ and respectively $(u_1)^{(8)}>0,(u_2)^{(8)}<0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_1)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_1)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},,(\bar{\nu}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{\nu}_1)^{(8)}>0,(\bar{\nu}_2)^{(8)}<0$ and respectively $(\bar{u}_1)^{(8)}>0,(\bar{u}_2)^{(8)}<0$ the<br>roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_2)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_2)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$<br>**Definition of** $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)}=(\nu_0)^{(8)},(m_1)^{(8)}=(\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)}<(\nu_1)^{(8)}$ | |

| | |
|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , $\boldsymbol{if}$ $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, $\boldsymbol{if}$ $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , $\boldsymbol{if}$ $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}$ $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\right)\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}] + G_{42}^0 e^{-(a'_{42})^{(8)} t})$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})} \left[ e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t} \right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})} [e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}] + G_{46}^0 e^{-(a'_{46})^{(9)}t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})} \left[ e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t} \right] + T_{46}^0 e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left( (a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t) \right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} \right) \le \frac{dv^{(1)}}{dt} \le -\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} \right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \le (\bar{\nu}_1)^{(2)}$$

| | |
|---|---|
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$ | 400 |

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25}, t)\right) - (a_{25}'')^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

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And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$(m_2)^{(4)} \leq v^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{v^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $v^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t) v^{(5)} - (a_{29})^{(5)} v^{(5)}$$

**Definition of** $v^{(5)}$ :- $\boxed{v^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)} v^{(5)} - (a_{28})^{(5)}\right) \leq \frac{dv^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)} v^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$v^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)} (\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right) t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right) t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq v^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get 409

$$\nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)}-(\nu_0)^{(5)}}{(\nu_0)^{(5)}-(\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \le \frac{(\nu_1)^{(5)}+(C)^{(5)}(\nu_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}} \le \nu^{(5)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \le (\bar{\nu}_1)^{(5)} \le \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \le \nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \le \nu^{(5)}(t) \le (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \le u^{(5)}(t) \le (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right) - (a_{33}^{''})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_2)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)\le\frac{dv^{(6)}}{dt}\le-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_1)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)},(v_0)^{(6)}$ :-

For $0<\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}<(v_1)^{(6)}<(\bar{v}_1)^{(6)}$

$$v^{(6)}(t)\ge\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}\quad,\quad\boxed{(C)^{(6)}=\frac{(v_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(v_2)^{(6)}}}$$

it follows $(v_0)^{(6)}\le v^{(6)}(t)\le(v_1)^{(6)}$

In the same manner , we get

$$v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(6)}=\frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)}\le v^{(6)}(t)\le(\bar{v}_1)^{(6)}$

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If $0<(v_1)^{(6)}<(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}<(\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)}\le\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}\le v^{(6)}(t)\le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(\bar{v}_1)^{(6)}$$

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If $0<(v_1)^{(6)}\le(\bar{v}_1)^{(6)}\le\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)}\le v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)}\le v^{(6)}(t)\le(m_1)^{(6)},\quad\boxed{v^{(6)}(t)=\frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)}\le u^{(6)}(t)\le(\mu_1)^{(6)},\quad\boxed{u^{(6)}(t)=\frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

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**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .
Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then
$(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

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In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

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$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get

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$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \le \frac{d\nu^{(9)}}{dt} \le -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \ge \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \le \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \le \nu^{(9)}(t) \le (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \le \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \le \nu^{(9)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \le (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0 \quad 427$$

$$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0 \quad 428$$

$$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0\ , \quad 429$$

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| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$ | 434 |

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| $with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |

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| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\,]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\,]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |

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| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(i) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:** | 488 |

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| (a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that | 494 |

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| there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)] + (b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right]+(b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right]+(b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ | |

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| Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^*$ such that $\varphi((G_{47})^*) = 0$ | |
| Finally we obtain the unique solution<br>$G_{14}^*$ given by $\varphi(G^*) = 0$ , $T_{14}^*$ given by $f(T_{14}^*) = 0$ and<br>$G_{13}^* = \frac{(a_{13})^{(1)} G_{14}^*}{[(a_{13}')^{(1)} + (a_{13}'')^{(1)}(T_{14}^*)]}$ , $G_{15}^* = \frac{(a_{15})^{(1)} G_{14}^*}{[(a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}^*)]}$<br>$T_{13}^* = \frac{(b_{13})^{(1)} T_{14}^*}{[(b_{13}')^{(1)} - (b_{13}'')^{(1)}(G^*)]}$ , $T_{15}^* = \frac{(b_{15})^{(1)} T_{14}^*}{[(b_{15}')^{(1)} - (b_{15}'')^{(1)}(G^*)]}$<br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^*$ given by $\varphi((G_{19})^*) = 0$ , $T_{17}^*$ given by $f(T_{17}^*) = 0$ and | 512 |
| $G_{16}^* = \frac{(a_{16})^{(2)} G_{17}^*}{[(a_{16}')^{(2)} + (a_{16}'')^{(2)}(T_{17}^*)]}$ , $G_{18}^* = \frac{(a_{18})^{(2)} G_{17}^*}{[(a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}^*)]}$ | 513 |
| $T_{16}^* = \frac{(b_{16})^{(2)} T_{17}^*}{[(b_{16}')^{(2)} - (b_{16}'')^{(2)}((G_{19})^*)]}$ , $T_{18}^* = \frac{(b_{18})^{(2)} T_{17}^*}{[(b_{18}')^{(2)} - (b_{18}'')^{(2)}((G_{19})^*)]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}')^{(3)} + (a_{20}'')^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}')^{(4)} + (a_{24}'')^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}')^{(4)} - (b_{24}'')^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}')^{(4)} - (b_{26}'')^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution | 518 |

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| $G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 523 A |

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| $T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}} (T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j} ((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)} G_{16}^{*} \mathbb{T}_{17}$ | 534 |

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| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}')^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}')^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}')^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}')^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}')^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(3)}$ *and* $(b_i'')^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{21}'')^{(3)}}{\partial T_{21}}(T_{21}^{*}) = (q_{21})^{(3)} \quad , \frac{\partial(b_i'')^{(3)}}{\partial G_j}((G_{23})^{*}) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}')^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}')^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}')^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}')^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}')^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}')^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(4)}$ *and* $(b_i'')^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 548 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^* ) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\big((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\big)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\big((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\big)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\big((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\big)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\big((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\big)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\big(s_{(24)(j)}T_{24}^*\mathbb{G}_j\big)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\big((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\big)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\big(s_{(25)(j)}T_{25}^*\mathbb{G}_j\big)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\big((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\big)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\big(s_{(26)(j)}T_{26}^*\mathbb{G}_j\big)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \ , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}(\,(G_{31})^*\,) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\big((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\big)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\big((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\big)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\big((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\big)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\big((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\big)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\big(s_{(28)(j)}T_{28}^*\mathbb{G}_j\big)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\big((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\big)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\big(s_{(29)(j)}T_{29}^*\mathbb{G}_j\big)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\big((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\big)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\big(s_{(30)(j)}T_{30}^*\mathbb{G}_j\big)$ | 562 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ $and$ $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b_i^{''})^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ $and$ $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 586 A |

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| **Proof:** Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \quad , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$ $+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ $+$ | |

$$(\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\Big)$$

$$+\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\Big)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\Big)\} = 0$$

$$+$$

$$(\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\Big)$$

$$+\left((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\Big)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\Big)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\Big]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\Big)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\Big]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\Big)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\Big)$$

$$+\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\Big)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(46)}T^{*}_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and**

| | |
|---|---|
| **this proves the theorem.** | |
| Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk. | |

## Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010) It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and

corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dx_i}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \tag{9}$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output,

creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so

on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 9. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\ (U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C :\ \omega\ (x) \neq \emptyset \text{ and } \omega\ (x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\ (B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\ (x)$ is continuous

**A2:** $A_i\ (x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\ (A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ $(\Gamma(A))$, *and* $\| A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N$ $(\Gamma(A))$. (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $\|B_i(t,x)F_i(x)\| \leq a\, exp(-\beta t)\|B_i(t,x)E_i(x)\|$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in*

$||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In \ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp \ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In \ B_i(t,x) \ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In \ B_i \ (t,x) \ u_i \ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\in C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp \ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In \ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \operatorname{limsup}_{t\to\infty}\frac{1}{t}\,\text{In } |B_i(t,x)x_i| = \operatorname{limsup}_{t\to\infty}\frac{1}{t}\,\text{In } |\,(x.t)_i\,|$$

$$\le \operatorname{limsup}_{t\to\infty}\frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\operatorname{limsup}_{t\to\infty}\frac{1}{t}In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x})$ =lim $_{t\to\infty}$ $\frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\,d\mu(x) - \int_c h(x)d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x)d\mu(x)|$ is

$$\leq |\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)\,d\mu_n(x)| + |\int_c h(x.t)\,d\mu_n(x) - \int_c h(x)\,d\mu(x)|$$

$$\leq \epsilon + \int_c |h(x.t) - h(x^n.t)|\,d\mu_n(x)| + |\int_c h(x^n.t)|\,d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 2\epsilon + |\int_c h(x^n.t) - h(x)\,d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 3\epsilon$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak*

compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\in\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\mu \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t c int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))c *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} A(x) & if\ i \epsilon I \\ A(x) - \frac{\delta}{2Idp\,(x)} & if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t$ $whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \tilde{V}.t \subset intV\ for\ all\ t > 0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K⊂ ∂ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K⊂ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i{}_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j(E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ ($E_1$)and $\lambda_3$ ($E_1$)are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \quad i = 1, \dots, k; \; j = 1, \dots, m$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\mathcal{A}$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $e_i^j = \sum_l d_i^{jl}$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \; \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \quad i = 1, \dots, k$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with \; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & \gamma \\ O & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification

doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration

dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.25) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.26) $D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.27) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & t>0 \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$ (1.4)

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 9. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'}\ - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_1 \text{x}\ k_2, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_2 \text{x}\ k_1, \text{R}^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) \; P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) \, exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

| |
|---|
| $P_1(s)=(p_1(s),p_2(s-t))$, |
| $P_2(s)=(p_1(s-t),p_2(s))$, |
| For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that |
| (2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$ |
| $t > a \geq, i = 1,2,$ |
| Where |
| (2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$ |
| Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows: |
| $M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$ |
| (2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$ |
| It will now follow from (1.1),(1.3),(2.2)-2.5)that |
| (2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$ |
| $t > 0, i = 1,2,$ |
| (2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$ |
| (2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$ |
| $+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$ |
| $i = 1, 2, t > 0$ |
| the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following. |
| THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of* |
| *(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).* |
| *conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are* |
| *defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] . |

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i\,(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i\,(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\pi_1(P_{1t}, P_{2t}), \pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in \left\{ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\} \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)\, da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)\, da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \right\} t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi_1(P_t^{(1)}) - \Pi_1(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi_1(P^{(1)}(t)) - \Pi_1(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+\left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\left. X\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma\right.$$

$$+\int_o^\infty \varphi(\sigma,o)\left[b_1\left(\sigma+a,P_t^{(1)}(a)\right)-b_1\left(\sigma+aP_t^{(2)}(a)\right)\right],L_1\left(\sigma,a,P_t^{(1)}(a)\right)d\sigma$$

$$+\int_o^\infty \varphi(\sigma,o)b_1\left(\sigma+a,P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma,P_t^{(1)}(a)\right)-b_1\left(\sigma+a,P_t^{(2)}(a)\right)\right|\le B_1\ \|P_t^{(1)}-P_t^{(2)}\|_{t^*} \text{ for } a\ \varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)|\le\beta_1\Phi_1(0)||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a[B_1^{(2)}(\sigma)\{b_1(a\text{-}\sigma,P_t^{(1)}(a))\text{-}b_1(a\text{-}\sigma,P_t^{(2)}(a))\}M_1(\sigma,a,P_t^{(1)}(a))$$

$$+b_1(a,\text{-}\sigma,P_t^{(2)}(a))\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0,P_t^{(2)}(a)\right)\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$\le\int_0^a[\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\beta_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}+\beta_1 d_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}(a\text{-}o)]do$$

$$\le\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\|P_t^{(1)}-P_t^{(2)}\|_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left|B_1^{(1)}(a)\text{-}B_1^{(2)}(a)\right|$$

$$\le\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|B_1^{*d}\sigma$$

$$+\Phi_1(0)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}\left\{B_1+B_2 d_1 a+B_1^*\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le\beta_1^*\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|do+\left(r^*+\|\Phi_t\|_{t*}\right)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}g_1^{(a)}$$

Where $g_1(a)\to\beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} \{ g_1(a)+0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a))-b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^* \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\ (P_t^{(1)} - \pi\ \left(P_t^{(2)}\right) \| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*)\ P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty b_i(a,\ P_1^*,\ P_2^*) P_i^*(a) da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o) \exp\left[ - \int_o^\infty f_i(s, P_1^*, P_2^*) ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}(P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i(s,P_1,P_2)ds\right]da,$$

$$i=1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}(0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0) < F_{2.}(C,0),$$

(3.8) $$F_1(0.C) < F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$$

$$i=1,2,\quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0$ *and* , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0,\quad i=1,2,$$

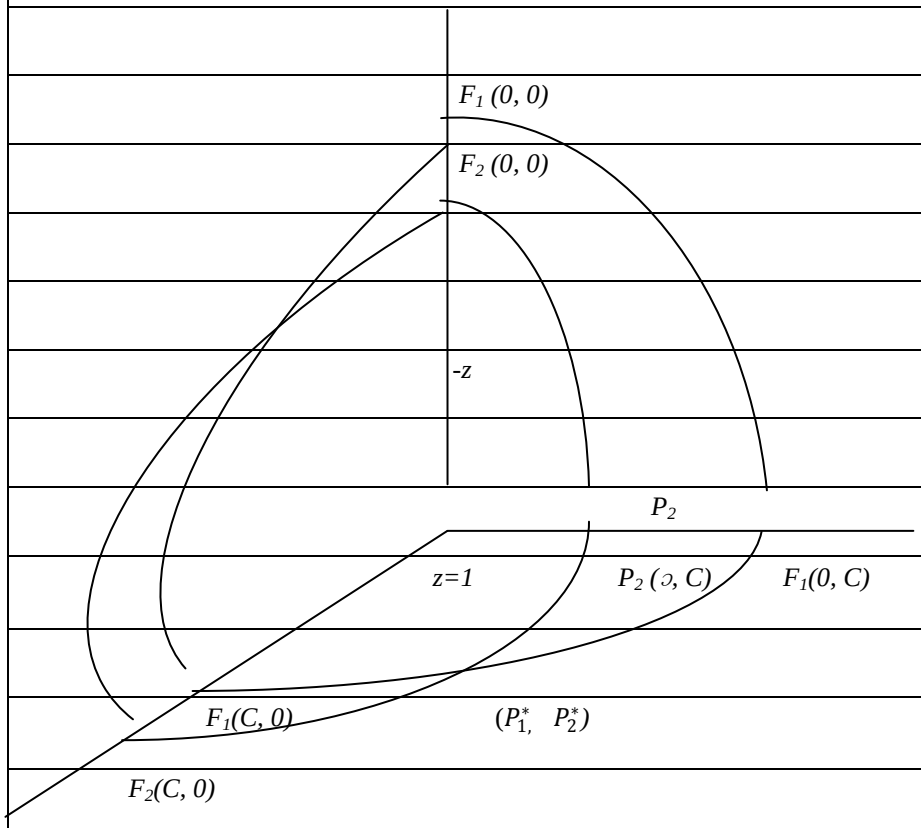
$j=1,2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p^*_1, p^*_2\right)_{\square}(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_{i\square}\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right)_{\square}(a)da + \int_0^\infty p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p^*_j}\, p^*_{i\square}\right\} da,$$

Where $i = 1, 2,$ and

$$p^*_{11} = p^*_1, \quad p^*_{12} = p^*_2 \exp\{-\lambda\tau\},$$

$$p^*_{21} = p^*_1 \exp\{-\lambda\tau\}, \quad p^*_{22} = p^*_2.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i(a, p_1^*, p_2^*)\, p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i(a, p_1^*, p_2^*)p_i^*(a)e^{-\lambda a}da, \quad i = 1, 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^*\exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^*\exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{i} = \int_0^{\infty} p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^{\infty}\sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^{\infty}\left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^{\infty}\sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)da,\ i, \square = 1,\ 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a,\ p_1^*, p_2^*\right) - f_i\left(a,\ p_1^*, p_2^*\right),\ i,\ = 1,\ 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - \left(A_{11} + g_{11}\langle\lambda\rangle\right) & -\left(A_{12} + g_{12}\right)e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right)e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}\langle\lambda\rangle\right)\end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21}\exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda,\tau) = \left[A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)\right]\exp\left[-2\lambda\tau\right]$

$$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ = 1,\ 2.)\ \left(see(A_1)\ and\ (A_2)\right)$;

$A_7.\ \left\{A_{11} + A_{22} + \|\,S_1(0)\,\|\right\}^2 < 4\left\{(A_{11}A_{22} - A_{12}A_{21}) - \|\,S_2(0)\,\|\right\}$,

$$\|\,S_1(0)\,\| = |\,g_{11}(0)| + |\,g_{22}(0)|\,,$$

$$\|\,S_2(0)\,\| = |\,A_{11}\,g_{12}(0)| + |\,A_{12}\,g_{21}(0)| + |\,g_{12}(0)\,g_{21}(0)|$$

$$+ |\,A_{11}\,g_{22}(0)| + |\,A_{22}\,g_{11}(0)| + |\,g_{11}(0)\,g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a,\ t) = \square(a)\exp\left[\lambda t\right] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda,\tau)$ and $F_2(\lambda,\tau)$ as follows:

$F_1(\lambda,\tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau];$

$F_2(\lambda,\tau) = \lambda S_1(\lambda) + S_2(\lambda,\tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda,\tau) - P_2(\lambda,\tau) \right|$$

$$\geq \left| F_1(\lambda,0) \right| - F_2(\lambda,\tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0,\tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0,T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda,\tau) = F_2(\lambda,\tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany

mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational.

It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection.

Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which

paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

# Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure

has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 17. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N≥0 then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \quad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) \right\} F(N(s))\, ds$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^1 f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that

"time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)\ (dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.} \tag{1.5}$$

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\big(N(t-\tau)\big) - m(N(t)) \tag{1.6}$$

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \tag{1.7}$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive

and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ *(i* = 1, 2) are continuous with continuous partial derivatives for all $x_1$ *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ *(i, j* = 1, 2), (2.2)

(xxv) $b_1(0) = 0, \quad m_1(0, x_2) \equiv 0, b_2(0) = 0, \quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xxvi) There exist positive constants $\delta_1$, $\delta_2$ such that

$$b_1(\delta_1) - m_1(\delta_1, x_2) < 0,$$

$$b_2(\delta_2) - m_2(x_1, \delta_2) < 0,$$

for $x_1 \geq 0, \quad x_2 \geq 0,$ (2.5)

(xxvii) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha, \beta) = 0 = b_2(\beta) - m_2(\alpha, \beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the ($x_1$ , $x_2$)- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\frac{dx_1(t)}{dt} = r_1 x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),$$

$$\frac{dx_2(t)}{dt} = r_2 x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),$$

(2.7)

Where $r$, $a_{ij}$ , $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i$ (0) > 0, $x_2$ (0)> 0 does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1$, $X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\frac{dX_1(t)}{dt} = \beta_{11} X_1(t) + \beta_{12} X_2(t),$$

$$\frac{dX_2(t)}{dt} = \beta_{21} X_1(t) + \beta_{22} X_2(t), \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda\,(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply $a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1\ (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1\ (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J\ (i,\ j = 1, 2; i \neq j)$.

## 18. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i,j=1,2)$ be a set of nonnegative constants with $\tau = \max\ \tau_{ij}\ , i,j=1,2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t-\tau_{11})\big) - m_1\big(x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t-\tau_{22})\big) - m\big(x_1(t-\tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i=1,2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s)\ \ 0, \qquad s\ \in [-\tau, 0], \qquad i=1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\{\psi_1^{(n)}(s)\}, s\in[-\tau,0], i=1,2; (n=1,2,\dots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s\in[-\tau,0], \qquad i=1,2 \qquad (3.3)$$

Let $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ be the solution of (3.1) corresponding to the initial condition

$$\{x_1^{(n)}(t), x_2^{(n)}\ (t)\} = \{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\}, \quad s\in[-\tau,0],. \qquad (3.4)$$

Consider the solution $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0,\tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i,j=1,2))$ for which

$$x_1^{(n)}(t^*) = 0 \ and \ x_1^{(n)}(t)(t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \qquad (3.5)$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0 \ on \ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [2\tau^*, 3\tau^*] \ etc$. Thus it will follow that so long as $(x_1^{(n)}(t), \ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t) \geq 0 \ x_2^{(n)}(t) \geq 0.$$

If we now consider the limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t), \ x_2^{(n)}(t)\right\} = \{x_1(t), x_2(t)\} \qquad (3.2)$$

With $x_1(t)$ , $x_2(t)$ where $(x_1(t), x_2(t))$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1 \ or \ x_2$ we have

$$\lim_{t\to t_1-} x_i(t) = \infty. \qquad (3.7)$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i(t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1(x_1(t_2 - \tau_{11})) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(x_1(t_2)) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(\delta_1) - m_1(\delta_1, x(t_2 - \tau_{12})) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ .

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1(t-\tau_{21}) - m_{22}Y_2(t-\tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i} \; at \; (\alpha, \beta). \tag{3.10}$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ 0 $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= f_1(y_1(t)) - g_1(y_1(t), y_2(t)), \\ \frac{dy_2(t)}{dt} &= f_2(y_1(t)) - y_2(t) - g_2(y_2(t)), \end{aligned}\right\} \tag{4.1}$$

Where $f_i \; and \; g_i \; i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i \; and \; g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i \; and \; g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0 \; (i = 1, 2); also \; we \; require$

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \; 0; \quad \frac{\partial g_1}{\partial y_2} \; 0, \\ &\frac{\partial f_2}{\partial y_1} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0, \end{aligned}\right\} \text{Where } y_1 \; 0, y_1 \; 0, \tag{4.2}$$

(xxxiii)

$$\left.\begin{aligned} f_1(0) &= 0, \quad g_1(0, y_2 \equiv 0, \\ f_2(0, y_2) &= 0 = f_2(y_1, 0); \quad g_2(0) = 0, \end{aligned}\right\} \tag{4.3}$$

(xxxiv) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xxxv) There exists a point $(\alpha^* \beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$\left.\begin{aligned} f_1(\alpha^*) - g_1(\alpha^*, \beta^*) &= 0, \\ f_2(\alpha^*, \beta^*) - g_2(\beta^*) &= 0, \end{aligned}\right\} \tag{4.5}$$

(xxxvi) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \geq 0,$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \quad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_l, a_{ij}\ (i, j = 1,2)$ are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t \geq 0$ for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all $t \geq 0$ let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l(0)$ satisfy $0 < y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0) \geq \eta_l and\ y_2(0) \geq \eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t > 0,$ ). Suppose $y_l(t)$ does not exist for all $t \geq 0$; then for some $t_l < \infty\ lim_{t \to t_l -} y_l(t) = \infty$ and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_{1,} y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t \geq 0$; if $y_2(t)$ is not defined for all $t \geq 0$, then for some $t_3 \geq \infty$ we will have $lim_{t - t_3 -} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2) \leq f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t \geq 0$.

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\frac{dU_1(t)}{dt} = \frac{\partial f_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_2}U_2(t),$$

$$\frac{dU_2(t)}{dt} = \frac{\partial f_2}{\partial y_1}U_1(t) - \frac{\partial f_2}{\partial y_2}U_2(t) - \frac{\partial g_2}{\partial y_2}U_2(t), \qquad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

Where $f_{ij} = \frac{\partial f_1}{\partial y_l}$, $g_{ij} = \frac{\partial g_1}{\partial y_j}$, at $(\alpha^*, \beta^*)$ (4.8)

$(i,j = 1, 2)$

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$

(4.9)

$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}.$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$

$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \qquad \textit{Evaluated at } (\alpha^*, \beta^*), \qquad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 13. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty\ ]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.2)$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t),$$

$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_2(t)N_1(t)\ , \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\frac{dN_1(t)}{dt} = rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t),$$
$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_1(t)N_2(t). \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\frac{dN_1(t)}{dt} = a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t),$$
$$\frac{dN_2(t)}{dt} = a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$
$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^{\infty} \delta(a)N_2(t-a)e^{-d*a}\,da,$$
(5.6)

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$
$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)),$$
(5.7)

Where $\tau_{ij}$ $(i,j=1,2)$ are nonnegative constants with $\tau=\max\tau_{ij}, i,j=1,2$; the birth and death rate functions $f_1,f_2,g_1,g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s)=\phi_1(s)>0,\quad s\in[-\tau,0].$$

$$N_1(s)=\phi_1(s)>0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1}>0,\frac{\partial f_2}{\partial N_2}>0, f_2(N_2,N_2)>0 \text{ whenever } N_1>0,\ N_2>0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s)>0$ and $N_2(s)>0$ for $s\in[-\tau,0]$, solution of (5.7) exist for all $t\geq 0$ and remain nonnegative for all $t\geq 0$. also $(\alpha^*,\beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*,\beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state $(\alpha^*,\beta^*)$ is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt}=-m_{11}Y_1(t)+b_{11}Y_1(t-\tau_{11})-m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt}=-m_{22}Y_1(t)+m_{21}Y_1(t-\tau_{21})-b_{22}Y_2(t-\tau_{22}),\quad t>0, \quad (6.1)$$

Where$m_{1J}, b_{i1}$ $(i,j=1,2)$ are positive constants while $\tau_{1J}(i,j=1,2)$ are non negative with $\tau=\max(\tau_{1J})(i,j=1,2.$as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt}=-m_{11}Z_1(t)+b_{11}Z_1(t-\tau_{11})-m_{12}Z_2(t-\tau_{12}),$$

$t > 0,$ (6.2)

$$\frac{dZ_2(t)}{dt} = -m_{22}\, Z_1(t) + m_{21} Z_1(t-\tau_{21}) - b_{22} Z_2(t-\tau_{22}),$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \ \ for\ s \le 0 \ \ for\ which\ Y_i(s)\ is\ defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad \big|\big|Z\big|\big| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \big|\big|B\big|\big| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_j [b_{jj} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jj} | ]$ $B=(b_{1j})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \le [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \le \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm ‖B‖ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \ or \ -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix}\lambda+m_{11} & 0\\ 0 & \lambda+22\end{bmatrix}-b_{11}^{e-\lambda T}{}_{11}\begin{pmatrix}-m_{12}^{e-\lambda T}{}_{12}\\ -m_{21}^{e-\lambda T}{}_{21} \quad b_{22}^{e-\lambda T}{}_{22}\end{pmatrix}$$

$=\lambda I-A-B(\lambda,T_{ij})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\boldsymbol{\alpha}<0.$

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\boldsymbol{\alpha}>0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\boldsymbol{\alpha}$ in (6.8)and define

$G(z-\boldsymbol{\alpha},T_{IJ})=M(z)=(z-a)I-A-B(z-\boldsymbol{\alpha},T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$Re(\lambda)<-a$

is the following :

$-\mu(aI+A)>\|B_1(a,T_{ij})\|,$

Where

$$B_1(a,T_{ij})=\begin{pmatrix}b_{11}^{e-\lambda T}{}_{11} & -m_{12}^{e-\lambda T}{}_{12}\\ -m_{21}^{e-\lambda T}{}_{21} & b_{22}^{e-\lambda T}{}_{22}\end{pmatrix}$$

Since $\mu(\boldsymbol{\alpha}\, I+A)=\boldsymbol{\alpha}+\boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\boldsymbol{\alpha}+\mu(A)+\|B_I(\boldsymbol{\alpha},T_{IJ})\|<0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as \; \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$$f(0, T_{Ij}) < 0 \;\; for \; \mathfrak{a} \in [0, \mathfrak{a}^{*})$$

And hence

$$f(\mathfrak{a}, T_{Ij}) = 0$$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root ofmax $b_{11}^{e-\lambda T}{}_{11} - m_{12}^{e-\lambda T}{}_{21} + \mathfrak{a}$-min $. m_{11} =0$

$b_{22}^{e-\lambda T}{}_{22} - m_{21}^{e-\lambda T}{}_{12} \qquad m_{22}$

Where the maximum in (6.18) is taken over $b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij} (i, = j \; 1,2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}. \qquad (6.19)$$

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^{*}$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^{*}$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more

species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows

from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement "all animal populations have the reproductive capacity to increase without limit, yet none di" and in his concluding remarks writes " this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated" More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution "Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist." In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self – regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^*\ y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^*\ y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta\, xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of

predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned}\frac{d|Y_1(t)|}{dt} &\le -m_{11}\,|Y_1(t)| + b_{11}|Y_1(t-\tau_{11})| + m_{12}\ Y_2(t-\tau_{12})|,\\ \frac{d|Y_1(t)|}{dt} &\le -m_{22}\,|Y_2(t)| + m_{21}|Y_1(t-\tau_{21})| + b_{22}\ |Y_2(t-\tau_{21})|,\end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\,|\,Y_1(t)\,| = \left\{\begin{aligned}&\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0,\\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0,\end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned}\frac{dZ_1(t)}{dt} &= -m_{11}\ Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}\ Z_2(t-\tau_{12}),\\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}\ Z_2(t-\tau_{22}),\end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$Z_1(s) > |Y_1(s)|, \quad s \in [-\tau, 0], i = 1,2.$ (A.5)

Then

$|Y_1(t)| < Z_1(t)$ for $t \geq 0$ and $i = 1,2.$ (A.6)

PROOF. Let $U_1(t) = Z_i(t) - |Y_1(s)|, \quad i = 1,2.$ $t \geq -\tau.$ Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\geq -m_{11}\ U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}\ U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\geq -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}\ U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad \text{(A.7)}$$

With

$U_1(s) = Z_i(s) - |Y_i(s)| > 0, \quad i = 1,2, s \in [-\tau, 0].$ (A.8)

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \geq U_1(0) \exp[-m_{11}t]$

$+\int_0^t [b_{11}U_1(s-\tau_{11}) + m_{12}\ U_2(s-\tau_{12})] \exp[-m_{11}(t-s)ds > 0,]$ (A.9)

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t)$

$+\int_{1-T}^t b_{11}\, Z_1(s)ds + \int_{1-T}^t m_{12}\, Z_2(s)ds$

$+\int_{1-T}^t m_{21}\, Z_1(s)ds + \int_{1-T}^t b_{22}\, Z_{22}(s)ds$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$0 \leq |Y_{1,}(t)| \leq Z_1(t) \to 0$ as $t \to \infty$,

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$= -[Z_1(t)\{m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{m_{22} - (b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\rightarrow 0$ as $t \rightarrow \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\rightarrow 0, Z_2(t)\rightarrow 0$ as $t\rightarrow \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s), s\in[0,\infty), i,j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$\int_0^\infty sk_{ij}(s)ds< \infty;\ \int_0^\infty k_{ij}(s)ds=1$, I,j=1,2.

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i,j = 1,2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where $d|Y_1(t)||/dt$ is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \ Z_i(t) \to \infty \leq as\ t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\left\{\begin{matrix} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{matrix}\right\}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+ m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

$+ m_{21}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+ b_{22}\int_0^\infty k_{22}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12))

$$\left(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]\right.$$

$< 0$(using (A.18)) for $Z_1(t) > 0\ and\ Z_2(t) > 0.$

By Lemma 6 below the result will follow

Lemma 6. Let $H(t)=H(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose

$$m_{11} > b_{11} + m_{21} \text{ and } m_{22} > b_{22} | m_{12}$$

Then

$$Z_1(t) + Z_2(t) \to 0 \text{ast} \to \infty.$$

PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence

$\{t_n\} \to \infty$ as n $\to \infty$ such that

$$[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon \qquad (A.22)$$

For some sufficiently small positive number $\varepsilon$ and hence

$$\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, \ldots.$$

By the continuity and differentiability of H with respect to t,

$$\mathrm{H}(t_n - \varepsilon) - \mathrm{H}(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$$

If $\varepsilon$ is small we will have

$$= \frac{dH(t^*)}{dt}\varepsilon/2$$

So that

$$\mathrm{H}(t_n) \leq \mathrm{H}(t_n - \varepsilon) - \varepsilon^2/2$$

$$\leq H(t_{n-1}) - \varepsilon^2/2$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2)$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2):$$

$$\leq H(t_0) - \mathrm{n}(\varepsilon^2/2): \mathrm{t}_0 = 0.$$

(A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$)

We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$

Where A and B are defined as in (6.3).

We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems

**Gopalsamy**

# Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 25. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific

interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 26. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0, \qquad \textbf{(2.1)}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

($H_2$) $f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

($H_3$) There exist positive constants $b_{il}$, $b_i^u$, $a_{il}$, $a_i^u$ ($l = 1, 2$) such that

$$b_{il} - a_{il} x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u / a_i^u, \qquad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} | h(t)|, h \in B_\omega^\tau \right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b_l = \min(b_{1l} / a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1 in\ \bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \; (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{matrix} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{matrix} \quad \text{for t> } 0, \qquad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t-s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+ \int_0^1 G_{12}(m, 0, t-s) g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) ds,$$

$$N_2(t, x_1, x_2) = \int_0^\omega G_{22}(m, \omega, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds \qquad \textbf{(2.6)}$$

$$+ \int_0^1 G_{21}(m, 0, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds,$$

For $t > 0$, with

$$g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) = mx_1(s) + x_1(s) f_1\left(s, x_1(s), x_2(s - \tau_{12})\right),$$

**(2.7)**

$$g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) = mx_2(s) + x_2(s) f_2\left(s, x_1(s - \tau_{21}) x_2, (s)\right),$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

**(2.8)**

$$G_{ij}(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2,$$

$$N(x_1,x_2)(t) = \begin{cases} N_1(n\omega+t,x_1,x_2) \\ N_2(n\omega+t,x_1,x_2) \end{cases} for\ t \in [-\tau,0], \qquad \textbf{(2.9)}$$

*m* being a suitably selected positive constant (more will be said below) while *n* is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ [illegible] and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ [illegible] then $N(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ [illegible] $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$, $t$ [illegible] $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = (x_1(t),x_2(t)) \quad and\ (x_1,x_2) \in K \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1,x_2)$ assuming that $N(x_1,x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1,x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1,x_2)(t)$. It will follow from (2.9) and the assumed periodicity of $N(x_1,x_2)$ $that\ N(x_1,x_2)$ is continuous

For $t$ [illegible] $-\tau$. Now we will verify the periodicity of $N(x_1,x_2)$ in $t$. A simple computation leads to

$$N_1(t+w,x_1,x_2) - N_1(t,x_1,x_2) = e^{-mt}F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m,\omega,t-s)\,g_1(m,s,x_1(s),x_2(s-\tau_{12}))ds$$

$$+\int_0^{\omega} G_{12}(m,0,t-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_{\omega}^{\omega+t} G_{12}(m,0,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$-\int_0^{\omega} G_{11}(m,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_0^{t} G_{12}(m,0,-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B,$ thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m,t,x_1(t),x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m,t,x_1(t-\tau_{21}),x_2(t)\right),$$

For any arbitrary real constant $m$ and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

| | |
|---|---|
| (xxv) | Either $x_1, x_2$ or both $x_1\ and\ x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1\ and\ x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1\ and\ x_2$ cannot be periodic. |
| (xxvi) | While $(x_1, x_2)\ \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0\ or\ dx_2(t^*)/dt < 0,$ which will imply that the values of $x_1(t^*)\ or\ x_2(t^*)$ cannot be repeated from below. |
| (xxvii) | Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}\ or\ 0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above. |

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1) and\ \lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}.$ *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq \alpha_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq \alpha_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda\, x_2\,(t) f_2(t, x_1(t-\tau_{21}), x_2\,(t)) + x_2\,(t) m(\lambda - 1)$$

For all $t$ such that $x_1(t) = b_1^u/a_1^u$ or $x_2(t) = b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ or $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \geq x_1^* < b_1^u/a_1^u$, $x_2(t) \geq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}\,(m, \omega, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}\,(m, 0, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds. \quad \textbf{(2.14)}$$

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for *m* for the case $N_1(t, x_1, x_2) \geq 0$ and then choose *m* = max $(m_1, m_2)$. Thus we have

$$N: K \cap (G_2/G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\,\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \rightarrow B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} \; for \;\; t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\, t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \rightarrow K$ be completely continuous and which satisfies the following conditions:*

***(q)*** *There exists* $(z_1, z_2) \in K \subset B, \|(z_1, z_2)\| = 1$ *and a* $\mu$ *satisfying*

$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(r) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that* $f_1, f_2$, *of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 27. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq$ *x1,x2≤C∗ where C∗some positive constant is. Assume the following:*

(y) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(z) N is monotonic nondecreasing on $K^*$.

(aa) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda(1 + \eta) N(x_1, x_2)$. Then N has at most one fixed point different from $(0,0)$ in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \geq \|x_1\| \geq b_1^u/a_1^u,\ b_{2l}/a_{2l} \geq \|x_2\| \geq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \quad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$\times \{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \quad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2(t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \geq 0 \text{ and } 0 \not\equiv x_2(s) - y_2(s) \geq 0;$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2) - N_2(t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$N(x_1,\ x_2)\ -N\ (y_1,y_2)$ for $(x_1-y_1,\ x_2-y_2) \in\ K^*-\{(0,0)\}$.

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0<\lambda<1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta)\ N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$N_1\ (t,\ \lambda x_1,\ \lambda x_2)-\lambda N_1\ (t,x_1,\ x_2)$

$=\lambda\int_0^{\omega} G_{11}\ (m,\omega,t-s)g_1\big(s,x_1(s),\ x_2(s-\tau_{12})\big)H_1(s,\lambda,\ x_1,\ x_2)ds$

$+\lambda\int_0^{t} G_{11}\ (m,0,t-s)g_1\big(s,x_1(s),\ x_2(s-\tau_{12})\big)H_1(s,\lambda,\ x_1,\ x_2)ds$

**(3.6)**

Where

$$H_1(s,\lambda,\ x_1,\ x_2)=\frac{m\ x_1(s)+x_1(s)f_1\big(s,\lambda\ x_1(s),\lambda\ x_2(s-\tau_{12})\big)}{m\ x_1(s)+x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big)}-1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s,\lambda,\ x_1,\ x_2)\ \eta_1$ for $0<\ \lambda<1, s>0, (x_1,\ x_2)\ \in\ K^*$, **(3.8)**

And hence

$N_1(t,\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta_1)N_1(t,x_1,\ x_2)$.

A similar argument will show that there exists an $\eta_2>0$ such that

$N_2(t,\lambda x_1,\ \lambda x_2)\ \lambda(1+\eta_2)N_2(t,x_1,\ x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ }?\text{ }-\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ }?\text{ }-\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ }?\text{ }-\tau, \tag{4.1}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t)X_1(t) - a_{12}(t)x_2^*(\mathrm{t} - \tau_{12})\, X_2(\mathrm{t} - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t)x_1^*(t - \tau_{21})x_1(t - \tau_{21}) - a_{22}(t)x_2^*(t)x_2(t), \tag{4.2}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t} - \tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t - \tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(t))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \tag{4.4}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(t-\tau_{12})|X_2(t-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(t-\tau_{21})|X_1(t-\tau_{12})| - b_{22}x_2^*(t)|X_2(t)|, \end{aligned} \tag{4.5}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \tag{4.6}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ of $(2.1) - (2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12}, \tag{4.7}$$

*Where the constants* $b_{ij}$ $(i, j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x_2^*(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x_1^*(u)|X_{1,}(u)|\mathrm{du}. \quad \textbf{(4.8)}$$

Since $x_1^*, x_2^*$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x_1^*(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x_2^*(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}. Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(\,|b_{11}| - b_{21})\, x_1^*(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x_2^*(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - v(t_m) \leq -\int_{t_m-\varepsilon}^{t_m} \frac{dv(T)}{dT} d\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$v(t_m - \varepsilon) - v(t_m)$ $\varepsilon^2/2$ and hence

$v(t_m)$ $v(t_m - \varepsilon) - (\varepsilon^2/2)$

$v(t_m - 1)$-$(\varepsilon^2/2)$

$v(t_m - 2)$-$(\varepsilon^2/2)$

$v(t_0)$-m $(\varepsilon^2/2)$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v$ $for$ $t$ 0.

this contradiction shows that $|X_{1,}(t)| + |X_2(t)| \to 0$ as $t \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1(t, x_1(t), x_2(t - T_{12}))1 \quad \left\{ \frac{x_1(t) + ax_2(t - T_{12})}{K_1(t)} \right\}$$

$$f_2(t, x_1(t - {}_{21})x_2(t)) = 1 - \left\{ \frac{\beta x_1(t - T_{21}) + x_2(t)}{K_2(t)} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left| \text{axm}_{t>T}\{-a_{11}(t)\} \right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}(-a_{21}(t)) \Big| = \min_{t} , \beta/K_2(t)$$

$$\max_{t T}\{-a_{22}(t)\} \Big| = \max_{t T}(1/K_2(t)) > \min_{t T}(-a_{12}(t)) \Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality 0< $\alpha,\beta$<1, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a

constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_1\left(\mathrm{t},x_1(\mathrm{t}),x_2(\mathrm{t}-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_2(\mathrm{t},x_1(\mathrm{t}-T_{21}),x_2(\mathrm{t}))$$

Since $\left(x_1(\mathrm{t}),x_2(\mathrm{t})\right)\rightarrow\left(\mathrm{x}_1^*(\mathrm{t}),\mathrm{x}_2^*(\mathrm{t})\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(t)}{dt}x_1(t)[1-x_1(t)+0.4x_2(t-\mathcal{T})/k_1(t)]$$

| |
|---|
| $\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$ |
| Where $k_1(t) = 20 + \sin\,(\pi t)$ and $k_2(t)$=2.5+sin (πt) |
| @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ |

There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of

"differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction,

projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

### Abstract

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction

rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 17. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

t>$t_0$;

$t_0 \in (-\infty,\infty); i = 1,2,\dots,n,$ (1.1)

where $b_1, a_{ij}$ $(i,j = 1,2,\dots.,n)$ are nonnegative almost periodic functions defined for $t \in (-\infty,\infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i,j = 1,2,\dots,n)$ by the following:

$$\inf_{t\in R} b_l(t) = b_i^l; \qquad \sup_{t\in R} b_l\,(t) = b_l^u, \qquad i,j = 1,2,\dots,n.$$

$$\inf_{t\in R} a_{lj}\,(t) = a_{lj}^l; \qquad \sup_{t\in R} a_{lj}\,(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u \left(b_j^u \,/\, a_{lj}^l\right). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \geq x_l\left\{b_l^u - a_{ll}^l x_i\right\}; \qquad t > t_0, i = 1, 2, \dots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \dots, n; t \geq t_0. \tag{1.5}$$

(1.9) And (1.5) together lead to

$$\frac{dx_1}{dt} \geq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \dots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \geq \left( b_i^l - \sum_{\substack{j=1 \\ J \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \geq x_i^l \ for\ t \geq t_0,\ i = 1, 2, \dots, n. \tag{1.7}$$

18. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \dots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R}_n | x_* \leq x_i \leq x^*; i = 1, 2, \dots, n\} \quad (1.8)$$

*Where*

$$x_* = \min_{1 \leq i \leq n} x_i^l ; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \quad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(177) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(178) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(179) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(180) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(181) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(182) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of

Southampton, UK. “This is a cool, new idea, and I’m up for trying to see it in experiments,” he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(183) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team’s work. Einstein’s theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth’s surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(184) Because of gravity’s effect on (e&eb) space-time, Pikovski’s team realised that variance in a molecule’s position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(185) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(186) “In most situations decoherence is due to (e) something external; here it’s as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself,” adds Pikovski.

**A practical limit**

(187) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(188) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(189) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

(190) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below).

(191) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints)

(192) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain.

(193) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy.

(194) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**.

(195) If gravity’s effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy.

| | |
|---|---|
| (196) | "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (197) | Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (198) | "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for

any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$||f(t+\tau,x) - f(t,x)|| = \sum_{l=1}^{m} |f_l(t+\tau,x) = f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1}\int_0^T f(t,x)\exp(-i\lambda_j t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or $\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times S, \{g(t+,x)\}$ is also uniformly convergent on $\boldsymbol{R} \times S$ then the module of $g(t+,x)$ is contained in the module of $f(t+,x)$.*

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number $\mu$ such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J\neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i$$

$$= 1,2,\dots,n. \qquad (2.3)$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say $w(t) = w_1(t),\dots,w_n(t), t \in \boldsymbol{R}$, whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l\left\{b_l(t) - \sum_{j=1}^{n} a_{lj}(t)x_j\right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \qquad (2.5)$$

Uniformly in $(x_1,\dots,x_n) \in S$ for $t \in (-\infty,\infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0),\dots,n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined for $t \in [\beta,\infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta,\infty), i = 1,2,\dots,n$ and the convergence is uniform on all compact subsets of $[\beta,\infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta,\infty)$.

Let $U$ be any compact subset of $\beta,\infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon,\beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon,\beta)$

and all $t \in (-\infty,\infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \quad (2.6)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \quad (2.7)$$

$$(2nx^*/x_*)\exp[-\mu x_*(\beta+\tau_k-t_0)] < (\varepsilon/2x^*) \quad (2.8)$$

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1,(s) - \log x_1(s+\tau_m+\tau_k)|, \quad x \in t_0, \quad (2.9)$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1,(t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \in v(s)$$

$$\in (1/x^*)\sum_{i=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)|. \quad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s) = \sum_{i=1}^{n} \{sign[x_i(s) - x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[\log x_i(s)]}{ds} - \frac{d[\log x_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\sum_{l=1}^{n} |b_l(s) - b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n} a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$-\mu x_* v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)\,|. \qquad (2.12)$$

Let us set $s-\tau_k = \sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)\,|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(\sigma+\tau_k)-a_{iJ}(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k),x(t+\tau_m)\big) \le v\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$[(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$ On using (2.8), (2.10) (2.14)

For $m \geq n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \leq x_i(t) \leq x^*, i = 1,2,\dots,n; t \geq t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \geq n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*,x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*,x^*], i = 1,2,\dots,n;$ for $t \geq t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t),\dots,x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt}\,x_i(t+\tau_k) - \frac{d}{dt}\,x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k),\dots,x_n\,(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\,(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k))$$

$$- f_l\big(t + \tau_m, x_1(t + \tau_m), \dots, x_n(t + \tau_m)\big)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t} + \tau_k) = x_1(t + \tau_k), \dots,\ x_n(t + \tau_k)\ \in \mathrm{S}$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ ? $k$ ? $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\big)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t + \tau_k)\ \in S$ and $x_l(t + \tau_m)\ \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ ? $k$ ? $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x(t + \tau_m), \dots, x_n(t + \tau_m)\big)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has

been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \geq n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ( (t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k \to \infty} \frac{d}{dt} x_1(t + \tau_k) \quad exists\ for\ i = 1, 2, \dots, n$$

Uniformly on all compact subsets of $(-\infty, \infty)$. Thus we have

$$\lim_{k \to \infty} \lim_{k \to \infty} \frac{x_i(t + \tau_k + h) - x_i(t + \tau_k)}{h}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{x_l(t + \tau_k + h) - x_l(t + \tau_k)}{h} \qquad (2.22)$$

$$= \lim_{h \to 0} \frac{w_1(t + h) - w_1(t)}{h}, \quad i = 1, 2, \dots, n,$$

Showing that (since the left side of (2.22) exists) $w_i(i = 1, 2, \dots, n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt} = \lim_{k\to\infty}[\, f_i\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n\,(t + \tau_k)\right)$$

$$-f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)$$

$$+ - f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)]$$

$$= -f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big); \qquad i = 1, 2, \dots, n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1, f_2, \dots, f_n$ on $R \times S$ and hence $w(t) = \big(w_1(t), \dots, w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t) = \big(w_1(t), \dots, w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t, x_1, \dots, x_n), f_2(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n)\}$ for $(x_1, x_2 \dots, x_n) \in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t + \tau_k, x_1, \dots, x_n), f_2(t, + \tau_k, x_1, \dots, x_n), \dots, f_n(t, + \tau_k, x_1, \dots, x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1\,(t, + \tau_k)\ \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l\,(t, + \tau_k) \to w_i(t + \tau\,), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ ☒$k$ ☐ *n0(ε)* then

$$\sum_{l=1}^{n} |b_l\,(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t+\tau_k) - a_{ij}(t+\tau_m)| < \left(\frac{\varepsilon}{4x^*}\right)\mu x_* \qquad (2.25)$$

$$n(2x^*/x_*)\exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s+\tau_m-\tau_k)\big)$ for $s \in [t, t+\tau_k]$ defined by

$$v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s+\tau_m-\tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{i=1}^{n}\{\text{sign}[w_l(s) - w_i(s+\tau_m-\tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds}\cdot\frac{1}{w_l(s)} - \frac{dw_l(s+\tau_m-\tau_k)}{ds}\,\frac{1}{w_l(s+\tau_m-\tau_k)}\right\}$$

$$\leq -\mu\sum_{l=1}^{n}|w_l(s) - w_l(s+\tau_m-\tau_k)| + \sum_{l=1}^{n}|b_i(s) - b_l(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{l=1}^{n}\sum_{J=1}^{n}\left|a_{lJ}(s) - a_{lJ}(s+\tau_m-\tau_k)\right| \qquad (2.28)$$

$$\leq -\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s) - b_l(s+\tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s) - a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k = \sigma$ then we have from (2.29),

$$D^{+}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^{*}\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\,(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)\right| \qquad (2.30)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_{*})/(4x^{*})]\ +[(\varepsilon\mu x_{*})/(4x^{*})]$$

(2.31)

Integrating (2.31) over $(t,t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called “shrinking operations”), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang’s model or to

Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during 0≤t<1. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$ e^{-\mu x^* T_k} v\big(w(t), w(t+T_m-T_k)\big) + (\varepsilon/2x^*)$

$ e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t+T_m-T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$<(\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} | w_1(t+T_k) - w_1(t+T_m)| < \varepsilon \; for \; m \; k \; n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty \; as \; k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\; as\; k \to \infty \; on$ **R** follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T >$ $0. Then\; if$ (1.2), (1.3)$and$ (2.3)$hold\; the\; system$ (1.1)$will\; have\; a\; periodic\; solution\; w(t) =$ $\{w, (t), \ldots\ldots, w_n(t)\} of\; period\; T\; such\; that$

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots) and\; note\; that$

$$x_1(t+KT) - w_1(t) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) - w_1(t) \to 0 \; as \; \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \ldots, w_n(t)\} of (1.1) is\; globally\; asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \ldots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \ldots, n$ is defined for all t□0 and satisfies

$$\lim_{t \to \infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \ldots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{J=1 \\ J\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu$ $Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= g|g:**R**→ $\boldsymbol{R}$ $and$ $g(t)\lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\,(t)y_1(t)\right\},$$

*i=1,2,…,n; t>*$t_0 \in R$,

Where

$$B_i \in H(b_i); (A_{ij}), \qquad i, j = 1,2, \dots, n,$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{ij}^{u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\dots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\dots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\dots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\dots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\dots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |\, x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1\,(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t\,) - y_i(t\,)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ \ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{aligned} &\beta_1 \leq x_i(t) \leq \beta_2, \\ &\beta_1 \leq y_i(t) \leq \beta_2, \end{aligned} \quad i = 1,2, \dots, n; t \geq t_0 \tag{3.7}$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \tag{3.8}$$

It is easy to see that

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1} \sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\ \big(x(t), y(t)\big) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\beta_1 V\big(x(t), y(t)\big) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+ \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t), y(t)) \leq V(x(t_0), y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right] \frac{1}{\mu\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} |x_1(t) - y_1(t)| \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} |x_1(t_0) - y_1(t_0)|$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)|\right.$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying “The cause is by cause,” which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics

would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} \mid a_{ij}(t) - A_{ij}(t) \mid \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Presciential Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of**

**dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t, a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ 0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ \infty,$$

$$\rho_1(t, 0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1 \left[\frac{dQ_2}{dt}\right]_1 - x_2 \left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3 \left[\frac{dQ_3}{dt}\right]_1 - x_4 \left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1P_1) + \big(f_2)_1(Q_1)\big)(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \le \rho_1(t, a_1) \le \rho_1^*(t, a_1),$$

$$0 \le Q_1(t) \le Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\, \rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\, \rho_1^*(t, a_1) da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt} = -b_1 Q_1^* + g_1(P_1^*)Q_1^*, \quad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*(t, a_1) da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \quad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a) e^{-\int_0^a \mu_1(\tau)dt} e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t, a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1(P_0)a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t, a_1) = \rho(t, a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1\,(t, a_1)da_1 = \int_0^\infty \rho_1(t, a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1 N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0'Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}')(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t,a_1) = \tilde{\mathrm{u}}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_1}\,da_1$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_1)$ gives,

$$\tilde{\mathrm{u}}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{\mathrm{u}}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})}\tau_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\int_0^a(\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1+\int_0^{\tau}\mu_1(s_1)ds}d\tau_1\,da_1$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}\int_0^a e^{f_{10}\tau+\int_0^{\tau}\mu(s)ds+y\tau}\alpha_1(\tau)d\tau\,da\right)w=0. (23)$$

Define $Y(\gamma,a)=e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta(a)Y(\gamma,a)da\right)u(0)$$

$$+w\int_0^{\infty}\beta(a)Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da=0, (24)$$

$$\left(\int_0^{\infty}Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w=0.$$

Equation (20) has a solution of the form of $u(t,a)=u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I(y,a)=\int_0^a\rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^{l}\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^{l}\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right)=0. \quad (25)$$

Since

$$\rho_0(a)=\rho_0(0)e^{-\int_0^a \mu(\tau)d\tau-f_{10}a}$$

$$I(\gamma,a)=\int_0^a \rho_0(\tau)e^{\gamma\tau+f_{10}\tau+\int_0^\tau \mu(s)ds}d\tau$$

$$=\frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a)=e^{-f_{10}a-\int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1,a_1)=E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma}\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^{1}]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\left(\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)$$

$$+\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1+\frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(a_1)da_1-\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1+\frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(a_1)da_1)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1=1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\int_0^a \alpha_1(\tau_1)Y^{-1}(\gamma_1,\tau_1)d\tau_1 da_1>0$. Therefore, the characteristic solution is

$$\gamma_1=-(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(a_1)da_1.$$

Since $f_1'(P_0)_1>0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1),0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey

population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*)\int_0^\infty \beta_1(a_1)p_1^*(a_J)p_1^*(a_1)da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0)e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1 \int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)\left([f_2^{'}]_1(z_1)\right)\int_0^{\infty} a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_1$$

$-h^{'}(z_1)\int_0^{\infty}\beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$$H_1(0) = 1 - \int_0^{\infty}\beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^{-}(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^{\infty}\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a} da_1$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R(f_1(P^*)) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - [p^*]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\,da + \frac{[h_*^1]_1}{[h_*]_1} p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $[f_{1*}^{'}]_1$ denoting $[f^{'}{}_1(P^*)]_1 [f_{2*}^{'}]_1$ denoting $[f_2]_1$ $(Q_1^*$, $h_{*1}$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $[g_*^{'}]_1$

Denoting $[g^{'}]_1(P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - [f'_{1*}]_1[p_1^*](a_1)M_1 - [f'_{2*}]_1 p_1^*(a_1)W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1)m_1(t,a_1)da_1 + \frac{[h_*^1]_1}{[h^*]_1}[p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1[Q_1^*]M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1)da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1)da_1 - \int_0^\infty \mu(a_1)m(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 \int_0^\infty p_1^*(a_1)da_1\right)M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1)da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}']_1 P_1^*\right)M_1 - [f_{2*}']_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^\infty ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + [f_{1*}']_1 P_1^*\right)M_1 + \left(\frac{[h_*']_1}{[h_*]^1}\, P_1^*(0) - f_{2*}' P_1^*\right)W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_{*}\right]_1 Q_1^* M_{.1}$$

By the mean value theorem,

$$\int_0^{\infty}([h_{"*}]_1\beta_1(a_1)-\mu_1(a_1))m_1(t,a_1)da_1=([h_*]_1\beta_1(\theta_1)-\mu_1(\theta_1))M_{,1}$$

Where $0< \theta_1=\theta_1(t)<\infty$.

And if $a_1^2[a_1^2]_1<4[a_2]_1<4a_2e$,

Re $\lambda_1[\lambda_1]_1=\mathrm{Re}\lambda_2[\lambda_2]_1=-\frac{1}{2}a_1[a2\ ]_1<0$.

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_1\beta_1(a_1)-\mu_1(a_1)\right|$. Then$|n_1(t)|$ ⊿∀t ⊿. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_1^2<4a_2e$ and $[a_1]>2\Delta$,or $[a_1^2]$ ⊿$[a_2]e$ and $(a_1-\sqrt{(a_1^2)_1-4(a_2)_1e})>2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+1}\right)$ and $\beta_1(a_1)=ae^{-ma}$ Then $\Delta=[\mu_0]+1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) \mathrm{e}^{-\int_0^{\mathrm{a}} \mu_1(\tau_1) + d\tau_1 \mathrm{da}} = \frac{1}{(\mathrm{m} + \mu_0 + 1)^2} + \frac{2}{(\mathrm{m} + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1) = K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1) = [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)$=$(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [\mathrm{K}][P_0])}{[m_1]_1 + (\mu_0)_1 + [\mathrm{K}][P_0])_1^3\ [\mathrm{K}]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$ i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*} \quad [h_*] = \frac{1+\omega_1 \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*}$

$[f_{1*}] = K_1\ [\mathrm{f}_{2*}^1] = \frac{\sigma_1}{(1+\mathrm{Q}_1^*)^2} [\mathrm{g}_*^1] = \frac{\sigma_1}{(1+\mathrm{P}_1^*)^2} \quad [\mathrm{h}_*^1] = \frac{\omega_1 - 1}{(1+\mathrm{Q}_1^*)^2}$

## AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

| |
|---|
| **Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model** |
| **Gravitational quantum wave equation:** |
| **Equation Time-dependent equation (Wikipedia)** |
| **(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]** |
| **(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.** |
| **(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).** |
| **(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2** |
| **(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.** |
| **(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.** |
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |

| |
|---|
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |

**(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]**

**Time-dependent Schrödinger equation**

**(Single non-relativistic particle)**

**(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)**

**(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").**

**(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.**

**(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.**

**(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.**

**(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).**

**(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)**

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in**

**atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⚿ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38) d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac**

**{(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1}+(m_1+[\mu_0]_1+\frac{K_1 b_1}{\delta_1-b_1}+\sigma_1)\,Q_1^*)^3$$

$=m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1+ Q^*)(1 + \omega_1 Q_1^*$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_1 a e^{mu}{}_1 \; [\mu_0]_1 - \frac{a_1}{a_1+1}$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}] P_1^*$$

$$[a_2]_1 = \left[f_{2*}^1\right] P_1^* - \frac{h_*^1}{h_*} P^*(0)$$

$$e=[\,g^{1}_{*}]_{\,1}\,Q^{*}_{1}$$

$$\eta_1(t) = \left[h_{*}\right]_{1}\beta_{1}\left(\theta_{1}(t)\right) - \mu_{1}\left(\theta(t)\right),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_{1}\right]M_1 - [a_2]_{1}W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_{1} - [a_2]_{1} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{1} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(t) = e^{A,t}[X]_{1}(0) + \int_{0} e^{At}$$

**Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models**

**(Module Three)**

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and

also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \quad P_3 \text{ } 0, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \text{ } \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_3(Q_3) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \quad (4)$$

$$\rho_3(t_3, 0) = h_3(Q_3) \int_0^{\infty} \beta_3(a_3)\rho_3(t, a_3)da, \quad (5)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \leq \rho_3(t, a_3) \leq \rho_3^*(t, a_3),$$

$$0 \leq Q_3(t) \leq Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \quad (7)$$

$$\rho_3^*(t, 0) = \int_0^{\infty} \beta_3(a_3)\,\rho_3^*(t, a_3)da_3 \quad (8)$$

$$\frac{dQ_3^*}{dt} = -b_3 Q_3^* + g_3(P_3^*)Q_3^*, \quad (9)$$

With $P_3^*(t) = \int_0^{\infty} \rho_3^*(t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_3(a) e^{-\int_0^a \mu_3(\tau)dt} e^{-ra}\, da_3 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(t)dt}\, da_3 > 1, \qquad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^\infty [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t, a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^\infty \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 \quad < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^{\infty}(\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da \quad = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t, a_3) = \rho(t, a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^{\infty} u_3\,(t, a_3)da_3 = \int_0^{\infty} \rho_3(t, a_3)da_3 - \int_0^{\infty} (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0'Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}')(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3 u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^\infty \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3 a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho \mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^\infty u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t,a_3) = \tilde{\mathrm{u}}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\,da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3, a_3)da_3\right)u_3(0)$$

$$+w\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0,$$

(24)

$$\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^{\infty}Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^{l}\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^{l})_3\int_0^{\infty}Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3)=(\rho_0)_3(0)e^{-\int_0^{a}\mu_3(\tau_3)d\tau_3-(f_{30})_3a_3}$$

$$I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)e^{\gamma_3\tau_3+(f_{30})_3\tau+\int_0^{\tau}\mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3) d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(a_3)da_3 - \int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1 + \frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^\infty E_3(a_3)da_3) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\,(\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(\,a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(\,p^*(a),Q^*\,)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(\,P_3^*) + [f_2]_3(Q_3^*)\,)P_3^* = 0, \qquad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^{\infty}\ \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \qquad (27)$$

$$g_3(P_3^*) - b_3 = 0, \qquad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3) \int_0^{\infty} \beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ \ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r$, $R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - [p^*]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3) M - [f_{2*}^3]_3 p_3^*(\mathrm{a}) W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $[f_{3*}']_3$ denoting $[f'_3(P^*)]_3 [f_{2*}']_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*3}$ denoting $h_*'$ denoting $h_3'$ $(Q_3^*)$, and $[g_*']_3$

Denoting $[g']_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - [f'_{3*}]_3 [p_3^*](a_3) M_3 - [f_{2*}']_3 p_3^*(a_3) W_3$$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t, a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3)da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3)da_3 - \int_0^\infty \mu(a_3)m(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f'_{3*}]_3 \int_0^\infty p_3^*(a_3)da_3\right)M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3)da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 - [\dot{f}_{2*}]_3 P_3^* W_{\cdot 3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 + \left(\frac{[h'_*]_3}{[h_*]^3}\, P_3^*(0) - f'_{2*}P_3^*\right)W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3\, Q_3^*\, M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2 [a_3^2]_3 < 4 [a_2]_3 < 4 a_2 e$,

$\mathrm{Re}\,\lambda_3 [\lambda_3]_3 = \mathrm{Re}\lambda_2 [\lambda_2]_3 = -\frac{3}{2} a_3 [a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty} \left| h_* [h_*]_3 \beta_3(a_3) - \mu_3(a_3) \right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4 a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0] \geq 0.5$. Assume $[f_3](x_3) = K_3 x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3) = [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3) = (1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])_{\,3}^{\,3}\ [\mathrm{K}]_{\,3}[P_0]_{\,3}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$ i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{\,*}\ [f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}_{2*}^{\,3}] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}_*^{\,3}] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}_*^{\,3}] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_{\,3} + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\ \mathrm{Q}_3^{\,*})^{\,3}$

$=m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q^*_3$

Now

$$\left[a^2_3\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q^*_3}{(\delta_3 - b_3)(1 + Q^*_3)} + \frac{\sigma^2_3 Q^{*2}_3}{(1 + Q^*_3)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q^*_3)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q^*_3}{\delta_3 (1 + \omega_3 Q^*_3)(1 + Q^*_3)} P * (0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae^{mu}_3 \quad [\mu_0]_3 - \frac{a_3}{a_3+3}$

Hence if $[a^2_3 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q^*_3}{1 + Q^*_3} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}] P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right] P^*_3 - \frac{h^3_*}{h_*} P^* (0)$$

$$e=[\,g^{3}_{*}]\ {}_{3}\,Q^{*}_{3}$$

$$\eta_3(t) = \left[h_{*}\right]_{3}\beta_{3}\left(\theta_{3}(t)\right) - \mu_{3}\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_{3}\right]M_3 - [a_2]_{3}W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_{3} - [a_2]_{3} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{3} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(t) = e^{A,t}[X]_{3}(0) + \int_0 e^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's**

**1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin**

**bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and**

**Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for**

**(e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole**

**merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of**

**(e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain**

**bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating**

**accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared**

**Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard**

**Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total**

**energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of**

**information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local**

**environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any ℓ-local Hamiltonian, where ℓ≥2, can be achieved using decoupling schemes of length at most O(nℓ−1logn). Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle**

**problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 73. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \; 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \; \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4P_4) + \big(f_2)_4(Q_4)\big)(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \le \rho_4(t, a_4) \le \rho_4^*(t, a_4),$$

$$0 \le Q_4(t) \le Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \tag{10}$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \tag{$H1$}$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4}\rho_4(t,a_4)\,da_4$$

$$\frac{dQ_4}{dt} = -b_4Q_4 + c_4P_4Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4P_4 - \lambda_4P_4Q_4$$

$$\frac{dQ_4}{dt} = b_4Q_4 - c_4P_4Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 74. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0'Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h'_{a_4})(\rho_0)_4 Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4 u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4 a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\, o\,)_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t,a_4)da_4$$

$$u_4(t,0) = \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t,a_4) = \tilde{\mathrm{u}}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_4}\, da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})\tau_4}\,(\alpha_4)_4(\tau_4)e^{\gamma_4 \tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4} e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds} d\tau_4\, da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\, da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4 - \int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds + y\tau}\alpha_4(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_4\,(a_4) Y_4(\gamma_4, a_4) da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4=0,$$

(24)

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^{a}\mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$= \frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4 a_4} - 1)$$

Let

$$E_4(a_4) = e^{-(f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4) d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4, a_4) = E_4(a_4)e^{-\gamma_4 a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma}\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\ (\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1 + \frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4 - \int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1 + \frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)da_4 = 1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4 \neq 0,$$

Since $\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^{a}\alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4 > 0.$ Therefore, the characteristic solution is

$$\gamma_4 = -(f_{40}^{'})_4(\rho_0)_4\int_0^{\infty}E_4(a_4)da_4.$$

Since $f_4^{'}(P_0)_4 > 0,$ we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^{\infty}\beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^*=\int_0^\infty p_4^*(a_4)\,da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^\infty \beta_4(a_4)\,e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}da_{,4}$$

And assume that

$$[k_3]_4\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4a_4}\,da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)([f_2^{'}]_4(z_4)) \int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_4$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} \quad da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} da_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r, R\left(f_4(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$$M_4(t) \equiv \int_0^\infty m_4(t, a_4) da_4 = P_4(t) - P_4^*,$$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - [p^*]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^\infty \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0) W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M,_4$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $[f_{4*}']_4$ denoting $[f'_4(P^*)]_4 [f_{2*}']_4$ denoting $[f_2]_4$ $(Q_4^*$, $h_4^*$ denoting $h_*'$ denoting $h_4'$ $(Q_4^*)$, and $[g_*']_4$

Denoting $[g']_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4 + [m_a]_4 = -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4 - [f'_{4*}]_4 [p_4^*](a_4)M_4 - [f_{2*}']_4 p_4^*(a_4)W_4$$

$$m_4(t,0) = [h_*]_4 \int_0^\infty \beta_4(a_4) m_4(t, a_4) da_4 + \frac{[h_*^4]_4}{[h^*]_4} [p^*]_4(0)\, W,_4 \qquad (30)$$

$$\frac{dW_4}{dt} = [g_*^4]_4 [Q_4^*] M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the

universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in

(eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way" point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in (eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing

by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to (e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a

complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed,

authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending

well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite

as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L) mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the

possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=)

the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as:

arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the

Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the

sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th]

(or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th]

(or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory**

**techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number: MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond

last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly, by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in

(e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~

(1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum**

**physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement: Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f^{'}{}_{4*}\right]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 - \left[f_{2*}^{'}\right]_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 + \left(\frac{[h_*^{'}]_4}{[h_*]^4}\, P_4^*(0) - f_{2*}^{'} P_4^*\right) W_4,$$

(34)

$$\frac{dW_4(t)}{dt}\left[g^{'}{}_*\right]_4 Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4 = ([h_*]_4 \beta_4(\theta_4) - \mu_4(\theta_4)) M_{,4}$$

Where $0 < \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2 [a_4^2]_4 < 4\, [a_2]_4 < 1\, a_2 e$,

$\mathrm{Re}\,\lambda_4 [\lambda_4]_4 = \mathrm{Re}\lambda_2 [\lambda_2]_4 = -\frac{4}{2} a_4 [a2\ ]_4 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_* [h_*]_4 \beta_4(a_4) - \mu_4(a_4)\right|$. Then $|n_4(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 < 4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2] \geq 4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4) + d\tau_4 da} = \frac{4}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ 0.5. Assume$[f_4](x_4)= K_4 x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4 x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [K][P_0])}{[m_4]_4 + (\mu_0)_4 + [K][P_0])\,{}_4^3\,[K]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\, P^*\, [f_{2*}] = \frac{\sigma Q_4^*}{4+Q_4^*}$ $[h_*] = \frac{4+\omega_4 Q_4^*}{4+Q_4^*}$

$[f_{4*}] = K_4\, [f_{2*}^4] = \frac{\sigma_4}{(4+Q_4^*)^2}[\,g_*^4] = \frac{\sigma_4}{(4+P_4^*)^2}$ $[h_*^4] = \frac{\omega_4 - 4}{(4+Q_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma)^2}+\frac{2\omega_4}{([m_4]_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma_4)^3}<1$$

And

$$\frac{1}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^2}+\frac{2\omega_4}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_4+[\mu_0]_4+1+\tfrac{K_4b_4}{\delta_4-b_4}+(m_4+[\mu_0]_4+\tfrac{K_4b_4}{\delta_4-b_4}+\sigma_4)\,Q_4^*)^3$$

$$=m_4+[\mu_0]_4+3+\tfrac{K_4b_4}{\delta_4-b_4}(4+Q^*)(1+\omega_4Q_4^*$$

Now

$$\left[a_4^2\right]-1\left[a_2\right]e=\left(\frac{2K_4b_4}{\delta_4-b_4}\right)^2+\frac{1K_4b_4\sigma_4\,Q_4^*}{(\delta_4-b_4)(1+Q_4^*)}+\frac{\sigma_4^2Q_4^{*2}}{(1+Q_4^*)^2}$$

$$\frac{1\,b_4\sigma_4\,(\delta_4-b)Q^*}{\delta_4(1+Q_4^*)^2}-\frac{1\,(\delta_4-b_4)^2(1-\omega_4)Q_4^*}{\delta_4(1+\omega_4Q_4^*)(1+Q_4^*)}P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae^{mu}_4 \ [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a^2_4 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q^*_4}{1 + Q^*_4} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P^*_4$$

$$[a_2]_4 = \left[f^4_{2*}\right] P^*_4 - \frac{h^4_*}{h_*} P^*(0)$$

$$e = [g^4_*]_4 Q^*_4$$

$$\eta_4(t) = \left[h_*\right]_4 \beta_4 \left(\theta_4(t)\right) - \mu_4(\theta(t)),$$

The system(34)can be written as the following system:

$$[M(t)]\left[a_4\right] M_4 - [a_2]_4 W_4 - [n(t)] M_4$$

$$w_4(t) = eM_4$$

Write

$$A = \begin{pmatrix} -[a_4]_4 - [a_2]_4 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_4 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}[X]_4(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum**

**systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th)**

**Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December 1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP**

**Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal**

**deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105**

**DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum**

**mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the**

**general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled**

**with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and**

**Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession**

**due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the**

**system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics.**

**Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc] for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures,**

**especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark**

**matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3**

**[astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley**

**National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero. Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon,"**

**Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of**

**Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two**

**dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show**

**up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of**

**polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 75. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \ P_6 \ \text{0}, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \ \text{?}\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \ \text{?}\infty,$$

$$\rho_6(t,0) = \int_0^{\infty} n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ 0 $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6)\int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \ \rho_6(t, a_6) \ \rho_6^*(t, a_6),$$

$$0 \ Q_6(t) \ Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\,\rho_6^*, \quad (7)$$

$$\rho_6^*(t,0) = \int_0^\infty \beta_6(a_6)\,\rho_6^*(t,a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6\,Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t,a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt}e^{-ra}\,da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\,da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6 \; h_6(Q_6) = 1, \qquad and \;\; g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 76. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t,0) = \int_0^\infty \beta_6(a_6)\rho_6(t,a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^\infty u_6(t, a_6)da_6 = \int_0^\infty \rho_6(t, a_6)da_6 - \int_0^\infty (\rho_0)_6(a_6)da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6'(P_0) = f_{60}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t,0) = \rho_6(t,0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0'Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}')(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6 u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^\infty \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6 a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\, o\,)_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho o]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t,a_6)da_6$$

$$u_6(t,0) = \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t,a_6) = \tilde{\mathrm{u}}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_6}\, da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})}\tau_6\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6 - \int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds + y\tau}\alpha_6(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_6\,(a_6)Y_6(\gamma_6, a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6=0,$$

(24)

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60}^{l}{}_6\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^{a}(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$= \frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6 a_6} - 1)$$

Let

$$E_6(a_6) = e^{-(f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6) d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(4169) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(4170) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(4171) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(4172) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(4173) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(4174) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(4175) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(4176) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(4177) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(4178) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort**

**of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

**The microscopic structure of spacetime**

(4179) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(4180) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(4181) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(4182) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

**Early cosmology**

(4183) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(4184) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(4185) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(4186) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(4187) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

**Black holes**

(4188) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(4189) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(4190) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(4191) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(4192) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

**Astrophysical effects**

(4193) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(4194) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(4195) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

**Main approaches**

(4196) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(4197) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(4198) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

**String theory**

(4199) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(4200) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(4201) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

| |
|---|
| (4202) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (4203) **However, when summed the entire series appears to be (=) divergent.** |
| (4204) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (4205) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (4206) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (4207) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

| |
|---|
| (4208) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (4209) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (4210) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (4211) **The granular properties of space can be (=) explicitly computed.** |
| (4212) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (4213) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (4214) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (4215) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (4216) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (4217) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

(4218) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.**

(4219) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.**

(4220) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.**

(4221) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.**

(4222) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.**

(4223) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.**

**Others**

(4224) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays**

(4225) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.**

(4226) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.**

(4227) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.**

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(4228) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.**

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| (4229) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (4230) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (4231) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (4232) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (4233) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (4234) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (4235) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (4236) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (4237) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (4238) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (4239) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |
| (4240) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (4241) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (4242) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (4243) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption**

**expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (4244) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (4245) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (4246) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (4247) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (4248) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (4249) **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (4250) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (4251) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (4252) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (4253) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (4254) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (4255) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (4256) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (4257) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (4258) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (4259) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only** |

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| **be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (4260) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (4261) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (4262) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (4263) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (4264) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (4265) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (4266) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (4267) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (4268) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (4269) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (4270) **This is a realistic scenario for (e) all known systems.** |
| (4271) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (4272) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (4273) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (4274) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (4275) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (4276) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |

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| (4277) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (4278) **So in those systems, the quantum phase transition does indeed exist.** |
| (4279) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (4280) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (4281) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (4282) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (4283) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (4284) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (4285) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (4286) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (4287) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (4288) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (4289) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (4290) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (4291) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |

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| (4292) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (4293) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (4294) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (4295) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (4296) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (4297) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (4298) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (4299) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (4300) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (4301) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (4302) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (4303) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (4304) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (4305) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (4306) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (4307) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (4308) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (4309) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (∼104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (4310) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (4311) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (4312) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (4313) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (4314) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (4315) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (4316) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (4317) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (4318) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (4319) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (4320) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |

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| (4321) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (4322) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (4323) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (4324) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (4325) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (4326) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (4327) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (4328) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (4329) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (4330) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (4331) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (4332) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (4333) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (4334) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (4335) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (4336) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (4337) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (4338) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (4339) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |

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| (4340) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (4341) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (4342) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (4343) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (4344) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (4345) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (4346) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (4347) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (4348) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (4349) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (4350) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (4351) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (4352) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (4353) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (4354) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (4355) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (4356) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (4357) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (4358) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (4359) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (4360) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (4361) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (4362) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (4363) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (4364) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (4365) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (4366) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (4367) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (4368) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (4369) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (4370) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (4371) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (4372) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (4373) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |

(4374) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.**

**Doomed satellite missions**

(4375) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.**

(4376) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.**

(4377) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.**

(4378) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.**

(4379) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.**

(4380) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people**

(4381) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."**

(4382) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.**

(4383) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(4384) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(4385) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

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| (4386) However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking. |
| (4387) Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses. |
| (4388) X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well. |
| (4389) The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma. |
| (4390) These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6. |
| (4391) Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus. |
| (4392) A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate. |
| (4393) Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure. |

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| Dark matter guides growth of supermassive black holes February 18, 2015 |

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| (4394) This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy. |
| (4395) Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole. |
| (4396) But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy. |
| (4397) A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole. |
| (4398) It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth. |
| (4399) "There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA). |
| (4400) This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies. |
| (4401) However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated. |

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| (4402) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (4403) **We know dark matter exists only from (e) its gravitational effects.** |
| (4404) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (4405) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (4406) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (4407) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (4408) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (4409) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (4410) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (4411) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (4412) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (4413) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (4414) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (4415) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (4416) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (4417) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the** |

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| **underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (4418) **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (4419) **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (4420) **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (4421) **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (4422) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (4423) **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (4424) **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (4425) **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (4426) **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (4427) **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (4428) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (4429) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (4430) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |

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| (4431) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (4432) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (4433) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (4434) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (4435) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (4436) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (4437) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (4438) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (4439) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (4440) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |
| (4441) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (4442) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (4443) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (4444) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |

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| (4445) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (4446) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (4447) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (4448) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (4449) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (4450) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (4451) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (4452) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (4453) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (4454) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (4455) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (4456) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (4457) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (4458) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e)** |

**electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.**

**Spheres on trees**

(4459) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.**

(4460) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.**

(4461) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.**

(4462) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.**

(4463) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.**

(4464) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.**

(4465) **As a result, it provides (eb) an important clue for finding the global minimum.**

(4466) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.**

(4467) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.**

**Thompson's problem 2**

(4468) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society**

(4469) **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.**

(4470) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.**

**Implications for Thomson's and Smale's problems**

(4471) **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.**

(4472) **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry**

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| | **community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (4473) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (4474) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (4475) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (4476) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (4477) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (4478) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (4479) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (4480) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (4481) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (4482) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(769)** | **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(770)** | **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |

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| **(771) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(772) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(773) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(774) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(775) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(776) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(777) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(778) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(779) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(780) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(781) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(782) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(783) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(784) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(785) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |
| **(786) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(787) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(788) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(789) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |

(790) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.

## Stationary black holes

(791) In the Newtonian theory, gravity is described by (e) the potential Φ.

(792) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.

(793) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.

(794) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.

(795) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.

(796) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.

(797) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.

(798) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.

(799) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.

(800) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.

(801) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.

(802) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.

(803) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.

(804) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.

(805) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.

(806) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.

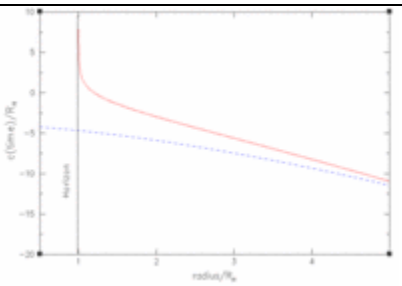
(807) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.

(808) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.

(809) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.

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| **(810) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(811) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(812) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(813) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(814) One of these coordinates is (=) called the "coordinate time."** |
| **(815) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(816) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(817) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(818) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(819) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |
| **(820) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(821) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(822) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(823) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(824) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |

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| **(825) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(826) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(827) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(828) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(829) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(830) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(831) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(832) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(833) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(834) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(835) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(836) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(837) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(838) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(839) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(840) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(841) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(842) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |

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| **(843) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(844) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(845) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(846) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(847) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(848) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(849) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(850) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(851) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(852) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(853) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(854) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(855) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(856) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(857) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(858) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(859) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(860) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |

**(861)** **But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.**

**(862)** **Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.**

**(863)** **The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.**

**(864)** **Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.**

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1 + \frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(a_6)da_6 - \int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6 = 1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6 \neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6 > 0$. Therefore, the characteristic solution is

$$\gamma_6 = -(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6^{'}(P_0)_6 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6),0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_6+(\mu_6(a_6)+[f_6]_6(P_6^*)+[f_2]_6(Q_6^*))P_6^*=0, \qquad (26)$$

$p_6^*(0) = h_6(Q_6^*) \int_0^\infty \beta_6(a_6) p_6^*(a_J) p_6^*(a_6) da_6$ (27)

$g_6(P_6^*) - b_6 = 0,$ (28)

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$p_6^*(a) = p_6^*(0) e^{\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*) \int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6} da_6\right) p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6) \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6} da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty} [H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,) \int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6} da_6$$

$-h^{'}(z_6) \int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\ da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-y_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0) W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[\dot{f_{6*}}\right]_6$ denoting $\left[\dot{f}_6(P^*)\right]_6 \left[\dot{f_{2*}}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,$ $h_{*_6}$ denoting $\dot{h_*}$ denoting $\dot{h_6}$ $(Q_6^*)$, and $\left[\dot{g_*}\right]_6$

Denoting $\left[\dot{g}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - [f'_{6*}]_6[p_6^*](a_6)M_6 - [f'_{2*}]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\, W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f'_{6*}]_6 \int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6 \int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\, m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 - [f_{2*}']_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^\infty ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 + \left(\frac{[h_*']_6}{[h_*]^6}\, P_6^*(0) - f_{2*}' P_6^*\right)W_6,$$

(36)

$$\frac{dW_6(t)}{dt}[g^{'}{}_{*}]_6 Q_6^* M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^\infty ([h_{*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6 < 4\,[a_2]_6 < 4\,a_2 e$,

$$\text{Re}\,\lambda_6[\lambda_6]_6 = \text{Re}\lambda_2[\lambda_2]_6 = -\frac{6}{2}a_6[a2\,]_6 < 0.$$

Let $\Delta = \max_{2<u<\infty}\left|h_*[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4\,a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)$=(1+$\omega_6 x_6$)/$(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [\mathrm{K}][P_0])}{[m_6]_{\,6} + (\mu_0)_{\,6} + [\mathrm{K}][P_0])\,{}^{3}_{6}\ [\mathrm{K}]_{\,6}[P_0]\,{}^{3}_{6}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_{\,6}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$ $[h_*] = \frac{6+\omega_6 \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$

$[f_{6*}] = K_6\ \left[\mathrm{f}\,{}^{6}_{2*}\right] = \frac{\sigma_6}{(6+\mathrm{Q}_6^*)^2}[\ \mathrm{g}\,{}^{6}_{*}] = \frac{\sigma_6}{(6+\mathrm{P}_6^*)^2}$ $\left[\mathrm{h}\,{}^{6}_{*}\right] = \frac{\omega_6 - 6}{(6+\mathrm{Q}_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^{\,2}} + \frac{2\omega_6}{([m_6]_{\,6} + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^{\,3}} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,2}} + \frac{2\omega_6}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}]\,P^*_6$$

$$[a_2]_6 = \left[f^6_{2*}\right]P^*_6 - \frac{h^6_*}{h_*}P^*(0)$$

$$e=[g^6_*]_6\,Q^*_6$$

$$\eta_6(t) = \left[h_*\right]_6\beta_6\left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right]M_6 - [a_2]_6 W_6 - [n(t)]M_6$$

$$w_6(t) = eM_6$$

Write

$$A=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(t) = e^{A,t}[X]_6(0) + \int_0 e^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case $\beta\rightarrow\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes

(eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal

anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for

cases $\omega_q \neq -2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb)

this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence,

authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the

dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R~1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They

also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual

field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2

[hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the

recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black**

**holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the

imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves

(e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise

mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top.

Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of

quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity.

**Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions.

The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques

are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity**

**QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP.**

**L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-**

**atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A,**

**75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the**

**sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain**

**are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized**

**ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study**

**the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor**

**connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they**

**interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold**

**dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct (“extreme”) types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, ‘the field’. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb)**

**that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ⁎This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum**

**interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are**

**in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-**

**0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x.**

**Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space**

**time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics**

**(math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 77. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \quad P_7 \; 1, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3 \left[\frac{dQ_3}{dt}\right]_7 - x_4 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5 \left[\frac{dQ_7}{dt}\right]_7 - x_6 \left[\frac{dQ_3}{dt}\right]_7$$

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| Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied. |
| On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$ |
| Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent. |
| Following equation throws light on important issues in the following cases |
| **Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA** |

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| (4483) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. |
| (4484) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. |
| (4485) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. |
| (4486) Such planets are all the "right" distance from (e) their star, and hopefully, also have water. |
| (4487) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. |
| (4488) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. |
| (4489) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. |
| (4490) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. |
| (4491) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, |

which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(4492) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(4493) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(4494) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(4495) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(4496) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(4497) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(4498) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(4499) The electronic and (e&eb) structural variability among them is extraordinary.

(4500) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(4501) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(4502) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(4503) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(4504) Nudge your bath tub just right and (eb) you might end up reversing time.

(4505) Don’t worry – you won’t end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a “time mirror”, which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

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| (4506) | To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse |
| (4507) | Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses. |
| (4508) | But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort. |
| (4509) | The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort. |
| (4510) | Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower |
| (4511) | Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower |

**Déjà vu**

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| (4512) | To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape. |
| (4513) | The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says. |
| (4514) | Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says. |
| (4515) | "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810 |

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

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| (4516) | Wave control is usually performed by (e) spatially engineering the properties of a medium. |
| (4517) | Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach. |

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| (4518) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors. |
| (4519) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source. |
| (4520) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation. |
| (4521) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption. |
| (4522) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control. |

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| **Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)** |

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| (4523) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice. |
| (4524) Quantum mechanics rules the dynamics of (e) light and matter. |
| (4525) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved. |
| (4526) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1]. |
| (4527) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest. |
| (4528) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (4529) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (4530) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (4531) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (4532) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |

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| (4533) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (4534) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (4535) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (4536) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (4537) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (4538) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (4539) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (4540) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (4541) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (4542) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (4543) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (4544) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (4545) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (4546) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (4547) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (4548) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (4549) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (4550) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (4551) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |

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| (4552) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (4553) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (4554) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (4555) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (4556) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (4557) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (4558) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (4559) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (4560) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (4561) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (4562) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (4563) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (4564) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (4565) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (4566) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (4567) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (4568) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (4569) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (4570) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |

(4571) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(4572) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ∞; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt}$$

$$= -b_7Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \leq \rho_7(t, a_7) \leq \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7\, Q_7^* + g_7(P_7^*) Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \quad (10)$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \qquad (H1)$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 78. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7e^{r_7t - \int_0^t (f_7)_7(P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau}\, e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7(P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7 N_7 + o_7(N_7), \quad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \quad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7(0) \quad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7 Q_7$$





The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{\mathrm{u}}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_7)$ gives,

$$\tilde{\mathrm{u}}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{\mathrm{u}}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a(\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds+y\tau}\alpha_7(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^a(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty} Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^a(\rho_0)_7(\tau_7)Y^{-7}(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty} Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^a\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7,a_7)=\int_0^a(\rho_0)_7(\tau_7)e^{\gamma_7\tau_7+(f_{70})_7\tau+\int_0^{\tau}\mu_7(s_7)ds_7}d\tau$$

$$=\frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7a_7}-1)$$

Let

$$E_7(a_7)=e^{-(f_{70})_7a_7-\int_0^a\mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)da_7 = 1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)\ \int_0^a \alpha_7\ (\tau_7)Y^{-7}(\gamma_7, \tau_7)d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7(\rho_0)_7 \int_0^{\infty} E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^{\infty} \beta_7(a_7)p_7^*(a_J)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^{\infty} p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7']_7(z_7))\int_0^\infty a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}da_{7}$$

$-h'(z_7)\int_0^{\infty}\beta_7(a_7)\,e^{\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7-[f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^-(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\,e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^\infty m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - [p^*]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M,_7$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}']_7$ denoting $[f'_7(P^*)]_7 [f_{7*}']_7$ denoting $[f_7]_7$ $(Q_7^*$, $h_{*7}$ denoting $h_*'$ denoting $h_7'$ $(Q_7^*)$, and $[g_*']_7$

Denoting $[g']_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7 [p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7) m_7(t, a_7) da_7 + \frac{[h_*^7]_7}{[h^*]_7} [p^*]_7(0)\, W,_7 \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7 [Q_7^*] M_7$$

Integrating $[m_7]_7(t, a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t, a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t, a_7)da_7 - \int_0^\infty \mu(a_7)m(t, a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$= m_7(t, 0) - \int_0^\infty \mu_7(a_7)\, m_7(t, a_7)da_7$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [\dot{f_{7*}}]_7 P_7^*\right)M_7 - [\dot{f_{7*}}]_7 P_7^* W_{\cdot 7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\, P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}[g'_*]_7\, Q_7^*\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{\cdot 7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $\frac{7}{7}$ [a $\frac{7}{7}$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_7 [\lambda_7]_7 = \text{Re}\lambda_7 [\lambda_7]_7 = -\frac{7}{7} a_7$[a2 ] $_7 < 0$.

Let ⊿=max$_{7<u<\infty}$ $|h_* [h_*]_7 \beta_7(a_7) - \mu_7(a_7)|$. Then$|n_7(t)|$ ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $\frac{7}{7}$ <4 a $_7$ e and [a $_7$] $> 1⊿$,or [a $\frac{7}{7}$] [a $_7$]e and (a $_7 - \sqrt{(\text{a}\frac{7}{7})_7 - 4(\text{a}_7)_7 \text{e}} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\text{a}}{\text{a}+7}\right)$ and $\beta_7(a_7) = \text{ae}^{-\text{ma}}$ Then ⊿=$[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_7(a_7)\text{e}^{-\int_0^{\text{a}} \mu_7(\tau_7) + d\tau_7 \text{da}} = \frac{1}{(\text{m} + \mu_0 + 1)^7} + \frac{1}{(\text{m} + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ .5. Assume$[f_7](x_7)= K_7 x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)$= $[0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)$=$(1+\omega_7 x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\text{K}][P_0])}{[m_7]_7 + (\mu_0)_7 + [\text{K}][P_0])\frac{3}{7}\ [\text{K}]_7 [P_0]\frac{3}{7}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_7$ i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^*\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^7] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^7] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma)^7} + \frac{1\omega_7}{([m_7]_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma_7)^3} < 1$$

And

$$\frac{1}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^7} + \frac{1\omega_7}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7}+(m_7+[\mu_0]_7+\frac{K_7 b_7}{\delta_7-b_7}+\sigma_7)\ \mathrm{Q}_7^*)^3$

$=m_7+[\mu_0]_7+3+\frac{K_7 b_7}{\delta_7-b_7}(1+\mathrm{Q}^*)(1+\omega_7 \mathrm{Q}_7^*$

Now

$$\left[\mathrm{a}\begin{matrix}1\\1\end{matrix}\right]-4\left[\mathrm{a}_{7}\right]\mathrm{e}=\left(\frac{1K_7b_7}{\delta_7-b_7}\right)^{7}+\frac{4K_7b_7\sigma_7\,\mathrm{Q}_{7}^{*}}{(\delta_7-b_7)(7+\mathrm{Q}_{7}^{*})}+\frac{\sigma_{7}^{7}\mathrm{Q}_{7}^{*7}}{\left(1+\mathrm{Q}_{7}^{*}\right)^{7}}$$

$$\frac{4\,b_7\sigma_7\,(\delta_{7}-\mathrm{b})\mathrm{Q}^{*}}{\delta_{7}(1+\mathrm{Q}_{7}^{*})^{7}}-\frac{4\,(\delta_{7}-b_7)^{7}(1-\omega_7)\mathrm{Q}_{7}^{*}}{\delta_{7}\left(1+\omega_7\mathrm{Q}_{7}^{*}\right)(1+\mathrm{Q}_{7}^{*})}\mathrm{P}^{*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{7}\,ae_{7}^{mu}\;\;[\mu_0]_{7}-\frac{a_7}{a_7+7}$$

Hence if $[\mathrm{a}_{7}^{7}>4(a_7)_{7}\mathrm{e}$ and

$$[a_7]=\frac{1K_7b_7}{\delta_7-b_7}+\frac{\sigma_7\,\mathrm{Q}_{7}^{*}}{1+\mathrm{Q}_{7}^{*}}>1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(783)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(784) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(785)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(786) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(787)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(788) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

(789)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States.

(790)Humans are not very good at understanding interacting many-body systems.

(791)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them.

(792)But what happens when we couple (e&eb) the particles strongly?

(793)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments.

(794)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water.

(795)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h.

(796) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1]

(797) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic

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| origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (798)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (799) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (800)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (801) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (802) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (803)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (804)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (805) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (806) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (807)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (808)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (809)When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (810)The energy levels of these quantized orbitals take on (e) discrete values |
| (811):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (812) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (813) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (814)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |

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| (815) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (816)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (817) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (818)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (819)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (820)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (821) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (822)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (823)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (824)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (825) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an "elementary particle." |
| (826)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (827)The remarkable thing is that solitons can become (=) "elementary," in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (828)An example of a duality, which we will use below, arises in (eb) Maxwell's theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (829)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (830)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (831)Another classic example of duality involves (e&eb) vortices. |
| (832) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (833)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (834) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (835)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (836)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |

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| (837) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (838)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (839)TIs are distinguished only by (e) their edge states. |
| (840) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (841)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (842) But the Dirac cone is not (e) the only possible signature. |
| (843) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (844)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (845)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (846)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (847) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (848). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (849)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (850)The bulk duality implies (eb) a duality between edge theories. |
| (851)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (852)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (853) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (854) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (855)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |

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| (856)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (857)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (858)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (859)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (860)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

| |
|---|
| (861) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, TH=$\hbar\kappa/2\pi$. |
| (862)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (863)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (864) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (865)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (866)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

| |
|---|
| IF we define |
| $$[a_7]_{\,7} = [f_{7*}]_{\,7} + [f_{7*}]_{\,7} + [f_{7*}]\,\mathrm{P}^{*}_{7}$$ |
| $$[a_7]_{\,7} = \left[\mathrm{f}^{\,7}_{7\,*}\right]\mathrm{P}^{*}_{7} - \frac{\mathrm{h}^{7}_{*}}{\mathrm{h}_{*}}\mathrm{P}^{*}(0)$$ |
| $$\mathrm{e}=[\,\mathrm{g}^{7}_{*}]_{\,7}\,\mathrm{Q}^{*}_{7}$$ |
| $$\eta_7(\mathrm{t}) = \left[\mathrm{h}_{*}\right]_{\,7}\beta_{\,7}\left(\theta_{\,7}(\mathrm{t})\right) - \mu_{\,7}(\theta(\mathrm{t})),$$ |

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{7}\right]M_7 - [a_7]_{7}W_7 - [\mathrm{n(t)}]M_7$$

$$w_7(\mathrm{t}) = \mathrm{e}M_7$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_7]_{7} & -[a_7]_{7} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{7} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{7}(0) + \int_0 \mathrm{e}^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality

function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 79. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(4573) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(4574) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(4575) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(4576) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(4577) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

(4578) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems.

(4579) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe.

(4580) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short.

(4581) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another).

(4582) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited.

(4583) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan.

(4584) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room

(4585) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them.

(4586) "The underlying physics is (=) chaos, essentially," he said.

(4587) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system.

(4588) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior.

(4589) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion.

(4590) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer.

(4591) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state.

(4592) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate.

(4593) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system.

(4594) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map.

(4595) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map.

(4596) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill.

(4597) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement.

(4598) It turns out that they are actually the driving forces behind (eb) thermalization.

(4599) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle."

(4600) The study's findings have fundamental implications for (e) quantum computing.

(4601) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool.

(4602) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

(4603) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box.

(4604) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction.

(4605) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed.

(4606) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer.

(4607) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box.

(4608) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance.

(4609) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction.

(4610) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device.

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(4611) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(4612) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(4613) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(4614) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(4615) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(4616) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(4617) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(4618) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(4619) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(4620) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(4621) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(4622) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(4623) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(4624) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(4625) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(4626) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(4627) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(4628) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(4629) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(4630) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(4631) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(4632) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(4633) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(4634) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(4635) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(4636) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(4637) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(4638) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(4639) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(4640) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until

genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(4641) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(4642) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(4643) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(4644) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(4645) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(4646) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(4647) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(4648) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(4649) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

### Black holes

(4650) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(4651) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(4652) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(4653) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(4654) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(4655) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(4656) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(4657) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(4658) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(4659) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(4660) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(4661) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(4662) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(4663) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(4664) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(4665) However, when summed the entire series appears to be (=) divergent.

(4666) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(4667) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(4668) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(4669) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(4670) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(4671) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(4672) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(4673) The granular properties of space can be (=) explicitly computed.

(4674) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(4675) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(4676) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(4677) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(4678) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(4679) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(4680) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(4681) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(4682) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(4683) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(4684) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(4685) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(4686) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(4687) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(4688) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(4689) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \quad P_8 \text{ 1}, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \text{ ?}\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \text{ ?}\infty,$$

$$\rho_8(t,0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 80. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^{\infty} u_8(t, a_8) da_8 = \int_0^{\infty} \rho_8(t, a_8) da_8 - \int_0^{\infty} (\rho_0)_8(a_8) da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t, 0) = \rho_8(t, 0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8\,(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8\,(0) \qquad (18)$$

Expanding

$$h_8(Q_8) = 1 + \ (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^{\infty} \beta_8\,(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\ o\ )_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^{8}]\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}da_8\left(\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^{\infty}E_8(a_8)da_8-\int_0^{\infty}E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^{\infty}E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)da_8=1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\neq 0,$$

Since $\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)\int_0^{a}\alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8>0$. Therefore, the characteristic solution is

$$\gamma_8=-(f_{80}^{'})_8(\rho_0)_8\int_0^{\infty}E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[\dot{g}\right]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - \left[f^{'}_{8*}\right]_8[p_8^*](a_8)M_8 - \left[\dot{f}_{8*}\right]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}_{8*}\right]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[\dot{f}_{8*}\right]_8 P_8^*\right)M_8 - \left[\dot{f}_{8*}\right]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8) + d\tau_8 \mathrm{da}} = \frac{1}{(\mathrm{m}+\mu_0+1)^8} + \frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8 x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8 x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])\,{}_8^3\,[\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8 - 8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^8} + \frac{1\omega_8}{([m_8]_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^3} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$$

$$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\,(\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, a e_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}] \, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8} \, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8} \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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|---|
| (535)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (536)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (537) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (538) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (539)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (540) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (541)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (542)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(543)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(544)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(545) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(546)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(547)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(548)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(549) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(550)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(551)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(552)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(553)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(554)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(555)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(556)Bringing them to thermal equilibrium requires (e) an external bath.

(557) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(558)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(559)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(560)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(561)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(562)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(563) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(564) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(565) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(566)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(567)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(568) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(569)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(570)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(571) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(572)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(573)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (574)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (575) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (576)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (577) This is a realistic scenario for (e) all known systems. |
| (578)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (579)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (580)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (581) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (582)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (583)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (584)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (585)So in those systems, the quantum phase transition does indeed exist. |
| (586)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (587)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 81. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; \text{0}, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \text{∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \text{∞},$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if a $^{9}_{9}$ [a $^{9}_{9}$]2<4 [a $_{9}$]2<4 a $_{9}$e,

Re $\lambda_{9}[\lambda_{9}]_{9} = \mathrm{Re}\lambda_{9}[\lambda_{9}]_{9} = -\frac{9}{9}a_9$[a2 ] $_{9} < 0$.

Let ⊿=max$_{9<u<\infty}$ $|$h $_{*}$ [h $_{*}$] $_{9}$β $_{9}(a_9) - \mu_9(a_9)|$. Then$|n_9(t)|$ ⊿∀t . Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^{*}$ (a), $\boldsymbol{Q}^{*}$) is asymptotically stable if either a $^{9}_{9}$ <4 a $_{9}$ e and [a $_{9}$] $> 1⊿$,or [a $^{9}_{9}$] [a $_{9}$]e and (a $_{9} - \sqrt{(\mathrm{a}\,^{9}_{9})_{9} - 4(\mathrm{a}_{9})_{9}\mathrm{e}} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\mathrm{a}}{\mathrm{a}+9}\right)$ and β $_{9}(a_9)$ = ae-$^{ma}$ Then ⊿=$[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_{9}(a_9)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_9(\tau_{9}) + d\tau_9 \mathrm{da}} = \frac{1}{(\mathrm{m} + \mu_0 + 1)^{9}} + \frac{1}{(\mathrm{m} + \mu_0 + 1)^{3}}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ .5. Assume$[f_9](x_9)$= $K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)$= $[0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9)$=$(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])_{\,9}^{\,3}\ [\mathrm{K}]_{\,9}[P_0]_{\,9}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{\,*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}_{9*}^{\,9}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\ [\mathrm{g}_*^{\,9}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}_*^{\,9}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{\,*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a_{\ 9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^{\ 9} Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_{\ 9} - b)Q^*}{\delta_{\ 9}(1 + Q_9^*)^9} - \frac{4\,(\delta_{\ 9} - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_{\ 9}(1 + \omega_9 Q_9^*)(1 + Q_9^*)} P^*(0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\ mu}_{\ 9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a\ {}^{9}_{9} > 4(a_9)_{\ 9}e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[\,g^9_*]\ _9\,Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION THREE A

## Mixed State Entanglement And Quantum Error Correction-2

### INTRODUCTION—VARIABLES USED

**Mixed State Entanglement and Quantum Error Correction Charles H. Bennett, David P. DiVincenzo, John A. Smolin, William K. Wootters (Submitted on 23 Apr 1996 (v1), last revised 8 Aug 1996 (this version, v2))**

(1) Authors compare the amount of entanglement E (M) required to (e) prepare a mixed state M by (e) local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal.
(2) While EPPs require (e) classical communica- tion, QECCs do not, and authors prove Q is not increased by (e) adding one-way classical communication.
(3) However, both D and Q can be increased by (e) adding two-way communication.
(4) They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for (e) reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available.
(5) Authors exhibit a family of codes based on (e) universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy.
(6) They also obtain a specific, simple 5-bit single- error-correcting quantum block code. They prove that {\em iff} a QECC results in (eb) high fidelity for the case of no error the QECC can be (=) recast into a form where (e) the encoder is the matrix inverse of the decoder.
(7) Comments: Subjects:Quantum Physics (quant-ph) Journal reference:Phys.Rev.A54:3824-3851,1996 DOI:10.1103/PhysRevA.54.3824 Cite as:arXiv:quant-ph/9604024 (or arXiv:quant-ph/9604024v2 for this version)

### NOTATION

**Module One**

Authors compare the amount of entanglement E (M) required to (e) prepare a mixed state M by (e) local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal

$G_{13}$ : Category one of prepare a mixed state M by (e) local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal

| | |
|---|---|
| $G_{14}$ : Category two of SAS<br><br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of amount of entanglement E (M)<br><br>$T_{14}$ : Category two of SAS<br><br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br><br>Authors compare the amount of entanglement E (M) required to prepare a mixed state M by (e) local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal | |
| $G_{16}$ : Category one oflocal actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of amount of entanglement E (M) required to prepare a mixed state M<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Authors compare the amount of entanglement E (M) required to prepare a mixed state M by local actions with (e&eb) the amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal | |
| $G_{20}$ : Category one of **amount of entanglement E (M) required to prepare a mixed state M by local actions;** amounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one ofamounts D1 (M) and D2 (M) that can be locally distilled from it by (e) EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal; **amount of entanglement E (M) required to prepare a mixed state M by local actions**<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |

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| **Module four**<br><br>Authors compare the amount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it by EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal | |
| $G_{24}$ : Category one of EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of compare the amount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors compare the amount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it by EPPs using one- and two-way classical communication respectively, and give (eb) an exact expression for E (M) when (e) M is Bell-diagonal | |
| $G_{28}$ : Category one ofamount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it by EPPs using one- and two-way classical communication respectively<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of exact expression for E (M) when (e) M is Bell-diagonal<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors compare the amount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it by EPPs using one- and two-way classical communication respectively, and give an exact expression for E (M) when (e) M is Bell-diagonal | |
| $G_{32}$ : Category one of M is Bell-diagonal<br><br>$G_{33}$ : Category two of SAS | |

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| $G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of amount of entanglement E (M) required to prepare a mixed state M by local actions with the amounts D1 (M) and D2 (M) that can be locally distilled from it by EPPs using one- and two-way classical communication respectively, and give an exact expression for E (M)<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>While EPPs require (e) classical communica- tion, QECCs do not, and authors prove Q is not increased by (e) adding one-way classical communication | |
| $G_{36}$ : Category one of classical communica- tion, QECCs do not, and authors prove Q is not increased by (e) adding one-way classical communication<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of EPPs<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>While EPPs require classical communica- tion, QECCs do not, and authors prove Q is not increased by (e) adding one-way classical communication | |
| $G_{40}$ : Category one of adding one-way classical communication<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of EPPs require classical communica- tion, QECCs do not, and authors prove Q is not increased<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>However, both D and Q can be increased by (e) adding two-way communication | |

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| $G_{44}$ : Category one of adding two-way communication<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of D and Q can be increased<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}\ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two** | |

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| **The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}), t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |

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| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |

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| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |
| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47}),t)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}((G_{47}),t)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47}),t)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}((G_{47}),t) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |

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| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a'_{18})^{(2)}\boxed{+(a''_{18})^{(2)}(T_{17},t)} & \boxed{+(a''_{15})^{(1,1,)}(T_{14},t)} & \boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18} \quad (63)$$

Where $\boxed{+(a''_{16})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b'_{16})^{(2)}\boxed{-(b''_{16})^{(2)}(G_{19},t)} & \boxed{-(b''_{13})^{(1,1,)}(G,t)} & \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{16} \quad (64)$$

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b'_{17})^{(2)}\boxed{-(b''_{17})^{(2)}(G_{19},t)} & \boxed{-(b''_{14})^{(1,1,)}(G,t)} & \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{17} \quad (65)$$

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b'_{18})^{(2)}\boxed{-(b''_{18})^{(2)}(G_{19},t)} & \boxed{-(b''_{15})^{(1,1,)}(G,t)} & \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{bmatrix} T_{18} \quad (66)$$

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

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| $-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c}(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t)+(a_{16}^{''})^{(2,2,2)}(T_{17},t)+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{44}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c}(a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t)+(a_{17}^{''})^{(2,2,2)}(T_{17},t)+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{45}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c}(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t)+(a_{18}^{''})^{(2,2,2)}(T_{17},t)+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)\\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)\\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)+(a_{46}^{''})^{(9,9,9)}(T_{45},t)\end{array}\right]G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3 | |

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| $+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}[-(b_{20}^{''})^{(3)}(G_{23},t)] & [-(b_{16}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{13}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{44}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{20}$ | 70 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}[-(b_{21}^{''})^{(3)}(G_{23},t)] & [-(b_{17}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{14}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{45}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{21}$ | 71 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}[-(b_{22}^{''})^{(3)}(G_{23},t)] & [-(b_{18}^{''})^{(2,2,2)}(G_{19},t)] & [-(b_{15}^{''})^{(1,1,1,)}(G,t)] \\ [-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)] & [-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)] & [-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)] \\ [-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)] & [-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)] & [-(b_{46}^{''})^{(9,9,9)}(G_{47},t)] \end{bmatrix} T_{22}$ | 72 |
| $-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*<br>$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*<br>$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*<br>$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3<br>$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3<br>$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)}[+(a_{24}^{''})^{(4)}(T_{25},t)] & [+(a_{28}^{''})^{(5,5,)}(T_{29},t)] & [+(a_{32}^{''})^{(6,6,)}(T_{33},t)] \\ [+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)] & [+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)] & [+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)] \\ [+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)] & [+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)] & [+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)] \end{bmatrix} G_{24}$ | 73 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)}[+(a_{25}^{''})^{(4)}(T_{25},t)] & [+(a_{29}^{''})^{(5,5,)}(T_{29},t)] & [+(a_{33}^{''})^{(6,6)}(T_{33},t)] \\ [+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)] & [+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)] & [+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)] \\ [+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)] & [+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)] & [+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)] \end{bmatrix} G_{25}$ | 74 |

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)}\boxed{+(a_{26}^{''})^{(4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{26} \quad (75)$$

$\boxed{(a_{24}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{25}^{''})^{(4)}(T_{25},t)}, \boxed{(a_{26}^{''})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* $1, 2\ 3$

$\boxed{+(a_{28}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5,5,)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{32}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* $1, 2$ *and* $3$

$\boxed{+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)}$
*are fourth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)}$
*are fifth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)}$
*are sixth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)}$
*are seventh augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)}$
*are eighth augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b_{24}^{'})^{(4)}\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b_{25}^{'})^{(4)}\boxed{-(b_{25}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b_{26}^{'})^{(4)}\boxed{-(b_{26}^{''})^{(4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad (78)$$

*Where* $\boxed{-(b_{24}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{25}^{''})^{(4)}(G_{27},t)}, \boxed{-(b_{26}^{''})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{28}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{29}^{''})^{(5,5,)}(G_{31},t)}, \boxed{-(b_{30}^{''})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{32}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{33}^{''})^{(6,6,)}(G_{35},t)}, \boxed{-(b_{34}^{''})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{13}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1)}(G,t)}$

| | |
|---|---|
| $are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)}$ $are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)}$ $are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)}$ $are\ seventh\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)}$ $are\ eighth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$ $\boxed{-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[\begin{matrix} (a_{28}^{'})^{(5)}\boxed{+(a_{28}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[\begin{matrix} (a_{29}^{'})^{(5)}\boxed{+(a_{29}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[\begin{matrix} (a_{30}^{'})^{(5)}\boxed{+(a_{30}^{''})^{(5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{30}$ | *81* |
| $Where\ \boxed{+(a_{28}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5)}(T_{29},t)}\ are\ first\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$ $And\ \boxed{+(a_{24}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,)}(T_{25},t)}\ are\ second\ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{32}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6)}(T_{33},t)}\ are\ third\ \ augmentation\ coefficient\ for\ category\ 1,2\ and\ 3$ $\boxed{+(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}^{''})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3 $\boxed{+(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3 $\boxed{+(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3 $\boxed{+(a_{40}^{''})^{(8,8\ ,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3 $\boxed{+(a_{46}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation | |

coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{array}{ccc} (b_{28}^{'})^{(5)} & -(b_{28}^{''})^{(5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{13}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{array}{ccc} (b_{29}^{'})^{(5)} & -(b_{29}^{''})^{(5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{14}^{''})^{(1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{array}{ccc} (b_{30}^{'})^{(5)} & -(b_{30}^{''})^{(5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,)}(G_{27},t) \\ -(b_{15}^{''})^{(1,1,1,1,1,)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t) \\ -(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t) \end{array}\right] T_{30} \quad (84)$$

*where* $-(b_{28}^{''})^{(5)}(G_{31},t)$ , $-(b_{29}^{''})^{(5)}(G_{31},t)$ , $-(b_{30}^{''})^{(5)}(G_{31},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{25}^{''})^{(4,4,)}(G_{27},t)$ , $-(b_{26}^{''})^{(4,4,)}(G_{27},t)$ *are second detrition coefficients for category* 1,2 *and* 3

$-(b_{32}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6)}(G_{35},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ , $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$ , $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[\begin{array}{ccc} (a_{32}^{'})^{(6)} & +(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{array}\right] G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[\begin{array}{ccc} (a_{33}^{'})^{(6)} & +(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) \\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t) \end{array}\right] G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}')^{(6)}\boxed{+(a_{34}'')^{(6)}(T_{33},t)} & \boxed{+(a_{30}'')^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{26}'')^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}'')^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{34} \quad (87)$$

$\boxed{+(a_{32}'')^{(6)}(T_{33},t)}, \boxed{+(a_{33}'')^{(6)}(T_{33},t)}, \boxed{+(a_{34}'')^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{28}'')^{(5,5,5)}(T_{29},t)}, \boxed{+(a_{29}'')^{(5,5,5)}(T_{29},t)}, \boxed{+(a_{30}'')^{(5,5,5)}(T_{29},t)}$ *are second augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{24}'')^{(4,4,4,)}(T_{25},t)}, \boxed{+(a_{25}'')^{(4,4,4,)}(T_{25},t)}, \boxed{+(a_{26}'')^{(4,4,4,)}(T_{25},t)}$ *are third augmentation coefficients for category* $1, 2$ *and* $3$

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{14}'')^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a_{15}'')^{(1,1,1,1,1,1)}(T_{14},t)}$ - are fourth augmentation coefficients

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}'')^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}'')^{(2,2,2,2,2,2)}(T_{17},t)}$ - fifth augmentation coefficients

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}'')^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}'')^{(3,3,3,3,3,3)}(T_{21},t)}$ sixth augmentation coefficients

$\boxed{+(a_{36}'')^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{37}'')^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a_{38}'')^{(7,7,7,7,7,7,7)}(T_{37},t)}$ seventh augmentation coefficients

$\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8)}(T_{41},t)}$ Eighth augmentation coefficients

$\boxed{+(a_{44}'')^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{45}'')^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a_{46}'')^{(9,9,9,9,9,9)}(T_{45},t)}$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}')^{(6)}\boxed{-(b_{32}'')^{(6)}(G_{35},t)} & \boxed{-(b_{28}'')^{(5,5,5)}(G_{31},t)} & \boxed{-(b_{24}'')^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}'')^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}')^{(6)}\boxed{-(b_{33}'')^{(6)}(G_{35},t)} & \boxed{-(b_{29}'')^{(5,5,5)}(G_{31},t)} & \boxed{-(b_{25}'')^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b_{14}'')^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{17}'')^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}'')^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}'')^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}'')^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}'')^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}')^{(6)}\boxed{-(b_{34}'')^{(6)}(G_{35},t)} & \boxed{-(b_{30}'')^{(5,5,5)}(G_{31},t)} & \boxed{-(b_{26}'')^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b_{15}'')^{(1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{18}'')^{(2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}'')^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}'')^{(7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}'')^{(8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}'')^{(9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{34} \quad (90)$$

$\boxed{-(b_{32}'')^{(6)}(G_{35},t)}, \boxed{-(b_{33}'')^{(6)}(G_{35},t)}, \boxed{-(b_{34}'')^{(6)}(G_{35},t)}$ *are first detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{28}'')^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{29}'')^{(5,5,5)}(G_{31},t)}, \boxed{-(b_{30}'')^{(5,5,5)}(G_{31},t)}$ *are second detrition coefficients for category* $1, 2$ *and* $3$

$\boxed{-(b_{24}'')^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{25}'')^{(4,4,4,)}(G_{27},t)}, \boxed{-(b_{26}'')^{(4,4,4,)}(G_{27},t)}$ *are third detrition coefficients*

| | |
|---|---|
| $for\ category$ 1,2 $and$ 3<br>$\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)}, \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}, \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}, \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}, \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}, \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}, \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3 | |
| $\frac{dG_{36}}{dt}$<br>$= (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}\boxed{+(a_{36}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *91* |
| $\frac{dG_{37}}{dt}$<br>$= (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}\boxed{+(a_{37}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *92* |
| $\frac{dG_{38}}{dt}$<br>$= (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}\boxed{+(a_{38}^{''})^{(7)}(T_{37},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *93* |
| Where $\boxed{(a_{36}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{37}^{''})^{(7)}(T_{37},t)}, \boxed{(a_{38}^{''})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}, \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}, \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)}, \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation | |

coefficient for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3 $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

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$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3 $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[\begin{matrix} (a_{40}^{'})^{(8)}\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[\begin{matrix} (a_{41}^{'})^{(8)}\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[\begin{matrix} (a_{42}^{'})^{(8)}\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix}\right] G_{15}$$

Where $\boxed{+(a_{40}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \begin{bmatrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \begin{bmatrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3



$$\frac{dG_{44}}{dt} = (a_{44})^{(9)} G_{45} - \left[ \begin{array}{ccc} (a'_{44})^{(9)} & +(a''_{44})^{(9)}(T_{45},t) & +(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) \quad +(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{array} \right] G_{13}$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)} G_{44} - \left[ \begin{array}{ccc} (a'_{45})^{(9)} & +(a''_{45})^{(9)}(T_{45},t) & +(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) \quad +(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{array} \right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)} G_{45} - \left[ \begin{array}{ccc} (a'_{46})^{(9)} & +(a''_{46})^{(9)}(T_{37},t) & +(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t) \quad +(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t) \\ +(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t) \end{array} \right] G_{15}$$

Where $+(a''_{44})^{(9)}(T_{45},t)$, $+(a''_{45})^{(9)}(T_{45},t)$, $+(a''_{46})^{(9)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$, $+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)$ are Seventh augmentation coefficient for category 1, 2 and 3

$+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ $+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)} T_{45} -$$

$$\begin{bmatrix} (b_{44}^{'})^{(9)}\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

| | |
|---|---|
| $(a_i^{''})^{(1)}(T_{14},t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$<br><br>$(b_i^{''})^{(1)}(G,t) \leq (r_i)^{(1)} \leq (b_i^{'})^{(1)} \leq (\hat{B}_{13})^{(1)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$<br><br>**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br><br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | 98 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(1)}(T_{14}^{'},t) - (a_i^{''})^{(1)}(T_{14},t)\rvert \leq (\hat{k}_{13})^{(1)}\lvert T_{14} - T_{14}^{'}\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\lvert(b_i^{''})^{(1)}(G^{'},t) - (b_i^{''})^{(1)}(G,t)\rvert < (\hat{k}_{13})^{(1)}\lvert\lvert G - G^{'}\rvert\rvert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and $(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i^{'})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i^{'})^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i^{'})^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i^{'})^{(2)}, (a_i^{''})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (b_i^{''})^{(2)} > 0, \quad i,j = 16,17,18$ | |
| The functions $(a_i^{''})^{(2)}, (b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}$, $(r_i)^{(2)}$: | |

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|---|---|
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(2)}(T_{17},t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i^{''})^{(2)}((G_{19}),t) = (r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br>Where $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}$ are positive constants and $i = 16,17,18$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'},t) - (a_i^{''})^{(2)}(T_{17},t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'},t) - (b_i^{''})^{(2)}((G_{19}),t)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\widehat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'},t)$ and $(a_i^{''})^{(2)}(T_{17},t)$ . $(T_{17}^{'},t)$ and $(T_{17},t)$ are points belonging to the interval $[(\hat{k}_{16})^{(2)}, (\widehat{M}_{16})^{(2)}]$ . It is to be noted that $(a_i^{''})^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\widehat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br>$\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\widehat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\widehat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i,j = 20,21,22$ | 112 |

| | |
|---|---|
| The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21},t) \le (p_i)^{(3)} \le (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23},t) \le (r_i)^{(3)} \le (b_i^{'})^{(3)} \le (\hat{B}_{20})^{(3)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(3)}(T_{21},t) = (p_i)^{(3)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | 113 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(3)}(T_{21}^{'},t) - (a_i^{''})^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}^{'}\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i^{''})^{(3)}(G_{23}{}^{'},t) - (b_i^{''})^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}{}^{'}\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(3)}(T_{21}^{'},t)$ and$(a_i^{''})^{(3)}(T_{21},t)$ . $(T_{21}^{'},t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i^{''})^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i^{''})^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i^{'})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i^{'})^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i^{'})^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i^{'})^{(4)}, (a_i^{''})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (b_i^{''})^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i^{''})^{(4)}, (b_i^{''})^{(4)}$ are positive continuous increasing and bounded. | 117 |

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|---|---|
| **Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i^{''})^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i^{''})^{(4)}((G_{27}),t) \leq (r_i)^{(4)} \leq (b_i^{'})^{(4)} \leq (\hat{B}_{24})^{(4)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(4)}(T_{25},t) = (p_i)^{(4)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(4)}((G_{27}),t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(4)}(T_{25}^{'},t) - (a_i^{''})^{(4)}(T_{25},t)\rvert \leq (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}^{'}\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i^{''})^{(4)}((G_{27})^{'},t) - (b_i^{''})^{(4)}((G_{27}),t)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})^{'}\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}, \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i^{'})^{(4)}, (b_i)^{(4)}, (b_i^{'})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i^{'})^{(4)} + (\hat{A}_{24})^{(4)} + (\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)} + (b_i^{'})^{(4)} + (\hat{B}_{24})^{(4)} + (\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i^{'})^{(5)}, (a_i^{''})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (b_i^{''})^{(5)} > 0, \quad i,j = 28,29,30$<br><br>The functions $(a_i^{''})^{(5)}, (b_i^{''})^{(5)}$ are positive continuous increasing and bounded. | 122 |

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|---|---|
| **Definition of** $(p_i)^{(5)}$, $(r_i)^{(5)}$:<br>$(a_i^{''})^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br>$(b_i^{''})^{(5)}\big((G_{31}),t\big) \le (r_i)^{(5)} \le (b_i^{'})^{(5)} \le (\hat{B}_{28})^{(5)}$ | |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t) = (p_i)^{(5)}$<br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t) = (r_i)^{(5)}$<br>**Definition of** $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :<br>Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t) - (a_i^{''})^{(5)}(T_{29},t)\rvert \le (\hat{k}_{28})^{(5)}\lvert T_{29} - T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t) - (b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert < (\hat{k}_{28})^{(5)}\lvert\lvert(G_{31}) - (G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and $(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ :<br>$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants<br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ :<br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i^{'})^{(5)}, (b_i)^{(5)}, (b_i^{'})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i^{'})^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$<br>$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i^{'})^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)}, (a_i^{'})^{(6)}, (a_i^{''})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (b_i^{''})^{(6)} > 0, \quad i,j = 32,33,34$ | 127 |

The functions $(a_i^{''})^{(6)}, (b_i^{''})^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)}$, $(r_i)^{(6)}$:

$(a_i^{''})^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$

$(b_i^{''})^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2 \to \infty}(a_i^{''})^{(6)}(T_{33}, t) = (p_i)^{(6)}$ 128

$\lim_{G \to \infty}(b_i^{''})^{(6)}\left((G_{35}), t\right) = (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'}, t) - (a_i^{''})^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'}, t) - (b_i^{''})^{(6)}\left((G_{35}), t\right)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'}, t)$ and$(a_i^{''})^{(6)}(T_{33}, t)$ . $(T_{33}^{'}, t)$ and $(T_{33}, t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33}, t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i^{'})^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i^{'})^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}(\hat{k}_{32})^{(6)}] < 1$

Where we suppose

(CCC) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(DDD) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}$, $(r_i)^{(7)}$:

$(a_i^{''})^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39}, t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(EEE)* $\lim_{T_2 \to \infty} (a_i^{''})^{(7)}(T_{37}, t) = (p_i)^{(7)}$

*(FFF)*

$\lim_{G \to \infty} (b_i^{''})^{(7)}((G_{39}), t) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)} |T_{37} - T_{37}^{'}| e^{-(\hat{M}_{36})^{(7)} t}$

$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)} ||(G_{39}) - (G_{39})^{'}|| e^{-(\hat{M}_{36})^{(7)} t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(GGG) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$, are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(HHH) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}} [(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$

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| $\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} (\hat{k}_{36})^{(7)}] < 1$ | |
| Where we suppose | |
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}$ are positive constants and $i = 40,41,42$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42,$ | |

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| Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i')^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}(\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with | |

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| $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$<br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>Definition of $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$<br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions | 152 |

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| **Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |

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| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{20})^{(3)} + a''_{20})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a'_{21})^{(3)} + (a''_{21})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[ (a_{22})^{(3)} G_{21}(s_{(20)}) - \left( (a'_{22})^{(3)} + (a''_{22})^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{22}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[ (b_{20})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{20})^{(3)} - (b''_{20})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{20}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[ (b_{21})^{(3)} T_{20}(s_{(20)}) - \left( (b'_{21})^{(3)} - (b''_{21})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{21}(s_{(20)}) \right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[ (b_{22})^{(3)} T_{21}(s_{(20)}) - \left( (b'_{22})^{(3)} - (b''_{22})^{(3)} (G_{23}(s_{(20)}), s_{(20)}) \right) T_{22}(s_{(20)}) \right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[ (a_{24})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{24})^{(4)} + a''_{24})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{24}(s_{(24)}) \right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[ (a_{25})^{(4)} G_{24}(s_{(24)}) - \left( (a'_{25})^{(4)} + (a''_{25})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[ (a_{26})^{(4)} G_{25}(s_{(24)}) - \left( (a'_{26})^{(4)} + (a''_{26})^{(4)} (T_{25}(s_{(24)}), s_{(24)}) \right) G_{26}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[ (b_{24})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{24})^{(4)} - (b''_{24})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{24}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[ (b_{25})^{(4)} T_{24}(s_{(24)}) - \left( (b'_{25})^{(4)} - (b''_{25})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{25}(s_{(24)}) \right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[ (b_{26})^{(4)} T_{25}(s_{(24)}) - \left( (b'_{26})^{(4)} - (b''_{26})^{(4)} (G_{27}(s_{(24)}), s_{(24)}) \right) T_{26}(s_{(24)}) \right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to$ | |

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| $\mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[ (a_{28})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{28})^{(5)} + a''_{28})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{28}(s_{(28)}) \right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[ (a_{29})^{(5)} G_{28}(s_{(28)}) - \left( (a'_{29})^{(5)} + (a''_{29})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[ (a_{30})^{(5)} G_{29}(s_{(28)}) - \left( (a'_{30})^{(5)} + (a''_{30})^{(5)} (T_{29}(s_{(28)}), s_{(28)}) \right) G_{30}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[ (b_{28})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{28})^{(5)} - (b''_{28})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{28}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[ (b_{29})^{(5)} T_{28}(s_{(28)}) - \left( (b'_{29})^{(5)} - (b''_{29})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{29}(s_{(28)}) \right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[ (b_{30})^{(5)} T_{29}(s_{(28)}) - \left( (b'_{30})^{(5)} - (b''_{30})^{(5)} (G_{31}(s_{(28)}), s_{(28)}) \right) T_{30}(s_{(28)}) \right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$ | |

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| $\bar{T}_{33}(t) = T_{33}^{0} + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^{0} + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\bar{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}^{'})^{(7)} + a_{36}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^{0} + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^{0} + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{\mathrm{T}}_{38}(\mathrm{t}) = \mathrm{T}_{38}^{0} + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |

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| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\bar{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[ (a_{42})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{42})^{(8)} + (a''_{42})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{42}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[ (b_{40})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{40})^{(8)} - (b''_{40})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{40}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[ (b_{41})^{(8)} T_{40}(s_{(40)}) - \left( (b'_{41})^{(8)} - (b''_{41})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{41}(s_{(40)}) \right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[ (b_{42})^{(8)} T_{41}(s_{(40)}) - \left( (b'_{42})^{(8)} - (b''_{42})^{(8)} (G_{43}(s_{(40)}), s_{(40)}) \right) T_{42}(s_{(40)}) \right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{44})^{(9)} + a''_{44})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[ (a_{45})^{(9)} G_{44}(s_{(44)}) - \left( (a'_{45})^{(9)} + (a''_{45})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[ (a_{46})^{(9)} G_{45}(s_{(44)}) - \left( (a'_{46})^{(9)} + (a''_{46})^{(9)} (T_{45}(s_{(44)}), s_{(44)}) \right) G_{46}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[ (b_{44})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{44})^{(9)} - (b''_{44})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{44}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[ (b_{45})^{(9)} T_{44}(s_{(44)}) - \left( (b'_{45})^{(9)} - (b''_{45})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{45}(s_{(44)}) \right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[ (b_{46})^{(9)} T_{45}(s_{(44)}) - \left( (b'_{46})^{(9)} - (b''_{46})^{(9)} (G_{47}(s_{(44)}), s_{(44)}) \right) T_{46}(s_{(44)}) \right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |

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| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)}\left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$ $\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that $(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 1 | 168 |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$ $\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$ | 169 |
| From which it follows that $(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$ $\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$ | 171 |
| From which it follows that $(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that $G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$ $\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$ | 173 |

From which it follows that

$$(G_{24}(t)-G_{24}^{0})e^{-(\bar{M}_{24})^{(4)}t} \leq \frac{(a_{24})^{(4)}}{(\bar{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^{0})e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^{0}}{G_{25}^{0}}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4 — 174

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \leq G_{28}^{0}+\int_0^t\left[(a_{28})^{(5)}\left(G_{29}^{0}+(\hat{P}_{28})^{(5)}e^{(\bar{M}_{28})^{(5)}s_{(28)}}\right)\right]ds_{(28)} =$$

$$\left(1+(a_{28})^{(5)}t\right)G_{29}^{0}+\frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left(e^{(\bar{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^{0})e^{-(\bar{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\bar{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^{0})e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^{0}}{G_{29}^{0}}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5 — 175

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 176

$$G_{32}(t) \leq G_{32}^{0}+\int_0^t\left[(a_{32})^{(6)}\left(G_{33}^{0}+(\hat{P}_{32})^{(6)}e^{(\bar{M}_{32})^{(6)}s_{(32)}}\right)\right]ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^{0}+\frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left(e^{(\bar{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^{0})e^{-(\bar{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\bar{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^{0})e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^{0}}{G_{33}^{0}}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6 — 177

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(j) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that — 178

$$G_{36}(t) \leq G_{36}^{0}+\int_0^t\left[(a_{36})^{(7)}\left(G_{37}^{0}+(\hat{P}_{36})^{(7)}e^{(\bar{M}_{36})^{(7)}s_{(36)}}\right)\right]ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^{0}+\frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\bar{M}_{36})^{(7)}}\left(e^{(\bar{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^{0})e^{-(\hat{M}_{36})^{(7)}t}\leq\frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^{0})e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^{0}}{G_{37}^{0}}\right)}+(\hat{P}_{36})^{(7)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t)\leq G_{40}^{0}+\int_0^t\left[(a_{40})^{(8)}\left(G_{41}^{0}+(\hat{P}_{40})^{(8)}e^{(\hat{M}_{40})^{(8)}s_{(40)}}\right)\right]ds_{(40)}=$$

$$\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right) \quad (180)$$

From which it follows that (181)

$$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$$

$$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$$

From which it follows that

$$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose (182)

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)} \quad (183)$$

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)} \quad (184)$$

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

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| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | 185 |
| Indeed if we denote<br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br>It results<br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}ds_{(13)}+$<br>$\int_0^t\{(a_{13}^{'})^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br>$G_{13}^{(2)}\left\|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br>Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t}\le$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i=13,14,15$ depend only on $T_{14}$ and respectively on $G(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$<br>From 19 to 24 it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0$ for t > 0 | |

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| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ :<br><br>**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br><br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a'_{14})^{(1)} G_{14}$ and by integrating<br><br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a'_{14})^{(1)}$<br><br>In the same way , one can obtain<br><br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a'_{15})^{(1)}$<br><br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | 187 |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(1)}(G(t),t)) = (b_{14}^{'})^{(1)}$ then $T_{14} \to \infty$.<br><br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br><br>Indeed let $t_1$ be so that for $t > t_1$<br><br>$(b_{14})^{(1)} - (b_i^{''})^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br><br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded. The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}^{''})^{(1)}(G(t),t) = (b_{15}^{'})^{(1)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br><br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying | 193 |

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| Equations into itself | |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}},\widetilde{T_{19}}$ : $\left(\widetilde{G_{19}},\widetilde{T_{19}}\right)=\mathcal{A}^{(2)}(G_{19},T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{16})^{(2)}\left\|G_{17}^{(1)}-G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)}+$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)-(a_{16}^{''})^{(2)}\left(T_{17}^{(2)},s_{(16)}\right)\right\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)}-(G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t}\le$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+(a_{16}^{'})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)},(T_{19})^{(1)};(G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(2)}-(a_i^{''})^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}ds_{(16)}\right]}\ge 0$ | 200 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(2)}t)} > 0$ for $t > 0$ | |
| **Definition of** $((\widehat{M}_{16})^{(2)})_1$, $((\widehat{M}_{16})^{(2)})_2$ and $((\widehat{M}_{16})^{(2)})_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq ((\widehat{M}_{16})^{(2)})_1 - (a_{17}')^{(2)}G_{17}$ and by integrating<br><br>$G_{17} \leq ((\widehat{M}_{16})^{(2)})_2 = G_{17}^0 + 2(a_{17})^{(2)}((\widehat{M}_{16})^{(2)})_1/(a_{17}')^{(2)}$<br><br>In the same way , one can obtain<br><br>$G_{18} \leq ((\widehat{M}_{16})^{(2)})_3 = G_{18}^0 + 2(a_{18})^{(2)}((\widehat{M}_{16})^{(2)})_2/(a_{18}')^{(2)}$<br><br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | 201 |
| **Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br><br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br><br>Indeed let $t_2$ be so that for $t > t_2$<br><br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br><br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}((G_{19})(t),t) = (b_{18}')^{(2)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br><br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + ((\widehat{P}_{20})^{(3)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[((\widehat{Q}_{20})^{(3)} + T_j^0)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |

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| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}')^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}'')^{(3)}(T_{21}^{(1)},s_{(20)})-(a_{20}'')^{(3)}(T_{21}^{(2)},s_{(20)})\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+(a_{20}')^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}'')^{(3)}$ and $(b_{20}'')^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(3)}$ and $(b_i'')^{(3)}, i=20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i')^{(3)}-(a_i'')^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$ | 216 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(3)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{20})^{(3)})_1, ((\widehat{M}_{20})^{(3)})_2$ *and* $((\widehat{M}_{20})^{(3)})_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq ((\widehat{M}_{20})^{(3)})_1 - (a_{21}')^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq ((\widehat{M}_{20})^{(3)})_2 = G_{21}^0 + 2(a_{21})^{(3)} ((\widehat{M}_{20})^{(3)})_1 / (a_{21}')^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq ((\widehat{M}_{20})^{(3)})_3 = G_{22}^0 + 2(a_{22})^{(3)} ((\widehat{M}_{20})^{(3)})_2 / (a_{22}')^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}((G_{23})(t),t)) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i'')^{(3)}((G_{23})(t),t) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}((G_{23})(t),t) = (b_{22}')^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + ((\widehat{P}_{24})^{(4)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[((\widehat{Q}_{24})^{(4)} + T_j^0)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$ | 228 |

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| $T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1/(a_{25}^{'})^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}^{'})^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(4)}((G_{27})(t),t)) = (b_{25}^{'})^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i^{''})^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}\,(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}^{''})^{(4)}\left((G_{27})(t),t\right) = (b_{26}^{'})^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |

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| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}\ and\ (\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(5)}$ and $(b_i'')^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0\ and\ T_i(t)=0$ | 239 |

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| it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(5)} - (a_i'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(5)} t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{28})^{(5)})_1, ((\widehat{M}_{28})^{(5)})_2$ *and* $((\widehat{M}_{28})^{(5)})_3$ :<br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq ((\widehat{M}_{28})^{(5)})_1 - (a_{29}')^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq ((\widehat{M}_{28})^{(5)})_2 = G_{29}^0 + 2(a_{29})^{(5)} ((\widehat{M}_{28})^{(5)})_1 / (a_{29}')^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq ((\widehat{M}_{28})^{(5)})_3 = G_{30}^0 + 2(a_{30})^{(5)} ((\widehat{M}_{28})^{(5)})_2 / (a_{30}')^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t), t)) = (b_{29}')^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}'')^{(5)}((G_{31})(t), t) = (b_{30}')^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |

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| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)}+\left((\hat{P}_{32})^{(6)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on | 249 |

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| $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i')^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}\ ((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33}\ \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t\ > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval [0, t]<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 260 |

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| suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(7)}((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\,(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}^{''})^{(7)}\left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |

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| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose $(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$ $\sup_i\{\max_{t\in\mathbb{R}_+} \left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+} \left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote <br> **Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results <br> $\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\, ds_{(40)} +$ <br> $\int_0^t \{(a'_{40})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$ <br> $(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$ <br> $G_{40}^{(2)}\left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t} \leq$ <br> $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)}, (T_{43})^{(1)}; (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ <br> And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 274 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ $and$ $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(8)}t\right)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ $and$ $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ $and$ $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}$ $or$ $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ $is$ bounded, from below, the same property holds for $G_{41}$ $and$ $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results | 279 |

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $(\widetilde{G_{47}}), (\widetilde{T_{47}}) : \left((\widetilde{G_{47}}), (\widetilde{T_{47}})\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by

$(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \le -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \le -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \le -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G, t) - (b''_{14})^{(1)}(G, t) \le -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$, $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$, $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$, $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$, $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

are defined

Then the solution of global equations satisfies the inequalities 285

$G^0_{13} e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G^0_{13} e^{(S_1)^{(1)} t}$

where $(p_i)^{(1)}$ is defined by equation

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| $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$ $\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}')^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}')^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(T_{17}, t) + (a_{17}'')^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}((G_{19}), t) - (b_{17}'')^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |

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| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$,<br>and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ | 315 |

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| $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}^{'})^{(2)}$<br>$(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)}=(b_{18}^{'})^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)}\le-(a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}-(a_{20}^{''})^{(3)}(T_{21},t)+(a_{21}^{''})^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)}\le-(b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{21}^{''})^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$ :<br>By $(v_1)^{(3)}>0$ , $(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0$ , $(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and<br>By $(\bar{v}_1)^{(3)}>0$ , $(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0$ , $(\bar{u}_2)^{(3)}<0$ the<br>roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)}=(v_0)^{(3)},(m_1)^{(3)}=(v_1)^{(3)}$, ***if*** $(v_0)^{(3)}<(v_1)^{(3)}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)}<(v_0)^{(3)}<(\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)}=\frac{G_{20}^{0}}{G_{21}^{0}}}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)}<(v_0)^{(3)}$ | 321 |

and analogously

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)},\ \boldsymbol{if}\ (u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}, \boldsymbol{if}\ (u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)},$ and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},\ \boldsymbol{if}\ (\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$

$(p_i)^{(3)}$ is defined by equation

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$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$

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$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$

$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a'_{22})^{(3)} t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)} t}\right)$

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$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$

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$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$

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$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b'_{22})^{(3)} t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)} t} \le T_{22}(t) \le$

$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$

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**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$

$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$

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**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

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| $-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)} + (b_{25}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27}),t\big) - (b_{25}^{''})^{(4)}\big((G_{27}),t\big) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)},, (\bar{v}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)}, (\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br><br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |

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| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$ $\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$ $(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$ $(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$ $(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 338 |
| **Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ : $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying $-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$ $-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 339 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 340 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ **Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :- If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by $(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$ $(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$, | |

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| and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)},$ ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},$ ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)},$ ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{28}(t) \leq G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{29}(t) \leq \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \leq G_{30}(t) \leq$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \leq T_{30}(t) \leq$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** | 349 |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying

$$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$$

$$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}), t\big) - (b''_{33})^{(6)}\big((G_{35}), t\big) \le -(\tau_1)^{(6)}$$

**Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ : 350

By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(6)}, , (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$

**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by

$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,
and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$

and analogously 352

$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,
and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$

Then the solution of global equations satisfies the inequalities 353

$$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$$

where $(p_i)^{(6)}$ is defined by equation

| | |
|---|---|
| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G_{34}^0 e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$ *(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}')^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}')^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}')^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}')^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}')^{(6)}$ $(R_2)^{(6)} = (b_{34}')^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$: $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}')^{(7)} + (a_{37}')^{(7)} - (a_{36}'')^{(7)}(T_{37}, t) + (a_{37}'')^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}')^{(7)} + (b_{37}')^{(7)} - (b_{36}'')^{(7)}((G_{39}), t) - (b_{37}'')^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$: By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the | 362 |

roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$

and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,

and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,

and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

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Then the solution of global equations satisfies the inequalities

$$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$$

where $(p_i)^{(7)}$ is defined by equation

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$$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$$

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$$\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le \right.$$

$$\left.\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})}\left[e^{(S_1)^{(7)}t} - e^{-(a_{38}')^{(7)}t}\right] + G_{38}^0 e^{-(a_{38}')^{(7)}t}\right)$$

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$$\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$$

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$$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$$

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$$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b_{38}')^{(7)}t}\right] + T_{38}^0 e^{-(b_{38}')^{(7)}t} \le T_{38}(t) \le$$

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|---|---|
| $\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | |
| **Definition of** $(S_1)^{(7)},(S_2)^{(7)},(R_1)^{(7)},(R_2)^{(7)}$:-<br>Where $(S_1)^{(7)}=(a_{36})^{(7)}(m_2)^{(7)}-(a_{36}')^{(7)}$<br>$(S_2)^{(7)}=(a_{38})^{(7)}-(p_{38})^{(7)}$<br>$(R_1)^{(7)}=(b_{36})^{(7)}(\mu_2)^{(7)}-(b_{36}')^{(7)}$<br>$(R_2)^{(7)}=(b_{38}')^{(7)}-(r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 371 |
| **Definition of** $(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)}\le-(a_{40}')^{(8)}+(a_{41}')^{(8)}-(a_{40}'')^{(8)}(T_{41},t)+(a_{41}'')^{(8)}(T_{41},t)\le-(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)}\le-(b_{40}')^{(8)}+(b_{41}')^{(8)}-(b_{40}'')^{(8)}((G_{43}),t)-(b_{41}'')^{(8)}((G_{43}),t)\le-(\tau_1)^{(8)}$ | 372 |
| **Definition of** $(\nu_1)^{(8)},(\nu_2)^{(8)},(u_1)^{(8)},(u_2)^{(8)},\nu^{(8)},u^{(8)}$ :<br>By $(\nu_1)^{(8)}>0,(\nu_2)^{(8)}<0$ and respectively $(u_1)^{(8)}>0,(u_2)^{(8)}<0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_1)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_1)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},,(\bar{\nu}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{\nu}_1)^{(8)}>0,(\bar{\nu}_2)^{(8)}<0$ and respectively $(\bar{u}_1)^{(8)}>0,(\bar{u}_2)^{(8)}<0$ the<br>roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_2)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_2)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$<br>**Definition of** $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)}=(\nu_0)^{(8)},(m_1)^{(8)}=(\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)}<(\nu_1)^{(8)}$ | |

| | |
|---|---|
| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , $\boldsymbol{if}$ $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, $\boldsymbol{if}$ $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , $\boldsymbol{if}$ $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}$ $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\right)\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a_{42}')^{(8)})}[e^{(S_1)^{(8)} t} - e^{-(a_{42}')^{(8)} t}] + G_{42}^0 e^{-(a_{42}')^{(8)} t})$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b_{42}')^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b_{42}')^{(8)} t}\right] + T_{42}^0 e^{-(b_{42}')^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a_{40}')^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^{0}e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}}G_{44}^{0}e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}}G_{44}^{0}e^{(S_1)^{(9)}t}$$

(

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t} \le G_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}^{'})^{(9)})}\left[e^{(S_1)^{(9)}t}-e^{-(a_{46}^{'})^{(9)}t}\right]+\ G_{46}^{0}e^{-(a_{46}^{'})^{(9)}t})$$

$$\boxed{T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}^{'})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b_{46}^{'})^{(9)}t}\right]+T_{46}^{0}e^{-(b_{46}^{'})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}^{'})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}^{'})^{(9)}$

$(R_2)^{(9)} = (b_{46}^{'})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain

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$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a_{13}^{'})^{(1)} - (a_{14}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14},t)\right) - (a_{14}^{''})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \le (\bar{\nu}_1)^{(2)}$$

| | |
|---|---|
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$ | 398 |
| **Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$ | 400 |

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

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And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$(m_2)^{(4)} \leq v^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{v^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $v^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)v^{(5)} - (a_{29})^{(5)}v^{(5)}$$

**Definition of** $v^{(5)}$ :- $\boxed{v^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)}\right) \leq \frac{dv^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$v^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq v^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get 409

$$\nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)}-(\nu_0)^{(5)}}{(\nu_0)^{(5)}-(\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \le \frac{(\nu_1)^{(5)}+(C)^{(5)}(\nu_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}} \le \nu^{(5)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \le (\bar{\nu}_1)^{(5)} \le \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \le \nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \le \nu^{(5)}(t) \le (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \le u^{(5)}(t) \le (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right) - (a_{33}^{''})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_2)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)\le\frac{dv^{(6)}}{dt}\le-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_1)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)},(v_0)^{(6)}$ :-

For $0<\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}<(v_1)^{(6)}<(\bar{v}_1)^{(6)}$

$$v^{(6)}(t)\ge\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}\quad,\quad\boxed{(C)^{(6)}=\frac{(v_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(v_2)^{(6)}}}$$

it follows $(v_0)^{(6)}\le v^{(6)}(t)\le(v_1)^{(6)}$

In the same manner , we get

$$v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(6)}=\frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)}\le v^{(6)}(t)\le(\bar{v}_1)^{(6)}$

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If $0<(v_1)^{(6)}<(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}<(\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)}\le\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}\le v^{(6)}(t)\le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(\bar{v}_1)^{(6)}$$

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If $0<(v_1)^{(6)}\le(\bar{v}_1)^{(6)}\le\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)}\le v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)}\le v^{(6)}(t)\le(m_1)^{(6)},\quad\boxed{v^{(6)}(t)=\frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)}\le u^{(6)}(t)\le(\mu_1)^{(6)},\quad\boxed{u^{(6)}(t)=\frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

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**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .
Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then
$(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

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In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

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$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get

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$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

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| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$ | 434 |

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| $with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |

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| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\,]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\,]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |

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| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(j) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:** | 488 |

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| (a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that | 494 |

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| there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)] + (b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right]+(b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right]+(b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ | |

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| Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^*$ such that $\varphi((G_{47})^*) = 0$ | |
| Finally we obtain the unique solution<br>$G_{14}^*$ given by $\varphi(G^*) = 0$ , $T_{14}^*$ given by $f(T_{14}^*) = 0$ and<br>$G_{13}^* = \frac{(a_{13})^{(1)} G_{14}^*}{[(a_{13}')^{(1)} + (a_{13}'')^{(1)}(T_{14}^*)]}$ , $G_{15}^* = \frac{(a_{15})^{(1)} G_{14}^*}{[(a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}^*)]}$<br>$T_{13}^* = \frac{(b_{13})^{(1)} T_{14}^*}{[(b_{13}')^{(1)} - (b_{13}'')^{(1)}(G^*)]}$ , $T_{15}^* = \frac{(b_{15})^{(1)} T_{14}^*}{[(b_{15}')^{(1)} - (b_{15}'')^{(1)}(G^*)]}$<br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^*$ given by $\varphi((G_{19})^*) = 0$ , $T_{17}^*$ given by $f(T_{17}^*) = 0$ and | 512 |
| $G_{16}^* = \frac{(a_{16})^{(2)} G_{17}^*}{[(a_{16}')^{(2)} + (a_{16}'')^{(2)}(T_{17}^*)]}$ , $G_{18}^* = \frac{(a_{18})^{(2)} G_{17}^*}{[(a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}^*)]}$ | 513 |
| $T_{16}^* = \frac{(b_{16})^{(2)} T_{17}^*}{[(b_{16}')^{(2)} - (b_{16}'')^{(2)}((G_{19})^*)]}$ , $T_{18}^* = \frac{(b_{18})^{(2)} T_{17}^*}{[(b_{18}')^{(2)} - (b_{18}'')^{(2)}((G_{19})^*)]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a_{20}')^{(3)} + (a_{20}'')^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a_{24}')^{(4)} + (a_{24}'')^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b_{24}')^{(4)} - (b_{24}'')^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b_{26}')^{(4)} - (b_{26}'')^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution | 518 |

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| $G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 523 A |

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| $T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}} (T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j} ((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)} G_{16}^{*} \mathbb{T}_{17}$ | 534 |

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| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a'_{17})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G^*_{17}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a'_{18})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G^*_{18}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b'_{16})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T^*_{16}\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b'_{17})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T^*_{17}\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b'_{18})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T^*_{18}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(3)}$ *and* $(b''_i)^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G^*_i + \mathbb{G}_i \quad , T_i = T^*_i + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{21})^{(3)}}{\partial T_{21}}(T^*_{21}) = (q_{21})^{(3)} \quad , \frac{\partial(b''_i)^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a'_{20})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G^*_{20}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a'_{21})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G^*_{21}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a'_{22})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G^*_{22}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b'_{20})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T^*_{20}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b'_{21})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T^*_{21}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b'_{22})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T^*_{22}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(4)}$ *and* $(b''_i)^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 548 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^* \ ) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \ , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}(\ (G_{31})^* \ ) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(8)}$ *and* $(b''_i)^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G^*_i + \mathbb{G}_i \quad , T_i = T^*_i + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)}$ , $\frac{\partial(b''_i)^{(8)}}{\partial G_j}((G_{43})^*) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 586 A |

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| **Proof:** Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \quad , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$ $+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ $+$ | |

$$(\lambda)^{(2)} + (b'_{18})^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a'_{18})^{(2)} + (p_{18})^{(2)})$$

$$\left[\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)\right]$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}^{*}_{17}\Big)$$

$$+\left((\lambda)^{(2)} + (a'_{17})^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}^{*}_{16} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17}\Big)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}^{*}_{16}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b'_{16})^{(2)} + (b'_{17})^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^{2} + \left((a'_{16})^{(2)} + (a'_{17})^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a'_{16})^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}^{*}_{17} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}^{*}_{16}\right)$$

$$\left((\lambda)^{(2)} + (b'_{16})^{(2)} - (r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}^{*}_{17} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}^{*}_{16}\Big)\} = 0$$

$$+$$

$$(\lambda)^{(3)} + (b'_{22})^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a'_{22})^{(3)} + (p_{22})^{(3)})$$

$$\left[\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)\right]$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(21)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(21)}T^{*}_{21}\Big)$$

$$+\left((\lambda)^{(3)} + (a'_{21})^{(3)} + (p_{21})^{(3)}\right)(q_{20})^{(3)}G^{*}_{20} + (a_{20})^{(3)}(q_{21})^{(1)}G^{*}_{21}\Big)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(20)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(20)}T^{*}_{20}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b'_{20})^{(3)} + (b'_{21})^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left((\lambda)^{(3)}\right)^{2} + \left((a'_{20})^{(3)} + (a'_{21})^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a'_{20})^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G^{*}_{21} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G^{*}_{20}\right)$$

$$\left((\lambda)^{(3)} + (b'_{20})^{(3)} - (r_{20})^{(3)}\right)s_{(21),(22)}T^{*}_{21} + (b_{21})^{(3)}s_{(20),(22)}T^{*}_{20}\Big)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\Big]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\Big)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\Big]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\Big)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^{2}+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^{2}+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left(\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^{2}+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^{2}+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left(\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\right)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\Big)$$

$$+\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\Big)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(46)}T^{*}_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and**

| | |
|---|---|
| **this proves the theorem.** | |
| Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future**. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk. | |

## Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010) It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and

corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(10) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output,

creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so

on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 10. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon$ $C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\ (x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega\ (U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C :\ \omega\ (x) \neq \emptyset \text{ and } \omega\ (x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s\ (B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\ (x)$ is continuous

**A2:** $A_i\ (x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\ (A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( Γ(A)), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *(Γ(A)).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ Γ(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in*

$||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\in C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\,|B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In}\,|B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In}\, b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}\,\text{In}\,|B_i(t,x)x_i| = \limsup_{t\to\infty}\frac{1}{t}\,\text{In}\,|\,(x.t)_i\,|$$

$$\le \limsup_{t\to\infty}\frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty}\frac{1}{t}In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x})$ =lim $_{t\to\infty}$ $\frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$- *almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\,d\mu(x) - \int_c h(x)d\mu_n(x)| \leq \epsilon$, $|\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)d\mu_n(x)| \leq \epsilon$, and $|h(x.t) - h(x^n.t)| \leq \epsilon$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x)d\mu(x)|$ is

$$\leq |\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)\,d\mu_n(x)| + |\int_c h(x.t)\,d\mu_n(x) - \int_c h(x)\,d\mu(x)|$$

$$\leq \epsilon + \int_c |h(x.t) - h(x^n.t)|\,d\mu_n(x)| + |\int_c h(x^n.t)|\,d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 2\epsilon + |\int_c h(x^n.t) - h(x)\,d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq 3\epsilon$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\epsilon > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\epsilon > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \epsilon$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \epsilon$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak*

compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\in\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\mu \in iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, … . $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t c int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))c *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ 0. $Hence, \Gamma(\tilde{A})$c *V* c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K⊂ ∂ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K⊂ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i{}_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria $E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ ($E_1$)and $\lambda_3$ ($E_1$)are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j (1 - a^j) > -\max_j (1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \quad i = 1, \dots, k; \; j = 1, \dots, m$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $J$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $e_i^j = \sum_l d_i^{jl}$

This guarantees that the set H= $\{x \in C: x_i^j = x_i^l \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \quad i = 1, \dots, k$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with \; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{matrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & O \\ \gamma & my-b(p) \end{matrix}$$

Since {0,($K$,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance $N$ and exhibits logistic dynamics f($N$)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification

doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration

dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.28) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.29) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a,t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.30) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ $[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 10. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\partial P_1, \partial f_1/\partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\partial P_1, \partial f_2/\partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ x\ k_1 x\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ x\ k_2 x\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$P_1(s)=(p_1(s),p_2(s-t)),$

$P_2(s)=(p_1(s-t),p_2(s)),$

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i\,(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i\,(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for\ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}), \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in \left\{ (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\} \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)\,da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)\,da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\pi(P_t) - \Phi_t\|_{t*} \le \left\{ \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^* \right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\pi_1(P_t^{(1)}) - \pi_1(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\pi_1(P^{(1)}(t)) - \pi_1(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a, t, P_t^{(2)}) da \right|$$

$$+\left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^{\infty} \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\left. X\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma\right.$$

$$+\int_o^\infty \varphi(\sigma,o)\left[b_1\left(\sigma+a,P_t^{(1)}(a)\right)-b_1\left(\sigma+aP_t^{(2)}(a)\right)\right],L_1\left(\sigma,a,P_t^{(1)}(a)\right)d\sigma$$

$$+\int_o^\infty \varphi(\sigma,o)b_1\left(\sigma+a,P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma,P_t^{(1)}(a)\right)-b_1\left(\sigma+a,P_t^{(2)}(a)\right)\right|\le B_1\ \|P_t^{(1)}-P_t^{(2)}\|_{t^*} \text{ for } a\ \varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)|\le\beta_1\Phi_1(0)||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a[B_1^{(2)}(\sigma)\{b_1(a\text{-}\sigma,P_t^{(1)}(a))\text{-}b_1(a\text{-}\sigma,P_t^{(2)}(a))\}M_1(\sigma,a,P_t^{(1)}(a))$$

$$+b_1(a,\text{-}\sigma,P_t^{(2)}(a))\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0,P_t^{(2)}(a)\right)\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$\le\int_0^a[\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\beta_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}+\beta_1 d_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}(a\text{-}o)]do$$

$$\le\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\|P_t^{(1)}-P_t^{(2)}\|_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left|B_1^{(1)}(a)\text{-}B_1^{(2)}(a)\right|$$

$$\le\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|B_1^{*d}\sigma$$

$$+\Phi_1(0)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}\left\{B_1+B_2 d_1 a+B_1^*\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le\beta_1^*\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|do+\left(r^*+\|\Phi_t\|_{t*}\right)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}g_1^{(a)}$$

Where $g_1(a)\to\beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} \{ g_1(a)+0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a))-b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^* \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\| \pi \left(P_t^{(1)} - \pi \left(P_t^{(2)}\right) \| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t^*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*)\, P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty b_i(a,\ P_1^*,\ P_2^*) P_i^*(a) da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o) \exp\left[ - \int_o^\infty f_i(s, P_1^*, P_2^*) ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}(P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i(s,P_1,P_2)ds\right]da,$$

$$i=1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}(0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0) < F_{2.}(C,0),$$

(3.8) $$F_1(0.C) < F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$$

$$i=1,2,\quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0$ *and* , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0,\quad i=1,2,$$

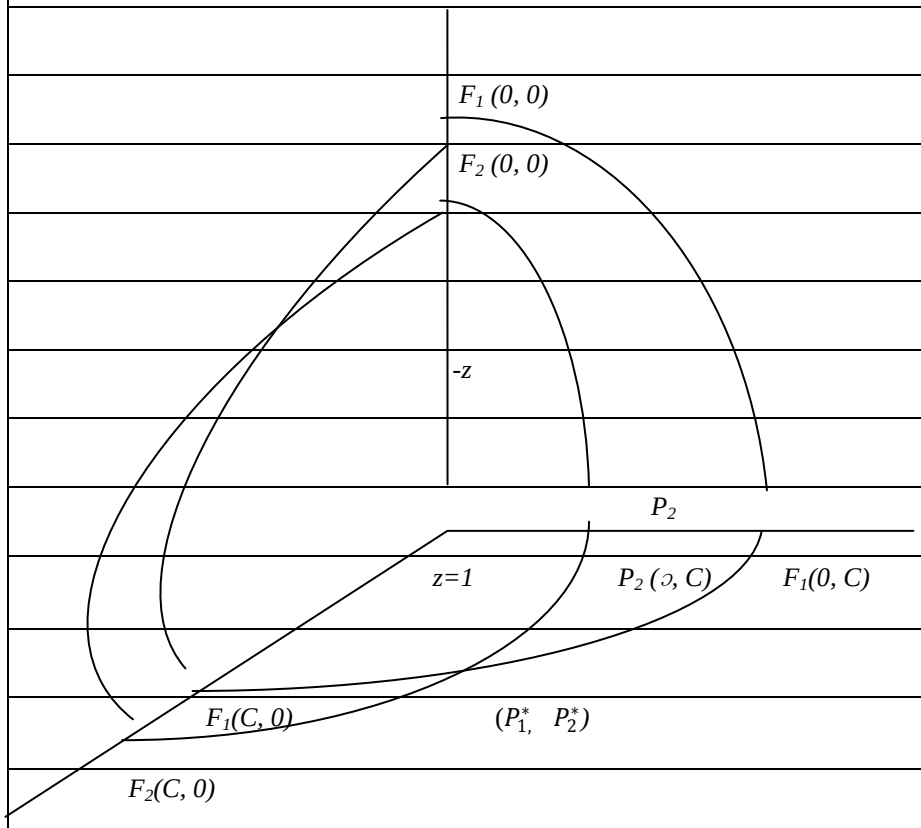
$j=1,2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p^*_1, p^*_2\right)_{\square}(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_{i\square}\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right)_{\square}(a)da + \int_0^\infty p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p^*_j}\, p^*_{i\square}\right\} da,$$

Where $i = 1, 2,$ and

$$p^*_{11} = p^*_1, \quad p^*_{12} = p^*_2 \exp\{-\lambda\tau\},$$

$$p^*_{21} = p^*_1, \exp\{-\lambda\tau\}, \; p^*_{22} = p^*_2.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i(a, p_1^*, p_2^*)\, p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i(a, p_1^*, p_2^*)p_i^*(a)e^{-\lambda a}da, \quad i = 1, 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^*\exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^*\exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{i} = \int_0^{\infty} p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^{\infty}\left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)da,\ i, \square = 1,\ 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a,\ p_1^*, p_2^*\right) - f_i\left(a,\ p_1^*, p_2^*\right),\ i,\ = 1,\ 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - \left(A_{11} + g_{11}\langle\lambda\rangle\right) & -\left(A_{12} + g_{12}\right)e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right)e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}\langle\lambda\rangle\right)\end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21}\exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda,\tau) = \left[A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)\right]\exp\left[-2\lambda\tau\right]$

$$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ = 1,\ 2.)\ \left(see(A_1)\ and\ (A_2)\right)$;

$A_7.\ \left\{A_{11} + A_{22} + \|\,S_1(0)\,\|\right\}^2 < 4\left\{(A_{11}A_{22} - A_{12}A_{21}) - \|\,S_2(0)\,\|\right\}$,

$$\|\,S_1(0)\,\| = |\,g_{11}(0)| + |\,g_{22}(0)|\,,$$

$$\|\,S_2(0)\,\| = |\,A_{11}\,g_{12}(0)| + |\,A_{12}\,g_{21}(0)| + |\,g_{12}(0)\,g_{21}(0)|$$

$$+ |\,A_{11}\,g_{22}(0)| + |\,A_{22}\,g_{11}(0)| + |\,g_{11}(0)\,g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a,\ t) = \square(a)\exp\left[\lambda t\right] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda,\tau)$ and $F_2(\lambda,\tau)$ as follows:

$F_1(\lambda,\tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau];$

$F_2(\lambda,\tau) = \lambda S_1(\lambda) + S_2(\lambda,\tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda,\tau) - P_2(\lambda,\tau) \right|$$

$$\geq \left| F_1(\lambda,0) \right| - F_2(\lambda,\tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0,\tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0,T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda,\tau) = F_2(\lambda,\tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany

mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational.

It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection.

Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which

paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

# Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure

has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 19. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N≥0 then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \quad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. .$$

(1.3)

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\}$$

(1.4)

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^1 f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that

"time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)\,(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.}$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\big(N(t-\tau)\big) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\big(t-s\big)N(s)ds - m(N(t))$$

Or equivalently

$$\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)N(t-s)ds - m(N(t))$$

(1.7)

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive

and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ *(i* = 1, 2) are continuous with continuous partial derivatives for all $x_1$ 0 *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ *(i, j* = 1, 2), (2.2)

(xxviii) $b_1(0) = 0, \quad m_1(0, x_2) \equiv 0, b_2(0) = 0, \quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xxix) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2 \quad (\delta_2) - m_2(x_1, \delta_2) < 0,$ for $x_1$ 0, $x_2$ 0, (2.5)

(xxx) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha, \beta) = 0 = b_2(\beta) - m_2(\alpha, \beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the ($x_1$ , $x_2$)- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\frac{dx_1(t)}{dt} = r_1 x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),$$

$$\frac{dx_2(t)}{dt} = r_2 x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),$$

(2.7)

Where $r$, $a_{ij}$ , $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i$ (0) > 0, $x_2$ (0)> 0 does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1$, $X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\frac{dX_1(t)}{dt} = \beta_{11} X_1(t) + \beta_{12} X_2(t),$$

$$\frac{dX_2(t)}{dt} = \beta_{21} X_1(t) + \beta_{22} X_2(t), \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} > \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} > \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda\,(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1\ (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1\ (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J\ (i,\ j = 1, 2; i \neq j\ )$.

## 20. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i,j=1,2)$ be a set of nonnegative constants with $\tau = \max\ \tau_{ij}\ , i,j=1,2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t-\tau_{11})\big) - m_1\big(x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t-\tau_{22})\big) - m\big(x_1(t-\tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i=1,2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s)\ \ 0, \qquad s\ \in [-\tau, 0], \qquad i=1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\{\psi_1^{(n)}(s)\}, s\in[-\tau,0], i=1,2; (n=1,2,\dots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s\in[-\tau,0], \qquad i=1,2 \qquad (3.3)$$

Let $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ be the solution of (3.1) corresponding to the initial condition

$$\{x_1^{(n)}(t), x_2^{(n)}\ (t)\} = \{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\}, \quad s\in[-\tau,0],. \qquad (3.4)$$

Consider the solution $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0,\tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i,j=1,2))$ for which

$$x_1^{(n)}(t^*) = 0 \ and \ x_1^{(n)}(t)(t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \qquad (3.5)$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0 \ on \ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [2\tau^*, 3\tau^*] \ etc$. Thus it will follow that so long as $(x_1^{(n)}(t), \ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t) \geq 0 \ x_2^{(n)}(t) \geq 0.$$

If we now consider the limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t), \ x_2^{(n)}(t)\right\} = \{x_1(t), x_2(t)\} \qquad (3.2)$$

With $x_1(t)$ 0,, $x_2(t)$ 0 where $(x_1(t), x_2(t))$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ 0; then there exists a $t_1 > 0$ such that at least for one of either of $x_1 \ or \ x_2$ we have

$$\lim_{t\to t_1-} x_i(t) = \infty. \qquad (3.7)$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i(t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1(x_1(t_2 - \tau_{11})) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(x_1(t_2)) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(\delta_1) - m_1(\delta_1, x(t_2 - \tau_{12})) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ 0.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1(t-\tau_{21}) - m_{22}Y_2(t-\tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i} \; at \; (\alpha, \beta). \tag{3.10}$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ 0 $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= f_1(y_1(t)) - g_1(y_1(t), y_2(t)), \\ \frac{dy_2(t)}{dt} &= f_2(y_1(t)) - y_2(t) - g_2(y_2(t)), \end{aligned}\right\} \tag{4.1}$$

Where $f_i \; and \; g_i \; i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i \; and \; g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i \; and \; g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0 \; (i = 1, 2); also \; we \; require$

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \; 0; \quad \frac{\partial g_1}{\partial y_2} \; 0, \\ &\frac{\partial f_2}{\partial y_1} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0, \end{aligned}\right\} \text{Where } y_1 \; 0, y_1 \; 0, \tag{4.2}$$

(xxxvii)

$$\left.\begin{aligned} f_1(0) &= 0, \quad g_1(0, y_2 \equiv 0, \\ f_2(0, y_2) &= 0 = f_2(y_1, 0); \quad g_2(0) = 0, \end{aligned}\right\} \tag{4.3}$$

(xxxviii) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xxxix) There exists a point $(\alpha^* \beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$\left.\begin{aligned} f_1(\alpha^*) - g_1(\alpha^*, \beta^*) &= 0, \\ f_2(\alpha^*, \beta^*) - g_2(\beta^*) &= 0, \end{aligned}\right\} \tag{4.5}$$

(xl) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \geq 0,$$
$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \quad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_l, a_{ij}\ (i, j = 1,2)$ are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t \geq 0$ for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all $t \geq 0$ let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l(0)$ satisfy $0 < y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0) \geq \eta_l and\ y_2(0) \geq \eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t > 0,$ ). Suppose $y_l(t)$ does not exist for all $t \geq 0$; then for some $t_l < \infty\ lim_{t \to t_l -} y_l(t) = \infty$ and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t \geq 0$; if $y_2(t)$ is not defined for all $t \geq 0$, then for some $t_3 \leq \infty$ we will have $lim_{t - t_3 -} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2) \leq f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t \geq 0$.

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\frac{dU_1(t)}{dt} = \frac{\partial f_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_2}U_2(t),$$

$$\frac{dU_2(t)}{dt} = \frac{\partial f_2}{\partial y_1}U_1(t) - \frac{\partial f_2}{\partial y_2}U_2(t) - \frac{\partial g_2}{\partial y_2}U_2(t), \qquad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

Where $f_{ij} = \frac{\partial f_1}{\partial y_l}$, $g_{ij} = \frac{\partial g_1}{\partial y_j}$, at $(\alpha^*, \beta^*)$ (4.8)

$(i,j = 1, 2)$

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$

(4.9)

$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}.$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$

$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \qquad \textit{Evaluated at } (\alpha^*, \beta^*), \qquad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 14. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.2)$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t),$$

$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_2(t)N_1(t), \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\frac{dN_1(t)}{dt} = rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t),$$
$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_1(t)N_2(t). \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\frac{dN_1(t)}{dt} = a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t),$$
$$\frac{dN_2(t)}{dt} = a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$
$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^{\infty} \delta(a)N_2(t-a)e^{-d*a}\,da,$$
(5.6)

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1(N_1(t-\tau_{11})) - g_1(N_1(t), N_2(t-\tau_{12})),$$
$$\frac{dN_2(t)}{dt} = f_2(N_1(t-\tau_{21})), N_2(t-\tau_{12}) - g_1(N_2(t)),$$
(5.7)

Where $\tau_{ij}$ $(i,j=1,2)$ are nonnegative constants with $\tau=\max\tau_{ij}, i,j=1,2$; the birth and death rate functions $f_1,f_2,g_1,g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s)=\phi_1(s)>0,\quad s\in[-\tau,0].$$

$$N_1(s)=\phi_1(s)>0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1}>0,\frac{\partial f_2}{\partial N_2}>0, f_2(N_2,N_2)>0 \text{ whenever } N_1>0,\ N_2>0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s)>0$ and $N_2(s)>0$ for $s\in[-\tau,0]$, solution of (5.7) exist for all $t\geq 0$ and remain nonnegative for all $t\geq 0$. also $(\alpha^*,\beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*,\beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state $(\alpha^*,\beta^*)$ is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt}=-m_{11}Y_1(t)+b_{11}Y_1(t-\tau_{11})-m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt}=-m_{22}Y_1(t)+m_{21}Y_1(t-\tau_{21})-b_{22}Y_2(t-\tau_{22}),\quad t>0, \quad (6.1)$$

Where$m_{1J}, b_{i1}$ $(i,j=1,2)$ are positive constants while $\tau_{1J}(i,j=1,2)$ are non negative with $\tau=\max(\tau_{1J})(i,j=1,2.$as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt}=-m_{11}Z_1(t)+b_{11}Z_1(t-\tau_{11})-m_{12}Z_2(t-\tau_{12}),$$

$t > 0,$ (6.2)

$$\frac{dZ_2(t)}{dt} = -m_{22}\, Z_1(t) + m_{21} Z_1(t-\tau_{21}) - b_{22} Z_2(t-\tau_{22}),$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \ \ for\ s \le 0 \ \ for\ which\ Y_i(s)\ is\ defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad ||Z|| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad ||B|| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_j [b_{jj} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jj} | ]$ $B=(b_{1j})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \le [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \le \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm ‖B‖ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \ or \ -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix}\lambda+m_{11} & 0\\ 0 & \lambda+22\end{bmatrix}-b_{11}^{e-\lambda T_{11}}\begin{pmatrix}-m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}}\end{pmatrix}$$

$=\lambda I-A-B(\lambda,T_{ij})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\boldsymbol{\alpha}<0.$

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\boldsymbol{\alpha}>0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\boldsymbol{\alpha}$ in (6.8)and define

$G(z-\boldsymbol{\alpha},T_{IJ})=M(z)=(z-a)I-A-B(z-\boldsymbol{\alpha},T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$Re(\lambda)<-a$

is the following :

$-\mu(aI+A)>\|B_1(a,T_{ij})\|,$

Where

$$B_1(a,T_{ij})=\begin{pmatrix}b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}}\\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}}\end{pmatrix}$$

Since $\mu(\boldsymbol{\alpha}\, I+A)=\boldsymbol{\alpha}+\boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\boldsymbol{\alpha}+\mu(A)+\|B_I(\boldsymbol{\alpha},T_{IJ})\|<0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a}f(\mathfrak{a},T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as \; \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$$f(0, T_{Ij}) < 0 \;\; for \; \mathfrak{a} \in [0, \mathfrak{a}^{*})$$

And hence

$$f(\mathfrak{a}, T_{Ij}) = 0$$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root ofmax $b_{11}^{e-\lambda T}{}_{11} - m_{12}^{e-\lambda T}{}_{21} + \mathfrak{a}$-min $m_{11}$ =0

$b_{22}^{e-\lambda T}{}_{22} - m_{21}^{e-\lambda T}{}_{12}$ $m_{22}$

Where the maximum in (6.18) is taken over $b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij} (i, = j\ 1,2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}. \qquad (6.19)$$

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^{*}$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^{*}$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more

species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows

from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement "all animal populations have the reproductive capacity to increase without limit, yet none di" and in his concluding remarks writes " this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated" More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution "Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist." In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self – regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^*\ y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^*\ y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta\, xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of

predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned}\frac{d|Y_1(t)|}{dt} &\le -m_{11}\,|Y_1(t)| + b_{11}|Y_1(t-\tau_{11})| + m_{12}\ Y_2(t-\tau_{12})|,\\ \frac{d|Y_1(t)|}{dt} &\le -m_{22}\,|Y_2(t)| + m_{21}|Y_1(t-\tau_{21})| + b_{22}\ |Y_2(t-\tau_{21})|,\end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\,|\,Y_1(t)\,| = \left\{\begin{aligned}&\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) \ge 0,\\ &\frac{dY_1(t)}{dt} \quad if \quad Y_1(t) < 0,\end{aligned}\right\} \quad i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned}\frac{dZ_1(t)}{dt} &= -m_{11}\ Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}\ Z_2(t-\tau_{12}),\\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}\ Z_2(t-\tau_{22}),\end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$Z_1(s) > |Y_1(s)|, \quad s \in [-\tau, 0], i = 1,2.$ (A.5)

Then

$|Y_1(t)| < Z_1(t)$ for $t \geq 0$ and $i = 1,2.$ (A.6)

PROOF. Let $U_1(t) = Z_i(t) - |Y_1(s)|, \quad i = 1,2.$ $t \geq -\tau.$ Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\geq -m_{11}\ U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}\ U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\geq -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}\ U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad \text{(A.7)}$$

With

$U_1(s) = Z_i(s) - |Y_i(s)| > 0, \quad i = 1,2, s \in [-\tau, 0].$ (A.8)

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \geq U_1(0) \exp[-m_{11}t]$

$+\int_0^t [b_{11}U_1(s-\tau_{11}) + m_{12}\ U_2(s-\tau_{12})] \exp[-m_{11}(t-s)ds > 0,]$ (A.9)

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t)$

$+\int_{1-T}^t b_{11}\, Z_1(s)ds + \int_{1-T}^t m_{12}\, Z_2(s)ds$

$+\int_{1-T}^t m_{21}\, Z_1(s)ds + \int_{1-T}^t b_{22}\, Z_{22}(s)ds$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$0 \leq |Y_{1,}(t)| \leq Z_1(t) \to 0$ as $t \to \infty$,

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$= -[Z_1(t)\{m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{m_{22} - (b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t)$,$Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\rightarrow 0$ as $t \rightarrow \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\rightarrow 0$,$Z_2(t)\rightarrow 0$ as $t\rightarrow \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i,j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$\int_0^\infty sk_{ij}(s)ds< \infty$; $\int_0^\infty k_{ij}(s)ds=1$, I,j=1,2.

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i,j = 1,2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \ Z_i(t) \to \infty \leq as\ t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\left\{\begin{matrix} M_{11} > b_{11}+m_{21}, \\ M_{22} > b_{22}+m_{12} \end{matrix}\right\}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)+b_{11}\int_0^\infty k_{11}(u)(\int_{t-u}^t Z_1(s)ds)du$

$+m_{12}\int_0^\infty k_{12}(u)(\int_{t-u}^t Z_2(s)ds)du$

$+m_{21}\int_0^\infty k_{12}(u)(\int_{t-u}^t Z_1(s)ds)du$

$+b_{22}\int_0^\infty k_{22}(u)(\int_{t-u}^t Z_2(s)ds)du$

Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12))

$$\left(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]\right.$$

$< 0$(using (A.18)) for $Z_1(t) > 0\ and\ Z_2(t) > 0.$

By Lemma 6 below the result will follow

Lemma 6. Let $H(t)=H(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose

$$m_{11} > b_{11} + m_{21} \text{ and } m_{22} > b_{22} | m_{12}$$

Then

$$Z_1(t) + Z_2(t) \to 0 \text{ast} \to \infty.$$

PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence

$\{t_n\} \to \infty$ as n $\to \infty$ such that

$$[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon \qquad (A.22)$$

For some sufficiently small positive number $\varepsilon$ and hence

$$\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, \ldots.$$

By the continuity and differentiability of H with respect to t,

$$\mathrm{H}(t_n - \varepsilon) - \mathrm{H}(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$$

If $\varepsilon$ is small we will have

$$= \frac{dH(t^*)}{dt}\varepsilon/2$$

So that

$$\mathrm{H}(t_n) \leq \mathrm{H}(t_n - \varepsilon) - \varepsilon^2/2$$

$$\leq H(t_{n-1}) - \varepsilon^2/2$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2)$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2):$$

$$\leq H(t_0) - \mathrm{n}(\varepsilon^2/2): \mathrm{t}_0 = 0.$$

(A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$)

We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$

Where A and B are defined as in (6.3).

We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems

**Gopalsamy**

# Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 28. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific

interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 29. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0, \qquad \textbf{(2.1)}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

($H_2$) $f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

($H_3$) There exist positive constants $b_{il}$, $b_i^u$, $a_{il}$, $a_i^u$ ($l = 1, 2$) such that

$$b_{il} - a_{il} x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u / a_i^u, \qquad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} {}^{|h(t)|, h \in B_\omega^\tau}\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b_l = \min(b_{1l} / a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1 in\ \bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \; (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{matrix} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{matrix} \quad \text{for t> } 0, \qquad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t-s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+ \int_0^1 G_{12}(m, 0, t-s) g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) ds,$$

$$N_2(t, x_1, x_2) = \int_0^\omega G_{22}(m, \omega, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds \qquad \textbf{(2.6)}$$

$$+ \int_0^1 G_{21}(m, 0, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds,$$

For $t > 0$, with

$$g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) = mx_1(s) + x_1(s) f_1\left(s, x_1(s), x_2(s - \tau_{12})\right),$$

**(2.7)**

$$g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) = mx_2(s) + x_2(s) f_2\left(s, x_1(s - \tau_{21}) x_2, (s)\right),$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

**(2.8)**

$$G_{ij}(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2,$$

$$N(x_1,x_2)(t) = \begin{cases} N_1(n\omega+t,x_1,x_2) \\ N_2(n\omega+t,x_1,x_2) \end{cases} for\ t \in [-\tau,0], \qquad \textbf{(2.9)}$$

*m* being a suitably selected positive constant (more will be said below) while *n* is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ [illegible] and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ [illegible] then $N(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ [illegible] $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$, $t$ [illegible] $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = (x_1(t),x_2(t)) \quad and\ (x_1,x_2) \in K \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1,x_2)$ assuming that $N(x_1,x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1,x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1,x_2)(t)$. It will follow from (2.9) and the assumed periodicity of $N(x_1,x_2)$ $that\ N(x_1,x_2)$ is continuous

For $t$ [illegible] $-\tau$. Now we will verify the periodicity of $N(x_1,x_2)$ in $t$. A simple computation leads to

$$N_1(t+w,x_1,x_2) - N_1(t,x_1,x_2) = e^{-mt}F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m,\omega,t-s)\,g_1(m,s,x_1(s),x_2(s-\tau_{12}))ds$$

$$+\int_0^{\omega} G_{12}(m,0,t-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_{\omega}^{\omega+t} G_{12}(m,0,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$-\int_0^{\omega} G_{11}(m,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_0^{t} G_{12}(m,0,-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B,$ thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m,t,x_1(t),x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m,t,x_1(t-\tau_{21}),x_2(t)\right),$$

For any arbitrary real constant $m$ and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

| | |
|---|---|
| (xxviii) | Either $x_1, x_2$ or both $x_1\ and\ x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1\ and\ x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1\ and\ x_2$ cannot be periodic. |
| (xxix) | While $(x_1, x_2)\ \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0\ or\ dx_2(t^*)/dt < 0,$ which will imply that the values of $x_1(t^*)\ or\ x_2(t^*)$ cannot be repeated from below. |
| (xxx) | Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}\ or\ 0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above. |

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1) and\ \lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}.$ *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq \alpha_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq \alpha_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda\, x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda\, x_2\,(t) f_2(t, x_1(t-\tau_{21}), x_2\,(t)) + x_2\,(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}\,(m, \omega, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}\,(m, 0, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds. \quad \textbf{(2.14)}$$

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for *m* for the case $N_1(t, x_1, x_2) \geq 0$ and then choose *m* = max $(m_1, m_2)$. Thus we have

$$N: K \cap (G_2/G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\,\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \rightarrow B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} \; for \;\; t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\, t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \rightarrow K$ be completely continuous and which satisfies the following conditions:*

(s) *There exists* $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ *and a* $\mu$ *satisfying*

$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2)$ *with* $0 < \lambda < \mu$ *satisfy* $(x_1, x_2) \notin \partial H_1$.

(t) *All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1$ *satisfy* $(x_1, x_2) \notin \partial H_2$.

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that* $f_1, f_2$, *of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \, ||x_1, x_2|| < \max(b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \, ||x_1, x_2|| < \min(b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 30. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq$ *x1,x2≤C∗ where C∗some positive constant is. Assume the following:*

(bb) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(cc) N is monotonic nondecreasing on $K^*$.

(dd) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda(1 + \eta) N(x_1, x_2)$. Then N has at most one fixed point different from $(0,0)$ in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \geq \|x_1\| \geq b_1^u/a_1^u,\ b_{2l}/a_{2l} \geq \|x_2\| \geq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2 / G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2 / G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \quad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$\times \{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \quad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2(t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \geq 0 \text{ and } 0 \not\equiv x_2(s) - y_2(s) \geq 0;$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2) - N_2(t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$N(x_1,\ x_2)\ - N\ (y_1, y_2)$ for $(x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}$.

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta)\, N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t-s) g_1\big(s, x_1(s),\ x_2(s-\tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t-s) g_1\big(s, x_1(s),\ x_2(s-\tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s-\tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s-\tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$$H_1(s, \lambda,\ x_1,\ x_2) \geq \eta_1 \text{ for } 0 < \lambda < 1, s > 0, (x_1,\ x_2)\ \in\ K^*, \qquad \textbf{(3.8)}$$

And hence

$$N_1(t, \lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta_1) N_1(t, x_1,\ x_2).$$

A similar argument will show that there exists an $\eta_2 >0$ such that

$$N_2(t, \lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta_2) N_2(t, x_1,\ x_2),$$

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ }?\text{ }-\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ }?\text{ }-\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ }?\text{ }-\tau, \tag{4.1}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t)X_1(t) - a_{12}(t)x_2^*(\mathrm{t} - \tau_{12})\, X_2(\mathrm{t} - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t)x_1^*(t - \tau_{21})x_1(t - \tau_{21}) - a_{22}(t)x_2^*(t)x_2(t), \tag{4.2}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t} - \tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t - \tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(t))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \tag{4.4}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(t-\tau_{12})|X_2(t-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(t-\tau_{21})|X_1(t-\tau_{12})| - b_{22}x_2^*(t)|X_2(t)|, \end{aligned} \tag{4.5}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \tag{4.6}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ of $(2.1) - (2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12}, \tag{4.7}$$

*Where the constants* $b_{ij}$ $(i, j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x_2^*(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x_1^*(u)|X_{1,}(u)|\mathrm{du}. \quad \textbf{(4.8)}$$

Since $x_1^*, x_2^*$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x_1^*(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x_2^*(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}. Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(\,|b_{11}| - b_{21})\, x_1^*(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x_2^*(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - v(t_m) \leq -\int_{t_m-\varepsilon}^{t_m} \frac{dv(T)}{dT} d\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$v(t_m - \varepsilon) - v(t_m)$ $\varepsilon^2/2$ and hence

$v(t_m)$ $v(t_m - \varepsilon) - (\varepsilon^2/2)$

$v(t_m - 1)$-$(\varepsilon^2/2)$

$v(t_m - 2)$-$(\varepsilon^2/2)$

$v(t_0)$-m $(\varepsilon^2/2)$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t$ 0. this contradiction shows that $|X_1(t)| + |X_2(t)| \to 0$ as $t \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1(t, x_1(t), x_2(t - T_{12}))1 \quad \frac{x_1(t) + ax_2(t - T_{12})}{K_1(t)}$$

$$f_2(t, x_1(t - {}_{21})x_2(t)) = 1 - \frac{\beta x_1(t - T_{21}) + x_2(t)}{K_2(t)}$$

Where $\alpha, \beta$ are positive constants and $k_1, k_2$, are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\text{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(t)\right)\Big| = \min_{t}, \beta/K_2(t)$$

$$\max_{t T}\{-a_{22}(t)\}\Big| = \max_{t T}(1/K_2(t)) > \min_{t T}\left(-a_{12}(t)\right)\Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_1, k_2$, to the inequality $0 < \alpha, \beta < 1$, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a

constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_1\left(\mathrm{t},x_1(\mathrm{t}),x_2(\mathrm{t}-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_2(\mathrm{t},x_1(\mathrm{t}-T_{21}),x_2(\mathrm{t}))$$

Since $\left(x_1(\mathrm{t}),x_2(\mathrm{t})\right)\rightarrow\left(\mathrm{x}_1^*(\mathrm{t}),\mathrm{x}_2^*(\mathrm{t})\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(t)}{dt}x_1(t)[1-x_1(t)+0.4x_2(t-\mathcal{T})/k_1(t)]$$

| |
|---|
| $\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$ |
| Where $k_1(t) = 20 + \sin(\pi t)$ and $k_2(t)=2.5+\sin(\pi t)$ |
| @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ |

There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of

"differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction,

projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

### Abstract

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction

rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 19. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

t>$t_0$;

$t_0 \in (-\infty, \infty); i = 1, 2, \dots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \dots., n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \dots, n)$ by the following:

$$\inf_{t\in R} b_l(t) = b_i^l; \qquad \sup_{t\in R} b_l\,(t) = b_l^u, \qquad i, j = 1, 2, \dots, n.$$

$$\inf_{t\in R} a_{lj}\,(t) = a_{lj}^l; \qquad \sup_{t\in R} a_{lj}\,(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{lj}^u (b_j^u / a_{lj}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l\{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \dots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \dots, n; t \geq t_0. \tag{1.5}$$

(1.10) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \dots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \leq \left( b_i^l - \sum_{\substack{j=1 \\ J\neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \geq x_i^l \ for\ t \geq t_0,\ i = 1, 2, \dots, n. \tag{1.7}$$

20. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \dots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R}_n | x_* \leq x_i \leq x^*; i = 1, 2, \dots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1 \leq i \leq n} x_i^l ; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(199) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(200) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(201) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(202) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(203) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(204) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of

Southampton, UK. “This is a cool, new idea, and I’m up for trying to see it in experiments,” he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(205) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team’s work. Einstein’s theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth’s surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(206) Because of gravity’s effect on (e&eb) space-time, Pikovski’s team realised that variance in a molecule’s position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(207) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(208) “In most situations decoherence is due to (e) something external; here it’s as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself,” adds Pikovski.

**A practical limit**

(209) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(210) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(211) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

(212) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below).

(213) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints)

(214) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain.

(215) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy.

(216) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**.

(217) If gravity’s effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy.

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|---|
| (218) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (219) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (220) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for

any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$||f(t+\tau,x) - f(t,x)|| = \sum_{l=1}^{m} |f_l(t+\tau,x) = f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1}\int_0^T f(t,x)\exp(-i\lambda_j t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or $\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times S, \{g(t+,x)\}$ is also uniformly convergent on $\boldsymbol{R} \times S$ then the module of $g(t+,x)$ is contained in the module of $f(t+,x)$.*

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number $\mu$ such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J\neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i$$

$$= 1,2,\dots,n. \qquad (2.3)$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say $w(t) = w_1(t),\dots,w_n(t), t \in \boldsymbol{R}$, whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l\left\{b_l(t) - \sum_{j=1}^{n} a_{lj}(t)x_j\right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \qquad (2.5)$$

Uniformly in $(x_1,\dots,x_n) \in S$ for $t \in (-\infty,\infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0),\dots,n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined for $t \in [\beta,\infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta,\infty), i = 1,2,\dots,n$ and the convergence is uniform on all compact subsets of $[\beta,\infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta,\infty)$.

Let $U$ be any compact subset of $\beta,\infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon,\beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon,\beta)$

and all $t \in (-\infty,\infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \quad (2.6)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \quad (2.7)$$

$$(2nx^*/x_*)\exp[-\mu x_*(\beta+\tau_k-t_0)] < (\varepsilon/2x^*) \quad (2.8)$$

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1,(s) - \log x_1(s+\tau_m+\tau_k)|, \quad x \in t_0, \quad (2.9)$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1,(t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \in v(s)$$

$$\in (1/x^*)\sum_{i=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)|. \quad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s) = \sum_{i=1}^{n} \{sign[x_i(s) - x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[\log x_i(s)]}{ds} - \frac{d[\log x_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\sum_{l=1}^{n} |b_l(s) - b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n} a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$-\mu x_* v\big(x(s),x(s+\tau_m-\tau_k)\big)+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)\,|. \qquad (2.12)$$

Let us set $s-\tau_k = \sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)\,|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|\,a_{iJ}(\sigma+\tau_k)-a_{iJ}(\sigma+\tau_m)|$$

$$<(\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k),x(t+\tau_m)\big) \le v\big(x(t_0),x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$[(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$ On using (2.8), (2.10) (2.14)

For $m \geq n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n} |logx_i(t+\tau_k) - logx_l(t+\tau_m)| < (\varepsilon/x^*) \quad (2.15)$$

And hence using $x_* \leq x_i(t) \leq x^*, i = 1,2,\dots,n; t \geq t_0$,

$$\sum_{i=1}^{n} |x_i(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \geq n_0(\varepsilon,\beta). \quad (2.16)$$

The existence of a function $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*,x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*,x^*], i = 1,2,\dots,n;$ for $t \geq t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n$.

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t),\dots,x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\leq \sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k),\dots,x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k)\right)| \quad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k))$$

$$- f_l\big(t + \tau_m, x_1(t + \tau_m), \dots, x_n(t + \tau_m)\big)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t} + \tau_k) = x_1(t + \tau_k), \dots, x_n(t + \tau_k) \in \mathrm{S}$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ $k$ $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\big)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t + \tau_k) \in S$ and $x_l(t + \tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ $k$ $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x(t + \tau_m), \dots, x_n(t + \tau_m)\big)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has

been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \geq n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ( (t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k \to \infty} \frac{d}{dt} x_1(t + \tau_k) \quad exists\ for\ i = 1, 2, \dots, n$$

Uniformly on all compact subsets of $(-\infty, \infty)$. Thus we have

$$\lim_{k \to \infty} \lim_{k \to \infty} \frac{x_i(t + \tau_k + h) - x_i(t + \tau_k)}{h}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{x_l(t + \tau_k + h) - x_l(t + \tau_k)}{h} \qquad (2.22)$$

$$= \lim_{h \to 0} \frac{w_1(t + h) - w_1(t)}{h}, \quad i = 1, 2, \dots, n,$$

Showing that (since the left side of (2.22) exists) $w_i(i = 1, 2, \dots, n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt} = \lim_{k\to\infty}[\, f_i\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n\,(t + \tau_k)\right)$$

$$-f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)$$

$$+ - f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)]$$

$$= -f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big); \qquad i = 1, 2, \dots, n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1, f_2, \dots, f_n$ on $R \times S$ and hence $w(t) = \big(w_1(t), \dots, w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t) = \big(w_1(t), \dots, w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t, x_1, \dots, x_n), f_2(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n)\}$ for $(x_1, x_2 \dots, x_n) \in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t + \tau_k, x_1, \dots, x_n), f_2(t, + \tau_k, x_1, \dots, x_n), \dots, f_n(t, + \tau_k, x_1, \dots, x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1\,(t, + \tau_k)\ \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l\,(t, + \tau_k) \to w_i(t + \tau\,), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ ☒k ☐ *n0(ε)* then

$$\sum_{l=1}^{n} |b_l\,(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t+\tau_k) - a_{ij}(t+\tau_m)| < \left(\frac{\varepsilon}{4x^*}\right)\mu x_* \qquad (2.25)$$

$$n(2x^*/x_*)\exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s+\tau_m-\tau_k)\big)$ for $s \in [t, t+\tau_k]$ defined by

$$v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s+\tau_m-\tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{i=1}^{n}\{\text{sign}[w_l(s) - w_i(s+\tau_m-\tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds}\cdot\frac{1}{w_l(s)} - \frac{dw_l(s+\tau_m-\tau_k)}{ds}\,\frac{1}{w_l(s+\tau_m-\tau_k)}\right\}$$

$$\leq -\mu\sum_{l=1}^{n}|w_l(s) - w_l(s+\tau_m-\tau_k)| + \sum_{l=1}^{n}|b_i(s) - b_l(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{l=1}^{n}\sum_{J=1}^{n}\left|a_{lJ}(s) - a_{lJ}(s+\tau_m-\tau_k)\right| \qquad (2.28)$$

$$\leq -\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s) - b_l(s+\tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s) - a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k = \sigma$ then we have from (2.29),

$$D^{+}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^{*}\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\,(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)\right| \qquad (2.30)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_{*})/(4x^{*})]\ +[(\varepsilon\mu x_{*})/(4x^{*})]$$

(2.31)

Integrating (2.31) over $(t,t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called “shrinking operations”), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang’s model or to

Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during 0≤t<1. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$ e^{-\mu x^* T_k} v\big(w(t), w(t+T_m-T_k)\big) + (\varepsilon/2x^*)$

$ e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t+T_m-T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$<(\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} | w_1(t+T_k) - w_1(t+T_m)| < \varepsilon \; for \; m \; k \; n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty \; as \; k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\; as\; k \to \infty \; on$ **R** follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T >$ $0. Then\; if$ (1.2), (1.3)$and$ (2.3)$hold\; the\; system$ (1.1)$will\; have\; a\; periodic\; solution\; w(t) =$ $\{w, (t), \ldots\ldots, w_n(t)\} of\; period\; T\; such\; that$

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots) and\; note\; that$

$$x_1(t+KT) - w_1(t) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) - w_1(t) \to 0 \; as \; \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \ldots, w_n(t)\} of (1.1) is\; globally\; asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \ldots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \ldots, n$ is defined for all t□0 and satisfies

$$\lim_{t \to \infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \ldots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{J=1 \\ J\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu$ $Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= g|g:**R**→ $\boldsymbol{R}$ $and$ $g(t)\lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}(t) y_1(t)\right\},$$

*i=1,2,…,n; t>*$t_0 \in R$,

Where

$$B_i \in H(b_i); (A_{ij}), \qquad i, j = 1,2, \dots, n,$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{ij}^{u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\dots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\dots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\dots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\dots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\dots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |\, x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1\,(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t\,) - y_i(t\,)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ \ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{aligned} &\beta_1 \leq x_i(t) \leq \beta_2, \\ &\beta_1 \leq y_i(t) \leq \beta_2, \end{aligned} \quad i = 1,2, \dots, n; t \geq t_0 \tag{3.7}$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \tag{3.8}$$

It is easy to see that

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1} \sum_{i=1}^{n} |x_i(t) - y_i(t)|, \quad t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\ \big(x(t), y(t)\big) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\beta_1 V\big(x(t), y(t)\big) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+ \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t), y(t)) \leq V(x(t_0), y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right] \frac{1}{\mu\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} |x_1(t) - y_1(t)| \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} |x_1(t_0) - y_1(t_0)|$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)|\right.$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying “The cause is by cause,” which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics

would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} \mid a_{ij}(t) - A_{ij}(t) \mid \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Presciential Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of**

**dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t, a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ 0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ \infty,$$

$$\rho_1(t, 0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1 \left[\frac{dQ_2}{dt}\right]_1 - x_2 \left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3 \left[\frac{dQ_3}{dt}\right]_1 - x_4 \left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1P_1) + \big(f_2)_1(Q_1)\big)(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \le \rho_1(t, a_1) \le \rho_1^*(t, a_1),$$

$$0 \le Q_1(t) \le Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\, \rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\, \rho_1^*(t, a_1) da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt} = -b_1 Q_1^* + g_1(P_1^*)Q_1^*, \quad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*(t, a_1) da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \quad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a) e^{-\int_0^a \mu_1(\tau)dt} e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t, a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1(P_0)a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t, a_1) = \rho(t, a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1\,(t, a_1)da_1 = \int_0^\infty \rho_1(t, a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1 N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0'Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}')(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t,a_1) = \tilde{\mathrm{u}}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_1}\,da_1$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_1)$ gives,

$$\tilde{\mathrm{u}}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{\mathrm{u}}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})}\tau_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\int_0^a(\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1+\int_0^{\tau}\mu_1(s_1)ds}d\tau_1\,da_1$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}\int_0^a e^{f_{10}\tau+\int_0^{\tau}\mu(s)ds+y\tau}\alpha_1(\tau)d\tau\,da\right)w=0. \ (23)$$

Define $Y(\gamma,a)=e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta\,(a)Y(\gamma,a)da\right)u(0)$$

$$+w\int_0^{\infty}\beta(a)Y\,(\gamma,a)\int_0^a\alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da=0, (24)$$

$$\left(\int_0^{\infty}Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w=0.$$

Equation (20) has a solution of the form of $u(t,a)=u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I(y,a)=\int_0^a\rho_0(\tau)Y^{-1}\,(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^{l}\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^{l}\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right)=0. \quad (25)$$

Since

$$\rho_0(a)=\rho_0(0)e^{-\int_0^a \mu(\tau)d\tau-f_{10}a}$$

$$I(\gamma,a)=\int_0^a \rho_0(\tau)e^{\gamma\tau+f_{10}\tau+\int_0^\tau \mu(s)ds}d\tau$$

$$=\frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a)=e^{-f_{10}a-\int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1,a_1)=E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma}\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^{1}]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\left(\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)$$

$$+\left(1-\int_0^{\infty}\beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1+\frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^{\infty}E_1(a_1)da_1-\int_0^{\infty}E_1(a_1)e^{-\gamma_1 a_1}da_1\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1+\frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^{\infty}E_1(a_1)da_1)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_1(a_1)Y_1(\gamma_1 a_1)da_1=1-\int_0^{\infty}\beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\neq 0,$$

Since $\int_0^{\infty}\beta_1(a_1)Y_1(\gamma_1 a_1)\int_0^{a}\alpha_1(\tau_1)Y^{-1}(\gamma_1,\tau_1)d\tau_1 da_1>0$. Therefore, the characteristic solution is

$$\gamma_1=-(f_{10}^{'})_1(\rho_0)_1\int_0^{\infty}E_1(a_1)da_1.$$

Since $f_1'(P_0)_1>0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1),0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey

population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*)\int_0^\infty \beta_1(a_1)p_1^*(a_J)p_1^*(a_1)da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0)e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1 \int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)\left([f_2^{'}]_1(z_1)\right)\int_0^{\infty} a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_1$$

$-h^{'}(z_1)\int_0^{\infty}\beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$$H_1(0) = 1 - \int_0^{\infty}\beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^{-}(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^{\infty}\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y_1^a}} da_1$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R(f_1(P^*)) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$$M_1(t) \equiv \int_0^\infty m_1(t, a_1) da_1 = P_1(t) - P_1^*,$$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - [p^*]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1) m_1(t, a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1} p_1^*(0) W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $[f_{1*}^{'}]_1$ denoting $[f^{'}{}_1(P^*)]_1 [f_{2*}^{'}]_1$ denoting $[f_2]_1$ $(Q_1^*$, $h_{*}_1$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $[g_*^{'}]_1$

Denoting $[g^{'}]_1(P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - [f'_{1*}]_1[p_1^*](a_1)M_1 - [f'_{2*}]_1 p_1^*(a_1)W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1)m_1(t,a_1)da_1 + \frac{[h_*^1]_1}{[h^*]_1}[p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1[Q_1^*]M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1)da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1)da_1 - \int_0^\infty \mu(a_1)m(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 \int_0^\infty p_1^*(a_1)da_1\right)M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1)da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 - [f'_{2*}]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^\infty ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 + \left(\frac{[h'_*]_1}{[h_*]^1}\, P_1^*(0) - f'_{2*}P_1^*\right)W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_{*}\right]_1 Q_1^* M_{.1}$$

By the mean value theorem,

$$\int_0^{\infty}([h_{"*}]_1\beta_1(a_1)-\mu_1(a_1))m_1(t,a_1)da_1=([h_*]_1\beta_1(\theta_1)-\mu_1(\theta_1))M_{,1}$$

Where $0< \theta_1=\theta_1(t)<\infty$.

And if $a_1^2[a_1^2]_1<4[a_2]_1<4a_2e$,

$\mathrm{Re}\,\lambda_1[\lambda_1]_1=\mathrm{Re}\lambda_2[\lambda_2]_1=-\frac{1}{2}a_1[a2\ ]_1<0$.

Let $\Delta=\max_{2<u<\infty}\left|h_*[h_*]_1\beta_1(a_1)-\mu_1(a_1)\right|$. Then$|n_1(t)|$ ⊿∀t ⊡. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2<4a_2e$ and $[a_1]>2\Delta$,or $[a_1^2]$ ⊡$[a_2]$e and $(a_1-\sqrt{(a_1^2)_1-4(a_2)_1e})>2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+1}\right)$ and $\beta_1(a_1)=ae^{-ma}$ Then $\Delta=[\mu_0]+1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1)\mathrm{e}^{-\int_0^a \mu_1(\tau_1)+d\tau_1}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^2}+\frac{2}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ ≥0.5. Assume$[f_1](x_1)= K_1x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)$=$(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [\mathrm{K}][P_0])}{[m_1]_1 + (\mu_0)_1 + [\mathrm{K}][P_0])\,{}^3_1\,[\mathrm{K}]_1[P_0]^3_1} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*}$ $[h_*] = \frac{1+\omega_1 \mathrm{Q}_1^*}{1+\mathrm{Q}_1^*}$

$[f_{1*}] = K_1\ [\mathrm{f}^1_{2*}] = \frac{\sigma_1}{(1+\mathrm{Q}_1^*)^2}[\ \mathrm{g}^1_*] = \frac{\sigma_1}{(1+\mathrm{P}_1^*)^2}$ $[\mathrm{h}^1_*] = \frac{\omega_1 - 1}{(1+\mathrm{Q}_1^*)^2}$

## AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

**(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.**

**(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.**

**(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff**

| |
|---|
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |

**(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]**

**Time-dependent Schrödinger equation**

**(Single non-relativistic particle)**

**(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)**

**(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").**

**(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.**

**(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.**

**(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.**

**(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).**

**(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)**

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in**

**atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⚿ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38) d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac**

**{(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1}+(m_1+[\mu_0]_1+\frac{K_1 b_1}{\delta_1-b_1}+\sigma_1)\,Q_1^*)^3$$

$=m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1+ Q^*)(1 + \omega_1 Q_1^*$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_1 a e^{mu}{}_1 \; [\mu_0]_1 - \frac{a_1}{a_1+1}$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}] P_1^*$$

$$[a_2]_1 = \left[f_{2*}^1\right] P_1^* - \frac{h_*^1}{h_*} P^*(0)$$

$$e=[\,g^{1}_{*}]_{\,1}\,Q^{*}_{1}$$

$$\eta_1(t) = \left[h_{*}\right]_{1}\beta_{1}\left(\theta_{1}(t)\right) - \mu_{1}\left(\theta(t)\right),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_{1}\right]M_1 - [a_2]_{1}W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_{1} - [a_2]_{1} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{1} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(t) = e^{A,t}[X]_{1}(0) + \int_{0} e^{At}$$

**Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models**

**(Module Three)**

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and

also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \quad P_3 \text{ } 0, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \text{ } \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_3(Q_3) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \quad (4)$$

$$\rho_3(t_3, 0) = h_3(Q_3) \int_0^\infty \beta_3(a_3)\rho_3(t, a_3)da, \quad (5)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \leq \rho_3(t, a_3) \leq \rho_3^*(t, a_3),$$

$$0 \leq Q_3(t) \leq Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \quad (7)$$

$$\rho_3^*(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*(t, a_3)da_3 \quad (8)$$

$$\frac{dQ_3^*}{dt} = -b_3 Q_3^* + g_3(P_3^*)Q_3^*, \quad (9)$$

With $P_3^*(t) = \int_0^\infty \rho_3^*(t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_3(a) e^{-\int_0^a \mu_3(\tau)dt} e^{-ra}\, da_3 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(t)dt}\, da_3 > 1, \qquad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^\infty [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t, a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^\infty \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^{\infty}(\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t, a_3) = \rho(t, a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^{\infty} u_3\,(t, a_3)da_3 = \int_0^{\infty} \rho_3(t, a_3)da_3 - \int_0^{\infty} (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0'Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}')(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3 u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^\infty \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3 a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho \mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^\infty u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t,a_3) = \tilde{\mathrm{u}}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\,da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3, a_3)da_3\right)u_3(0)$$

$$+w\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0,$$

(24)

$$\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^{\infty}Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^{l}\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^{l})_3\int_0^{\infty}Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3)=(\rho_0)_3(0)e^{-\int_0^{a}\mu_3(\tau_3)d\tau_3-(f_{30})_3a_3}$$

$$I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)e^{\gamma_3\tau_3+(f_{30})_3\tau+\int_0^{\tau}\mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3) d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(a_3)da_3 - \int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1 + \frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^\infty E_3(a_3)da_3) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\,(\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(\,a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(\,p^*(a),Q^*\,)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(\,P_3^*) + [f_2]_3(Q_3^*)\,)P_3^* = 0, \qquad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^{\infty}\ \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \qquad (27)$$

$$g_3(P_3^*) - b_3 = 0, \qquad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3) \int_0^{\infty} \beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ \ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^-(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r$, $R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - [p^*]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3) M - [f_{2*}^3]_3 p_3^*(\mathrm{a}) W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $[f_{3*}']_3$ denoting $[f'_3(P^*)]_3 [f_{2*}']_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*3}$ denoting $h_*'$ denoting $h_3'$ $(Q_3^*)$, and $[g_*']_3$

Denoting $[g']_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - [f'_{3*}]_3 [p_3^*](a_3) M_3 - [f_{2*}']_3 p_3^*(a_3) W_3$$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t, a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3)da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3)da_3 - \int_0^\infty \mu(a_3)m(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3)da_3\right)M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3)da_3 W_3$$

$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3)da_3$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 - [\dot{f}_{2*}]_3 P_3^* W_{\cdot 3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right)W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g^{'}{}_*]_3\, Q_3^*\, M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2 [a_3^2]_3 < 4 [a_2]_3 < 4 a_2 e$,

$\mathrm{Re}\,\lambda_3 [\lambda_3]_3 = \mathrm{Re}\lambda_2 [\lambda_2]_3 = -\frac{3}{2} a_3 [a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty} \left| h_* [h_*]_3 \beta_3(a_3) - \mu_3(a_3) \right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4 a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0] \geq 0.5$. Assume $[f_3](x_3) = K_3 x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3) = [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3) = (1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])_{\,3}^{\,3}\ [\mathrm{K}]_{\,3}[P_0]_{\,3}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$ i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{*}\ [f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}_{2*}^{\,3}] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}_*^{\,3}] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}_*^{\,3}] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the following equation:

$(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_{\,3} + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\ \mathrm{Q}_3^{*})^{\,3}$

$=m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$

Now

$$\left[a_3^2\right] - 4\left[a_2\right]e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3+3}$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}] P_3^*$$

$$[a_2]_3 = \left[f_{2*}^3\right] P_3^* - \frac{h_*^3}{h_*} P^*(0)$$

$$e=[\,g_{*}^{3}]_{\,3}\,Q_{3}^{*}$$

$$\eta_3(t) = \left[h_{*}\right]_{3}\beta_{3}\left(\theta_{3}(t)\right) - \mu_{3}\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_{3}\right]M_3 - [a_2]_{\,3}W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_{\,3} - [a_2]_{\,3} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{\,3} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(t) = e^{A,t}[X]_{\,3}(0) + \int_0 e^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's**

**1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin**

**bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and**

**Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for**

**(e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole**

**merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of**

**(e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain**

**bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating**

**accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared**

**Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard**

**Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total**

**energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of**

**information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local**

**environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any ℓ-local Hamiltonian, where ℓ≥2, can be achieved using decoupling schemes of length at most O(nℓ−1logn). Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle**

**problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 82. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \; 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \; \infty,$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4P_4) + \big(f_2)_4(Q_4)\big)(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \le \rho_4(t, a_4) \le \rho_4^*(t, a_4),$$

$$0 \le Q_4(t) \le Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \tag{10}$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \tag{$H1$}$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4}\rho_4(t,a_4)\,da_4$$

$$\frac{dQ_4}{dt} = -b_4Q_4 + c_4P_4Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4P_4 - \lambda_4P_4Q_4$$

$$\frac{dQ_4}{dt} = b_4Q_4 - c_4P_4Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 83. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0'Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h'_{a_4})(\rho_0)_4 Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4 u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4 a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\, o\,)_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t,a_4)da_4$$

$$u_4(t,0) = \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t,a_4) = \tilde{\mathrm{u}}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_4}\, da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})\tau_4}\,(\alpha_4)_4(\tau_4)e^{\gamma_4 \tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4} e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds} d\tau_4\, da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\, da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4 - \int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds + y\tau}\alpha_4(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_4\,(a_4) Y_4(\gamma_4, a_4) da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4=0,$$

(24)

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^{a}\mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$= \frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4 a_4} - 1)$$

Let

$$E_4(a_4) = e^{-(f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4) d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4, a_4) = E_4(a_4)e^{-\gamma_4 a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma}\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\ (\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1 + \frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4 - \int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1 + \frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)da_4 = 1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4 \neq 0,$$

Since $\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^{a}\alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4 > 0.$ Therefore, the characteristic solution is

$$\gamma_4 = -(f_{40}^{'})_4(\rho_0)_4\int_0^{\infty}E_4(a_4)da_4.$$

Since $f_4^{'}(P_0)_4 > 0,$ we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4),0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^{\infty}\beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^*=\int_0^\infty p_4^*(a_4)\,da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^\infty \beta_4(a_4)\,e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}da_{,4}$$

And assume that

$$[k_3]_4\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4a_4}\,da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)([f_2^{'}]_4(z_4)) \int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_4$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} \quad da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} da_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r, R\left(f_4(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$$M_4(t) \equiv \int_0^\infty m_4(t, a_4) da_4 = P_4(t) - P_4^*,$$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - [p^*]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4) M - [f_{2*}^4]_4 p_4^*(\mathrm{a}) W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^\infty \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0) W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M,_4$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $[f_{4*}']_4$ denoting $[f'_4(P^*)]_4 [f_{2*}']_4$ denoting $[f_2]_4$ $(Q_4^*$, $h_{*}_4$ denoting $h_*'$ denoting $h_4'$ $(Q_4^*)$, and $[g_*']_4$

Denoting $[g']_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4 + [m_a]_4 = -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4 - [f'_{4*}]_4 [p_4^*](a_4) M_4 - [f_{2*}']_4 p_4^*(a_4) W_4$$

$$m_4(t,0) = [h_*]_4 \int_0^\infty \beta_4(a_4) m_4(t, a_4) da_4 + \frac{[h_*^4]_4}{[h^*]_4} [p^*]_4(0)\, W,_4 \qquad (30)$$

$$\frac{dW_4}{dt} = [g_*^4]_4 [Q_4^*] M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the

universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in

(eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in (eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing

by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to (e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a

complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed,

authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending

well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite

as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L) mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the

possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=)

the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as:

arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the

Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the

sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th]

(or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th]

(or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory**

**techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number: MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond

last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly, by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in

(e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~

(1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum**

**physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement: Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f^{'}{}_{4*}\right]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 - \left[f_{2*}^{'}\right]_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^{\infty} ([h_*]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 + \left(\frac{[h_*^{'}]_4}{[h_*]^4} P_4^*(0) - f_{2*}^{'} P_4^*\right) W_4,$$

(34)

$$\frac{dW_4(t)}{dt} \left[g^{'}{}_*\right]_4 Q_4^* M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4 = ([h_*]_4 \beta_4(\theta_4) - \mu_4(\theta_4)) M_{,4}$$

Where $0 < \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2 [a_4^2]_4 < 4 [a_2]_4 < 1\, a_2 e$,

$\mathrm{Re}\,\lambda_4 [\lambda_4]_4 = \mathrm{Re}\lambda_2 [\lambda_2]_4 = -\frac{4}{2} a_4 [a2\ ]_4 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_* [h_*]_4 \beta_4(a_4) - \mu_4(a_4)\right|$. Then $|n_4(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 < 4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2] \geq 4[a_2] e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4) + d\tau_4 da} = \frac{4}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ 0.5. Assume$[f_4](x_4)= K_4 x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4 x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [K][P_0])}{[m_4]_4 + (\mu_0)_4 + [K][P_0])\,{}_4^3\,[K]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\, P^*\, [f_{2*}] = \frac{\sigma Q_4^*}{4+Q_4^*}$ $[h_*] = \frac{4+\omega_4 Q_4^*}{4+Q_4^*}$

$[f_{4*}] = K_4\, [f_{2*}^4] = \frac{\sigma_4}{(4+Q_4^*)^2}[\, g_*^4] = \frac{\sigma_4}{(4+P_4^*)^2}$ $[h_*^4] = \frac{\omega_4 - 4}{(4+Q_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma)^2}+\frac{2\omega_4}{([m_4]_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma_4)^3}<1$$

And

$$\frac{1}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^2}+\frac{2\omega_4}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_4+[\mu_0]_4+1+\tfrac{K_4b_4}{\delta_4-b_4}+(m_4+[\mu_0]_4+\tfrac{K_4b_4}{\delta_4-b_4}+\sigma_4)\,Q_4^*)^3$$

$$=m_4+[\mu_0]_4+3+\tfrac{K_4b_4}{\delta_4-b_4}(4+Q^*)(1+\omega_4Q_4^*$$

Now

$$\left[a_4^2\right]-1\left[a_2\right]e=\left(\frac{2K_4b_4}{\delta_4-b_4}\right)^2+\frac{1K_4b_4\sigma_4\,Q_4^*}{(\delta_4-b_4)(1+Q_4^*)}+\frac{\sigma_4^2Q_4^{*2}}{(1+Q_4^*)^2}$$

$$\frac{1\,b_4\sigma_4\,(\delta_4-b)Q^*}{\delta_4(1+Q_4^*)^2}-\frac{1\,(\delta_4-b_4)^2(1-\omega_4)Q_4^*}{\delta_4(1+\omega_4Q_4^*)(1+Q_4^*)}P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae^{mu}_4 \ [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a^2_4 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q^*_4}{1 + Q^*_4} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P^*_4$$

$$[a_2]_4 = \left[f^4_{2*}\right] P^*_4 - \frac{h^4_*}{h_*} P^*(0)$$

$$e = [g^4_*]_4 Q^*_4$$

$$\eta_4(t) = \left[h_*\right]_4 \beta_4 \left(\theta_4(t)\right) - \mu_4(\theta(t)),$$

The system(34)can be written as the following system:

$$[M(t)]\left[a_4\right] M_4 - [a_2]_4 W_4 - [n(t)] M_4$$

$$w_4(t) = eM_4$$

Write

$$A = \begin{pmatrix} -[a_4]_4 - [a_2]_4 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_4 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}[X]_4(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum**

**systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th)**

**Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December 1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP**

**Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal**

**deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105**

**DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum**

**mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the**

**general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled**

**with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and**

**Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession**

**due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the**

**system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics.**

**Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc] for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures,**

**especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark**

**matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3**

**[astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley**

**National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero. Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon,"**

**Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of**

**Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two**

**dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show**

**up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of**

**polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 84. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \ P_6 \ 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \ \infty,$$

$$\rho_6(t,0) = \int_0^{\infty} n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ 0 $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6)\int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \ \rho_6(t, a_6) \ \rho_6^*(t, a_6),$$

$$0 \ Q_6(t) \ Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\,\rho_6^*, \quad (7)$$

$$\rho_6^*(t,0) = \int_0^\infty \beta_6(a_6)\,\rho_6^*(t,a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6\,Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t,a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt}e^{-ra}\,da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\,da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6 \; h_6(Q_6) = 1, \qquad and \;\; g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 85. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t,0) = \int_0^\infty \beta_6(a_6)\rho_6(t,a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^\infty u_6(t, a_6)da_6 = \int_0^\infty \rho_6(t, a_6)da_6 - \int_0^\infty (\rho_0)_6(a_6)da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t,0) = \rho_6(t,0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6 u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^\infty \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6 a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\, o\,)_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho o]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t,a_6)da_6$$

$$u_6(t,0) = \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t,a_6) = \tilde{\mathrm{u}}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_6}\, da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6 - \int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds + y\tau}\,\alpha_6(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_6\,(a_6)Y_6(\gamma_6, a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6=0,$$

(24)

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60}^{l}{}_6\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^{a}(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$= \frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6 a_6} - 1)$$

Let

$$E_6(a_6) = e^{-(f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6) d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(4690) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(4691) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(4692) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(4693) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(4694) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(4695) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(4696) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(4697) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(4698) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(4699) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort**

**of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

**The microscopic structure of spacetime**

(4700) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(4701) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(4702) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(4703) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

**Early cosmology**

(4704) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(4705) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(4706) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(4707) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(4708) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

**Black holes**

(4709) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(4710) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(4711) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(4712) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(4713) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

**Astrophysical effects**

(4714) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(4715) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(4716) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

**Main approaches**

(4717) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(4718) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(4719) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

**String theory**

(4720) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(4721) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(4722) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

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| (4723) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (4724) **However, when summed the entire series appears to be (=) divergent.** |
| (4725) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (4726) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (4727) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (4728) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

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| (4729) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (4730) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (4731) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (4732) **The granular properties of space can be (=) explicitly computed.** |
| (4733) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (4734) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (4735) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (4736) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (4737) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (4738) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (4739) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (4740) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (4741) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (4742) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (4743) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (4744) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

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| **Others** |

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| (4745) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (4746) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (4747) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (4748) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (4749) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |

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| (4750) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (4751) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (4752) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (4753) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (4754) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (4755) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (4756) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (4757) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (4758) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (4759) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (4760) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |
| (4761) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (4762) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (4763) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (4764) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption**

**expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (4765) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (4766) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (4767) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (4768) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (4769) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (4770) **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (4771) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (4772) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (4773) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (4774) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (4775) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (4776) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (4777) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (4778) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (4779) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (4780) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only** |

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| **be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (4781) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (4782) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (4783) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (4784) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (4785) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (4786) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (4787) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (4788) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (4789) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (4790) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (4791) **This is a realistic scenario for (e) all known systems.** |
| (4792) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (4793) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (4794) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (4795) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (4796) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (4797) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |

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| (4798) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (4799) **So in those systems, the quantum phase transition does indeed exist.** |
| (4800) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (4801) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (4802) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (4803) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (4804) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (4805) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (4806) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (4807) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (4808) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (4809) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (4810) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (4811) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (4812) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |

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| (4813) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (4814) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (4815) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (4816) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (4817) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (4818) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (4819) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (4820) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (4821) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (4822) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (4823) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (4824) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (4825) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (4826) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (4827) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (4828) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (4829) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (4830) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (∼104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (4831) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (4832) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (4833) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (4834) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (4835) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (4836) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (4837) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (4838) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (4839) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (4840) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (4841) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |

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| (4842) | **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (4843) | **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (4844) | **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (4845) | **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (4846) | **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (4847) | **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (4848) | **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (4849) | **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (4850) | **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (4851) | **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (4852) | **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (4853) | **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (4854) | **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (4855) | **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (4856) | **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (4857) | **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (4858) | **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (4859) | **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (4860) | **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |

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| (4861) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (4862) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (4863) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (4864) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (4865) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (4866) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (4867) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (4868) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (4869) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (4870) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (4871) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (4872) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (4873) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (4874) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (4875) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (4876) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (4877) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (4878) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (4879) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (4880) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (4881) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (4882) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (4883) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (4884) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (4885) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (4886) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (4887) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (4888) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (4889) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (4890) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (4891) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (4892) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (4893) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (4894) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |

(4895) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.**

**Doomed satellite missions**

(4896) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.**

(4897) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.**

(4898) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.**

(4899) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.**

(4900) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.**

(4901) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people**

(4902) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."**

(4903) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.**

(4904) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(4905) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(4906) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

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| (4907) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.** |
| (4908) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.** |
| (4909) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.** |
| (4910) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.** |
| (4911) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.** |
| (4912) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.** |
| (4913) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.** |
| (4914) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.** |

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| **Dark matter guides growth of supermassive black holes February 18, 2015** |

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| (4915) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.** |
| (4916) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.** |
| (4917) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.** |
| (4918) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.** |
| (4919) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (4920) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (4921) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (4922) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |

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| (4923) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (4924) **We know dark matter exists only from (e) its gravitational effects.** |
| (4925) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (4926) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (4927) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (4928) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (4929) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (4930) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (4931) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (4932) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (4933) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (4934) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (4935) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (4936) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (4937) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (4938) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the** |

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| **underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (4939) **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (4940) **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (4941) **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (4942) **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (4943) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (4944) **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (4945) **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (4946) **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (4947) **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (4948) **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (4949) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (4950) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (4951) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |

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| (4952) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (4953) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (4954) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (4955) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (4956) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (4957) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (4958) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (4959) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (4960) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (4961) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |
| (4962) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (4963) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (4964) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (4965) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |

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| (4966) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (4967) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (4968) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (4969) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (4970) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (4971) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (4972) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (4973) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (4974) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (4975) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (4976) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (4977) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (4978) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (4979) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e)** |

**electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.**

**Spheres on trees**

(4980) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.**

(4981) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.**

(4982) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.**

(4983) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.**

(4984) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.**

(4985) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.**

(4986) **As a result, it provides (eb) an important clue for finding the global minimum.**

(4987) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.**

(4988) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.**

**Thompson's problem 2**

(4989) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society**

(4990) **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.**

(4991) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.**

**Implications for Thomson's and Smale's problems**

(4992) **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.**

(4993) **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry**

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| | **community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (4994) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (4995) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (4996) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (4997) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (4998) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (4999) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (5000) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (5001) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (5002) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (5003) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(865)** | **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(866)** | **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |

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| **(867) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(868) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(869) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(870) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(871) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(872) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(873) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(874) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(875) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(876) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(877) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(878) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(879) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(880) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(881) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |
| **(882) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(883) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(884) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(885) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |

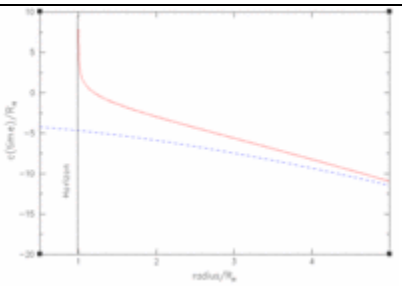
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| **(886) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(887) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(888) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(889) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(890) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(891) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(892) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(893) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(894) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(895) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(896) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(897) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(898) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(899) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(900) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |
| **(901) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(902) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(903) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(904) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(905) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |

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| **(906) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(907) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(908) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(909) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(910) One of these coordinates is (=) called the "coordinate time."** |
| **(911) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(912) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(913) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(914) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(915) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |
| **(916) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(917) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(918) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(919) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(920) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |

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| **(921) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(922) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(923) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(924) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(925) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(926) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(927) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(928) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(929) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(930) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(931) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(932) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(933) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(934) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(935) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(936) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(937) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(938) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |

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| **(939) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(940) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(941) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(942) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(943) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(944) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(945) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(946) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(947) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(948) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(949) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(950) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(951) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(952) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(953) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(954) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(955) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(956) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |

**(957) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.**

**(958) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.**

**(959) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.**

**(960) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.**

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1 + \frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(a_6)da_6 - \int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6 = 1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6 \neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6 > 0$. Therefore, the characteristic solution is

$$\gamma_6 = -(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6^{'}(P_0)_6 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6),0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_6+(\mu_6(a_6)+[f_6]_6(P_6^*)+[f_2]_6(Q_6^*))P_6^*=0, \qquad (26)$$

$p_6^*(0) = h_6(Q_6^*) \int_0^\infty \beta_6(a_6) p_6^*(a_J) p_6^*(a_6) da_6$ (27)

$g_6(P_6^*) - b_6 = 0,$ (28)

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b),$

From (26),

$p_6^*(a) = p_6^*(0) e^{\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*) \int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6} da_6\right) p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6) \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6} da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty} [H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,) \int_0^{\infty} a_6 \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6} da_6$$

$-h^{'}(z_6) \int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-y_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0) W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6 \left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,$ $h_{*}_6$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - [f'_{6*}]_6[p_6^*](a_6)M_6 - [f'_{2*}]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\, W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f'_{6*}]_6 \int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6 \int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\, m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}^{'}]_6 P_6^*\right)M_6 - [f_{2*}^{'}]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^\infty ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f_{6*}^{'}]_6 P_6^*\right)M_6 + \left(\frac{[h_*^{'}]_6}{[h_*]^6}\, P_6^*(0) - f_{2*}^{'} P_6^*\right)W_6,$$

(36)

$$\frac{dW_6(t)}{dt}[g^{'}_{*}]_6 Q_6^* M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^\infty ([h_{*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2 [a_6^2]_6 < 4 [a_2]_6 < 4 a_2 e$,

$$\text{Re}\,\lambda_6[\lambda_6]_6 = \text{Re}\lambda_2[\lambda_2]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0.$$

Let $\Delta = \max_{2<u<\infty} |h_* [h_*]_6 \beta_6(a_6) - \mu_6(a_6)|$. Then $|n_6(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_6^2 < 4 a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2] \geq 4[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [\mathrm{K}][P_0])}{[m_6]_{\,6} + (\mu_0)_{\,6} + [\mathrm{K}][P_0])\,{}^{3}_{6}\ [\mathrm{K}]_{\,6}[P_0]\,{}^{3}_{6}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_{\,6}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*} \quad [h_*] = \frac{6+\omega_6 \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$

$[f_{6*}] = K_6\ \left[\mathrm{f}\,{}^{6}_{2*}\right] = \frac{\sigma_6}{(6+\mathrm{Q}_6^*)^2}[\ \mathrm{g}\,{}^{6}_{*}] = \frac{\sigma_6}{(6+\mathrm{P}_6^*)^2} \quad \left[\mathrm{h}\,{}^{6}_{*}\right] = \frac{\omega_6 - 6}{(6+\mathrm{Q}_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^{\,2}} + \frac{2\omega_6}{([m_6]_{\,6} + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^{\,3}} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,2}} + \frac{2\omega_6}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}]P^*_6$$

$$[a_2]_6 = \left[f^6_{2*}\right]P^*_6 - \frac{h^6_*}{h_*}P^*(0)$$

$$e=[g^6_*]_6 Q^*_6$$

$$\eta_6(t) = \left[h_*\right]_6 \beta_6\left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right]M_6 - [a_2]_6 W_6 - [n(t)]M_6$$

$$w_6(t) = eM_6$$

Write

$$A=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(t) = e^{A,t}[X]_6(0) + \int_0 e^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case $\beta\to\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes

(eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal

anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case ωq=−2/3 authors derive analytic expressions of (e) critical physical quantities, while (e) for

cases $\omega_q \neq -2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb)

this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence,

authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the

dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R~1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They

also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual

field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2

[hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the

recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black**

**holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the

imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves

(e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise

mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top.

Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of

quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity.

**Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions.

The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques

are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity**

**QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

# ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-**

**atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A,**

**75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the**

**sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain**

**are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized**

**ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study**

**the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor**

**connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they**

**interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold**

**dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct (“extreme”) types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, ‘the field’. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb)**

**that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ⁎This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum**

**interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are**

**in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-**

**0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x.**

**Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space**

**time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics**

**(math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

**86. MODEL FORMULATION**

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \quad P_7 \; 0, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3 \left[\frac{dQ_3}{dt}\right]_7 - x_4 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5 \left[\frac{dQ_7}{dt}\right]_7 - x_6 \left[\frac{dQ_3}{dt}\right]_7$$

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| Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied. |
| On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$ |
| Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent. |
| Following equation throws light on important issues in the following cases |
| **Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA** |

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| (5004) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. |
| (5005) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. |
| (5006) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. |
| (5007) Such planets are all the "right" distance from (e) their star, and hopefully, also have water. |
| (5008) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. |
| (5009) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. |
| (5010) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. |
| (5011) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. |
| (5012) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, |

which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(5013) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(5014) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(5015) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(5016) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(5017) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(5018) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(5019) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(5020) The electronic and (e&eb) structural variability among them is extraordinary.

(5021) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(5022) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(5023) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(5024) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(5025) Nudge your bath tub just right and (eb) you might end up reversing time.

(5026) Don’t worry – you won’t end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a “time mirror”, which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

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| (5027) | To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse |
| (5028) | Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses. |
| (5029) | But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort. |
| (5030) | The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort. |
| (5031) | Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower |
| (5032) | Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower |

**Déjà vu**

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| (5033) | To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape. |
| (5034) | The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says. |
| (5035) | Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says. |
| (5036) | "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810 |

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

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| (5037) | Wave control is usually performed by (e) spatially engineering the properties of a medium. |
| (5038) | Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach. |

(5039) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(5040) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(5041) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(5042) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(5043) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(5044) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(5045) Quantum mechanics rules the dynamics of (e) light and matter.

(5046) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(5047) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(5048) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(5049) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit.

(5050) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations.

(5051) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5].

(5052) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates.

(5053) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable.

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| (5054) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (5055) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (5056) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (5057) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (5058) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (5059) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (5060) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (5061) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (5062) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (5063) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (5064) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (5065) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (5066) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (5067) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (5068) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (5069) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (5070) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (5071) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (5072) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |

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| (5073) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (5074) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (5075) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (5076) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (5077) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (5078) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (5079) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (5080) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (5081) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (5082) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (5083) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (5084) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (5085) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (5086) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (5087) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (5088) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (5089) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (5090) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (5091) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |

(5092) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(5093) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big(f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt}$$

$$= -b_7Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \leq \rho_7(t, a_7) \leq \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7\, Q_7^* + g_7(P_7^*) Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \quad (10)$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \qquad (H1)$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 87. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7e^{r_7t - \int_0^t (f_7)_7(P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau}\, e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7(P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7 N_7 + o_7(N_7), \quad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \quad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7(0) \quad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7 Q_7$$





The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{\mathrm{u}}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_7)$ gives,

$$\tilde{\mathrm{u}}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{\mathrm{u}}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a(\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds+y\tau}\alpha_7(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^a(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty} Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty} Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)e^{\gamma_7\tau_7+(f_{70})_7\tau+\int_0^{\tau}\mu_7(s_7)ds_7}d\tau$$

$$=\frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7a_7}-1)$$

Let

$$E_7(a_7)=e^{-(f_{70})_7a_7-\int_0^{a}\mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)da_7 = 1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)\ \int_0^a \alpha_7\ (\tau_7)Y^{-7}(\gamma_7, \tau_7)d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7(\rho_0)_7 \int_0^{\infty} E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^{\infty} \beta_7(a_7)p_7^*(a_J)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^{\infty} p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7']_7(z_7))\int_0^\infty a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}da_{7}$$

$-h'(z_7)\int_0^{\infty}\beta_7(a_7)\,e^{\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7-[f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^-(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\,e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^\infty m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - [p^*]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M,_7$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}']_7$ denoting $[f'_7(P^*)]_7 [f_{7*}']_7$ denoting $[f_7]_7$ $(Q_7^*$, $h_{*7}$ denoting $h_*'$ denoting $h_7'$ $(Q_7^*)$, and $[g_*']_7$

Denoting $[g']_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7 [p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7) m_7(t, a_7) da_7 + \frac{[h_*^7]_7}{[h^*]_7} [p^*]_7(0)\, W,_7 \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7 [Q_7^*] M_7$$

Integrating $[m_7]_7(t, a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t, a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t, a_7)da_7 - \int_0^\infty \mu(a_7)m(t, a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$= m_7(t, 0) - \int_0^\infty \mu_7(a_7)\, m_7(t, a_7)da_7$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [\dot{f}_{7*}]_7 P_7^*\right)M_7 - [\dot{f}_{7*}]_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f'_{7*}]_7 P_7^*\right)M_7 + \left(\frac{[h'_*]_7}{[h_*]^7}\, P_7^*(0) - f'_{7*}P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}[g'_*]_7\, Q_7^*\, M_{.7}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $\frac{7}{7}$ [a $\frac{7}{7}$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_7 [\lambda_7]_7 = \text{Re}\lambda_7 [\lambda_7]_7 = -\frac{7}{7} a_7$[a2 ] $_7 < 0$.

Let ⊿=max$_{7<u<\infty}$ $|h_* [h_*]_7 \beta_7(a_7) - \mu_7(a_7)|$. Then$|n_7(t)|$ ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $\frac{7}{7}$ <4 a $_7$ e and [a $_7$] $> 1⊿$,or [a $\frac{7}{7}$] [a $_7$]e and (a $_7 - \sqrt{(\text{a}\frac{7}{7})_7 - 4(\text{a}_7)_7 \text{e}} > 1⊿$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\text{a}}{\text{a}+7}\right)$ and $\beta_7(a_7) = \text{ae}^{-\text{ma}}$ Then ⊿=$[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_7(a_7)\text{e}^{-\int_0^{\text{a}} \mu_7(\tau_7) + d\tau_7 \text{da}} = \frac{1}{(\text{m} + \mu_0 + 1)^7} + \frac{1}{(\text{m} + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ .5. Assume$[f_7](x_7)= K_7 x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)$= $[0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)$=$(1+\omega_7 x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\text{K}][P_0])}{[m_7]_7 + (\mu_0)_7 + [\text{K}][P_0])\frac{3}{7}\ [\text{K}]_7 [P_0]\frac{3}{7}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_7$ i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^*\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\ 7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7} [\ \mathrm{g}_*^{\ 7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\ 7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\ 7}} + \frac{1\omega_7}{([m_7]_{\ 7} + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\ 3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\ 7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_7 + [\mu_0]_{\ 7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_{\ 7} + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\ \mathrm{Q}_7^*)^{\ 3}$

$= m_7 + [\mu_0]_{\ 7} + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + \mathrm{Q}^*)(1 + \omega_7 \mathrm{Q}_7^*$

Now

$$\left[\mathrm{a}\begin{matrix}1\\1\end{matrix}\right]-4\left[\mathrm{a}_{7}\right]\mathrm{e}=\left(\frac{1K_7b_7}{\delta_7-b_7}\right)^{7}+\frac{4K_7b_7\sigma_7\,\mathrm{Q}_{7}^{*}}{(\delta_7-b_7)(7+\mathrm{Q}_{7}^{*})}+\frac{\sigma_{7}^{7}\,\mathrm{Q}_{7}^{*7}}{\left(1+\mathrm{Q}_{7}^{*}\right)^{7}}$$

$$\frac{4\,b_7\sigma_7\,(\delta_{7}-\mathrm{b})\mathrm{Q}^{*}}{\delta_{7}\,(1+\mathrm{Q}_{7}^{*})^{7}}-\frac{4\,(\delta_{7}-b_7)^{7}\,(1-\omega_7)\mathrm{Q}_{7}^{*}}{\delta_{7}\left(1+\omega_7\mathrm{Q}_{7}^{*}\right)(1+\mathrm{Q}_{7}^{*})}\mathrm{P}^{*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{7}\,ae_{7}^{mu}\;\;[\mu_0]_{7}-\frac{a_7}{a_7+7}$$

Hence if $[\mathrm{a}_{7}^{7}>4(a_7)_{7}\mathrm{e}$ and

$$[a_7]=\frac{1K_7b_7}{\delta_7-b_7}+\frac{\sigma_7\,\mathrm{Q}_{7}^{*}}{1+\mathrm{Q}_{7}^{*}}>1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(867)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(868) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(869)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(870) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(871)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(872) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

(873)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States.

(874)Humans are not very good at understanding interacting many-body systems.

(875)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them.

(876)But what happens when we couple (e&eb) the particles strongly?

(877)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments.

(878)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water.

(879)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h.

(880) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1]

(881) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic

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| origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (882)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (883) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (884)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (885) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (886) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (887)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (888)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (889) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (890) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (891)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (892)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (893)When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (894)The energy levels of these quantized orbitals take on (e) discrete values |
| (895):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (896) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (897) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (898)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |

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| (899) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (900)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (901) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (902)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (903)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (904)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (905) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (906)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (907)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (908)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (909) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an "elementary particle." |
| (910)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (911)The remarkable thing is that solitons can become (=) "elementary," in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (912)An example of a duality, which we will use below, arises in (eb) Maxwell's theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (913)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (914)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (915)Another classic example of duality involves (e&eb) vortices. |
| (916) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (917)Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (918) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (919)This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (920)Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |

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| (921) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (922)The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (923)TIs are distinguished only by (e) their edge states. |
| (924) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (925)A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (926) But the Dirac cone is not (e) the only possible signature. |
| (927) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (928)Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (929)When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (930)However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (931) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (932). But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (933)The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (934)The bulk duality implies (eb) a duality between edge theories. |
| (935)The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (936)One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (937) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (938) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (939)Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |

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| (940)Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (941)Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (942)The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (943)This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (944)They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (945) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (946)This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (947)This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (948) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (949)Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (950)The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

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| IF we define |
| $$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,P^*_7$$ |
| $$[a_7]_7 = \left[f^7_{7*}\right]P^*_7 - \frac{h^7_*}{h_*}P^*(0)$$ |
| $$e=[g^7_*]_7\,Q^*_7$$ |
| $$\eta_7(t) = \left[h_*\right]_7\beta_7\left(\theta_7(t)\right) - \mu_7(\theta(t)),$$ |

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{7}\right]M_7 - [a_7]_{7}W_7 - [\mathrm{n(t)}]M_7$$

$$w_7(\mathrm{t}) = \mathrm{e}M_7$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_7]_{7} & -[a_7]_{7} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{7} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{7}(0) + \int_0 \mathrm{e}^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality

function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 88. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(5094) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(5095) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(5096) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(5097) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(5098) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

(5099) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems.

(5100) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe.

(5101) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short.

(5102) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another).

(5103) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited.

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| (5104) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (5105) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (5106) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (5107) "The underlying physics is (=) chaos, essentially," he said. |
| (5108) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (5109) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (5110) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (5111) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (5112) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (5113) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (5114) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (5115) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (5116) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (5117) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |
| (5118) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (5119) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (5120) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (5121) The study's findings have fundamental implications for (e) quantum computing. |
| (5122) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |

(5123) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

(5124) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box.

(5125) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction.

(5126) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed.

(5127) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer.

(5128) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box.

(5129) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance.

(5130) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction.

(5131) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device.

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(5132) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(5133) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(5134) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(5135) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(5136) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(5137) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(5138) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(5139) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(5140) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(5141) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(5142) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(5143) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(5144) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(5145) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(5146) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(5147) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(5148) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(5149) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(5150) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(5151) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(5152) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(5153) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(5154) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(5155) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(5156) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(5157) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(5158) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(5159) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(5160) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(5161) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until

genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(5162) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(5163) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(5164) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(5165) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(5166) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(5167) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(5168) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(5169) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(5170) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

### Black holes

(5171) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(5172) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(5173) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(5174) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(5175) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(5176) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(5177) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(5178) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(5179) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(5180) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(5181) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(5182) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(5183) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(5184) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(5185) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(5186) However, when summed the entire series appears to be (=) divergent.

(5187) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(5188) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(5189) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(5190) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(5191) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(5192) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(5193) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(5194) The granular properties of space can be (=) explicitly computed.

(5195) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(5196) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(5197) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(5198) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(5199) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(5200) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(5201) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(5202) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(5203) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(5204) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(5205) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(5206) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(5207) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(5208) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(5209) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(5210) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \quad P_8 \text{ } 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \text{ } \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \text{ ?}\infty,$$

$$\rho_8(t,0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 89. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^{\infty} u_8(t, a_8) da_8 = \int_0^{\infty} \rho_8(t, a_8) da_8 - \int_0^{\infty} (\rho_0)_8(a_8) da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t, 0) = \rho_8(t, 0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8\,(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8\,(0) \qquad (18)$$

Expanding

$$h_8(Q_8) = 1 + \ (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^{\infty} \beta_8\,(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\ o\ )_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^{8}]\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}da_8\left(\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^{\infty}E_8(a_8)da_8-\int_0^{\infty}E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^{\infty}E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)da_8=1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\neq 0,$$

Since $\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)\int_0^{a}\alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8>0$. Therefore, the characteristic solution is

$$\gamma_8=-(f_{80}^{'})_8(\rho_0)_8\int_0^{\infty}E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8) d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[g^{'}\right]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - \left[f^{'}_{8*}\right]_8[p_8^*](a_8)M_8 - \left[f^{'}_{8*}\right]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}_{8*}\right]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}_{8*}\right]_8 P_8^*\right)M_8 - \left[f^{'}_{8*}\right]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if $a_8^8\,[a_8^8]2<4\ [a_{\ 8}]2<4\ a_{\ 8}e$,

$\mathrm{Re}\,\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=\mathrm{Re}\lambda_{\ 8}\left[\lambda_{\ 8}\right]_{\ 8}=-\frac{8}{8}a_8[a2\ ]_{\ 8}<0.$

Let $\Delta=\max_{8<u<\infty}\left|h_{\ *}\,[h_{\ *}]_{\ 8}\beta_{\ 8}(a_8)-\mu_8(a_8)\right|$. Then$|n_8(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{\ *}(a),\boldsymbol{Q}^{\ *})$ is asymptotically stable if either $a_8^8<4\ a_{\ 8}e$ and $[a_{\ 8}]>1\Delta$,or $[a_8^8]$ ⊠4$[a_{\ 8}]e$ and $(a_{\ 8}-\sqrt{\left(a_8^8\right)_{\ 8}-4(a_{\ 8})_{\ 8}e})>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+8}\right)$ and $\beta_{\ 8}(a_8)=ae^{-ma}$ Then $\Delta=[\mu_0]+2$ Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)=(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])\,{}_8^3\,[\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\,[f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\,\mathrm{g}_*^{8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{8}] = \frac{\omega_8-8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, ae_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}]\, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8}\, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8}\, \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} - [a_8]_{\,8} \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

| |
|---|
| (588)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (589)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (590) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (591) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (592)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (593) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (594)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (595)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(596)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(597)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(598) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(599)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(600)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(601)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(602) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(603)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(604)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(605)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(606)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(607)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(608)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(609)Bringing them to thermal equilibrium requires (e) an external bath.

(610) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(611)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(612)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(613)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(614)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(615)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(616) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(617) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(618) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(619)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(620)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(621) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(622)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(623)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(624) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(625)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(626)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

| |
|---|
| (627)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (628) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (629)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (630) This is a realistic scenario for (e) all known systems. |
| (631)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (632)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (633)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (634) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (635)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (636)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (637)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (638)So in those systems, the quantum phase transition does indeed exist. |
| (639)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (640)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

### MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. 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They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 90. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \; P_9 \; 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \; \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ∞; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a) e^{-\int_0^a \mu_9(\tau)dt} e^{-ra}\, da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(t)dt}\, da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t, 0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9} \rho_9(t, a_9)\, da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9\,(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^{a}\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^{a}\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_9^*$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2 < 4 [a_9] 2 < 4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} \left| h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9) \right|$. Then $|n_9(t)| \ 2 \Delta \forall t \ 0$. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 < 4\, a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]\ 4 [a_9] e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]\ 0.5$. Assume $[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$ and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$ (See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])\,{}^{3}_{9}\ [\mathrm{K}]_{\,9}[P_0]\,{}^{3}_{9}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}\,{}^{9}_{9*}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\,[\,\mathrm{g}\,{}^{9}_{*}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}\,{}^{9}_{*}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the

following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a_{\ 9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_{\ 9} - b)Q^*}{\delta_{\ 9}(1 + Q_9^*)^9} - \frac{4\,(\delta_{\ 9} - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_{\ 9}(1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\ mu}_{\ 9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a\ {}^{9}_{9} > 4(a_9)_{\ 9}e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION THREE B

## Mixed State Entanglement And Quantum Error Correction-3

### INTRODUCTION—VARIABLES USED

**Mixed State Entanglement and Quantum Error Correction Charles H. Bennett, David P. DiVincenzo, John A. Smolin, William K. Wootters (Submitted on 23 Apr 1996 (v1), last revised 8 Aug 1996 (this version, v2))**

(1) They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for (e) reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available.
(2) Authors exhibit a family of codes based on (e) universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy.
(3) They also obtain a specific, simple 5-bit single- error-correcting quantum block code. They prove that {\em iff} a QECC results in (eb) high fidelity for the case of no error the QECC can be (=) recast into a form where (e) the encoder is the matrix inverse of the decoder.
(4) Comments: Subjects:Quantum Physics (quant-ph) Journal reference:Phys.Rev.A54:3824-3851,1996 DOI:10.1103/PhysRevA.54.3824 Cite as:arXiv:quant-ph/9604024 (or arXiv:quant-ph/9604024v2 for this version)

### NOTATION

**Module One**

They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for (e) reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available

$G_{13}$ : Category one of reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of certain noisy quantum channels, for example a 50% depolarizing channel

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

| | |
|---|---|
| They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for reliable transmission of quantum states if (e) two-way communication is available, but cannot be used if (e) only one-way communication is available | |
| $G_{16}$ : Category one of two-way communication is available, but cannot be used if (e) only one-way communication is available<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of certain noisy quantum channels, for example a 50% depolarizing channel, can be used for reliable transmission of quantum states<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>They show that certain noisy quantum channels, for example a 50% depolarizing channel, can be used for reliable transmission of quantum states if two-way communication is available, but cannot be used if (e) only one-way communication is available | |
| $G_{20}$ : Category one of only one-way communication is available<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of certain noisy quantum channels, for example a 50% depolarizing channel, can be used for reliable transmission of quantum states if two-way communication is available, but cannot be used<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors exhibit a family of codes based on (e) universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy | |
| $G_{24}$ : Category one of universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of family of codes<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |

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| **Module five**<br><br>Authors exhibit a family of codes based on universal hash- ing able toachieve (eb) an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy | |
| $G_{28}$ : Category one of family of codes based on universal hash- ing able toachieve<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>Authors exhibit a family of codes based on universal hash- ing able toachieve an asymptotic Q (or D) of 1-S for (e) simple noise models, where (e) S is the error entropy | |
| $G_{32}$ : Category one of simple noise models, where (e) S is the error entropy<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of family of codes based on universal hash- ing able toachieve an asymptotic Q (or D) of 1-S<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>Authors exhibit a family of codes based on universal hash- ing able toachieve an asymptotic Q (or D) of 1-S for simple noise models, where (e) S is the error entropy | |
| $G_{36}$ : Category one of error entropy<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of family of codes based on universal hash- ing able toachieve an asymptotic Q (or D) of 1-S for simple noise models<br><br>$T_{37}$ : Category two of SAS | |

| | |
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| $T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>They also obtain a specific, simple 5-bit single- error-correcting quantum block code.<br><br>They prove that {\em iff} a QECC results in high fidelity for the case of no error the QECC can be (=) recast into a form where (e) the encoder is the matrix inverse of the decoder | |
| $G_{40}$ : Category one of{\em iff} a QECC results in high fidelity for the case of no error the QECC<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of recast into a form where (e) the encoder is the matrix inverse of the decoder<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>They prove that {\em iff} a QECC results in high fidelity for the case of no error the QECC can be recast into a form where (e) the encoder is the matrix inverse of the decoder | |
| $G_{44}$ : Category one of encoder is the matrix inverse of the decoder<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of{\em iff} a QECC results in high fidelity for the case of no error the QECC can be recast into a form<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)},$<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$ | |

| | |
|---|---|
| are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}, \ (b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)} \ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19}), t)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}((G_{19}), t)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19}), t)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}((G_{19}), t) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |

| | |
|---|---|
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23},t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23},t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23},t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21},t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23},t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25},t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25},t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25},t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}),t)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}),t)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}),t)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25},t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}((G_{27}),t) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29},t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29},t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31}),t)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}((G_{31}),t)\right]T_{29}$ | 29 |
| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a_{28}^{''})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b_{28}^{''})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a_{32}^{''})^{(6)}(T_{33},t) =$ **First augmentation factor** | |

**Module Numbered Seven:**

**The differential system of this model is now (Seventh Module)**

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37},t)\right]G_{36} \quad (37)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37},t)\right]G_{37} \quad (38)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37},t)\right]G_{38} \quad (39)$$

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{36} \quad (40)$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{37} \quad (41)$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}\left((G_{39}),t\right)\right]T_{38} \quad (42)$$

$+(a_{36}^{''})^{(7)}(T_{37},t) =$ **First augmentation factor**

**Module Numbered Eight**

The differential system of this model is now

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right]G_{40} \quad (43)$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41},t)\right]G_{41} \quad (44)$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41},t)\right]G_{42} \quad (45)$$

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{40} \quad (46)$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{41} \quad (47)$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}\left((G_{43}),t\right)\right]T_{42} \quad (48)$$

**Module Numbered Nine**

The differential system of this model is now

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45},t)\right]G_{44} \quad (49)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45},t)\right]G_{45} \quad (50)$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46} \quad (51)$$

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44} \quad (52)$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45} \quad (53)$$

$$\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46} \quad (54)$$

$+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor**

$-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor**

$$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[\begin{array}{ccc} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{13} \quad (55)$$

$$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[\begin{array}{ccc} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{array}\right]G_{14} \quad (56)$$

$$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (57)$$

Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (58)$$

$$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14} \quad (59)$$

$$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15} \quad (60)$$

Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients

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| for category 1, 2 and 3 $-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)$ , $-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)$ are fifth detrition coefficients for category 1, 2 and 3 $-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)$ , $-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)$ are sixth detrition coefficients for category 1, 2 and 3 $-(b_{37}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{36}^{''})^{(7,7,)}(G_{39},t)$ , $-(b_{38}^{''})^{(7,7,)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3 $-(b_{40}^{''})^{(8,8)}(G_{43},t)$ $-(b_{41}^{''})^{(8,8)}(G_{43},t)$ $-(b_{42}^{''})^{(8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3 $-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ , $-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[ \begin{array}{c} (a_{16}^{'})^{(2)} +(a_{16}^{''})^{(2)}(T_{17},t) +(a_{13}^{''})^{(1,1,)}(T_{14},t) +(a_{20}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7)}(T_{37},t) +(a_{40}^{''})^{(8,8,8)}(T_{41},t) +(a_{44}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[ \begin{array}{c} (a_{17}^{'})^{(2)} +(a_{17}^{''})^{(2)}(T_{17},t) +(a_{14}^{''})^{(1,1,)}(T_{14},t) +(a_{21}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7)}(T_{37},t) +(a_{41}^{''})^{(8,8,8)}(T_{41},t) +(a_{45}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[ \begin{array}{c} (a_{18}^{'})^{(2)} +(a_{18}^{''})^{(2)}(T_{17},t) +(a_{15}^{''})^{(1,1,)}(T_{14},t) +(a_{22}^{''})^{(3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t) +(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t) +(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7)}(T_{37},t) +(a_{42}^{''})^{(8,8,8)}(T_{41},t) +(a_{46}^{''})^{(9,9)}(T_{45},t) \end{array} \right] G_{18}$ | *63* |
| Where $+(a_{16}^{''})^{(2)}(T_{17},t)$ , $+(a_{17}^{''})^{(2)}(T_{17},t)$ , $+(a_{18}^{''})^{(2)}(T_{17},t)$ are first augmentation coefficients for category 1, 2 and 3 $+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3 $+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3 $+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$ , $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3 $+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3 $+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3 $+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3 $+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3 | |

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| $+(a_{44}^{''})^{(9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \begin{bmatrix} (b_{16}^{'})^{(2)} & -(b_{16}^{''})^{(2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,)}(G,t) & -(b_{20}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{36}^{''})^{(7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{16}$ | 64 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \begin{bmatrix} (b_{17}^{'})^{(2)} & -(b_{17}^{''})^{(2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,)}(G,t) & -(b_{21}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{37}^{''})^{(7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{17}$ | 65 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \begin{bmatrix} (b_{18}^{'})^{(2)} & -(b_{18}^{''})^{(2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,)}(G,t) & -(b_{22}^{''})^{(3,3,3,)}(G_{23},t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) & \\ -(b_{38}^{''})^{(7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9)}(G_{47},t) & \end{bmatrix} T_{18}$ | 66 |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$, $-(b_{17}^{''})^{(2)}(G_{19},t)$, $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3<br>$-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)} & +(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)} & +(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) & \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) & \end{bmatrix} G_{21}$ | 68 |

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)}\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)} & \boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)} \end{bmatrix} G_{22} \quad (69)$$

$\boxed{+(a_{20}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{44}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{20} \quad (70)$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)}\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)}\boxed{-(b_{22}^{''})^{(3)}(G_{23},t)} & \boxed{-(b_{18}^{''})^{(2,2,2)}(G_{19},t)} & \boxed{-(b_{15}^{''})^{(1,1,1,)}(G,t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9)}(G_{47},t)} \end{bmatrix} T_{22} \quad (72)$$

$\boxed{-(b_{20}^{''})^{(3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b_{16}^{''})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b_{13}^{''})^{(1,1,1,)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ & +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ & +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ & +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation coefficient for category* 1, 2 *and* 3

$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ *are third augmentation coefficient for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$
*are fourth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$
*are fifth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$
*are sixth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$
*are seventh augmentation coefficients for category* 1, 2 *and* 3

$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$
*are eighth augmentation coefficients for category* 1, 2 *and* 3

$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)}-(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)}-(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)}-(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

*Where* $-(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ *are first detrition coefficients for category* $1,2$ *and* $3$

$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ *are second detrition coefficients for category* $1,2$ *and* $3$

$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ *are third detrition coefficients for category* $1,2$ *and* $3$

$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$ *are fourth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$ *are fifth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$ *are sixth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$ *are seventh detrition coefficients for category* $1,2$ *and* $3$

$-(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t)$ *are eighth detrition coefficients for category* $1,2$ *and* $3$

$-(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt}=(a_{28})^{(5)}G_{29}-\begin{bmatrix}(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,)}(T_{25},t) & +(a_{32}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{28} \quad (79)$$

$$\frac{dG_{29}}{dt}=(a_{29})^{(5)}G_{28}-\begin{bmatrix}(a_{29}^{'})^{(5)}+(a_{29}^{''})^{(5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,)}(T_{25},t) & +(a_{33}^{''})^{(6,6,6)}(T_{33},t)\\ +(a_{14}^{''})^{(1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{29} \quad (80)$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)}\boxed{+(a''_{30})^{(5)}(T_{29},t)} & \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} & \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad (81)$$

*Where* $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b'_{28})^{(5)}\boxed{-(b''_{28})^{(5)}(G_{31},t)} & \boxed{-(b''_{24})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28} \quad (82)$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b'_{29})^{(5)}\boxed{-(b''_{29})^{(5)}(G_{31},t)} & \boxed{-(b''_{25})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b'_{30})^{(5)}\boxed{-(b''_{30})^{(5)}(G_{31},t)} & \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} & \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

*where* $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)$ , $-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)$ are fourth detrition coefficients for category 1,2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1,2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1,2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2, and 3

$-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$, $-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1,2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2, and 3

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$$\frac{dG_{32}}{dt}=(a_{32})^{(6)}G_{33}-\begin{bmatrix}(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33},t) & +(a_{28}^{''})^{(5,5,5)}(T_{29},t) & +(a_{24}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{32}$$

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$$\frac{dG_{33}}{dt}=(a_{33})^{(6)}G_{32}-\begin{bmatrix}(a_{33}^{'})^{(6)}+(a_{33}^{''})^{(6)}(T_{33},t) & +(a_{29}^{''})^{(5,5,5)}(T_{29},t) & +(a_{25}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{33}$$

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$$\frac{dG_{34}}{dt}=(a_{34})^{(6)}G_{33}-\begin{bmatrix}(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33},t) & +(a_{30}^{''})^{(5,5,5)}(T_{29},t) & +(a_{26}^{''})^{(4,4,4,)}(T_{25},t)\\ +(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)\\ +(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{34}$$

$+(a_{32}^{''})^{(6)}(T_{33},t)$, $+(a_{33}^{''})^{(6)}(T_{33},t)$, $+(a_{34}^{''})^{(6)}(T_{33},t)$ *are first augmentation coefficients for category* 1, 2 *and* 3

$+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3

$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3

$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients

$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients

$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients

$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients

$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$

Eighth augmentation coefficients $+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt}=(b_{32})^{(6)}T_{33}-\begin{bmatrix}(b_{32}^{'})^{(6)}-(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{32} \quad (88)$$

$$\frac{dT_{33}}{dt}=(b_{33})^{(6)}T_{32}-\begin{bmatrix}(b_{33}^{'})^{(6)}-(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{33} \quad (89)$$

$$\frac{dT_{34}}{dt}=(b_{34})^{(6)}T_{33}-\begin{bmatrix}(b_{34}^{'})^{(6)}-(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t)\\ -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)\\ -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)\end{bmatrix}T_{34} \quad (90)$$

$-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3

$-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3

$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3

$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3

$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3

$-(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt}=(a_{36})^{(7)}G_{37}-\begin{bmatrix}(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)\\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)\\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)\end{bmatrix}G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a'_{37})^{(7)}\boxed{+(a''_{37})^{(7)}(T_{37},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a'_{38})^{(7)}\boxed{+(a''_{38})^{(7)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15} \quad (93)$$

Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad (94)$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)} G_{41} - \begin{bmatrix} (a'_{42})^{(8)} & +(a''_{42})^{(8)}(T_{41},t) & +(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t) & +(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t) \\ +(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t) & +(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t) & +(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t) & \\ +(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t) & +(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t) & +(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t) & \end{bmatrix} G_{15}$$

Where $+(a''_{40})^{(8)}(T_{41},t)$, $+(a''_{41})^{(8)}(T_{41},t)$, $+(a''_{42})^{(8)}(T_{41},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$, $+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$, $+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$, $+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$, $+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$, $+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ $+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for 1,2,3

$+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$, $+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)$ are eighth augmentation coefficient for 1,2,3

$+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$, $+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)} T_{41} - \begin{bmatrix} (b'_{40})^{(8)} & -(b''_{40})^{(8)}(G_{43},t) & -(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b''_{32})^{(6,6,6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)} T_{40} - \begin{bmatrix} (b'_{41})^{(8)} & -(b''_{41})^{(8)}(G_{43},t) & -(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t) & -(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t) \\ -(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t) & -(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t) & -(b''_{33})^{(6,6,6,6,6,6,6,6)}(G_{35},t) & \\ -(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t) & -(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t) & -(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t) & \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a'_{44})^{(9)}\boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13} \quad (96)$$

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a'_{45})^{(9)}\boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a'_{46})^{(9)}\boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b'_{44})^{(9)}\boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \begin{bmatrix} (b'_{45})^{(9)}\boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b'_{46})^{(9)}\boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a''_i)^{(1)}, (b''_i)^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$$(a''_i)^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$$

$$(b''_i)^{(1)}(G,t) \le (r_i)^{(1)} \le (b'_i)^{(1)} \le (\hat{B}_{13})^{(1)}$$

$lim_{T_2\to\infty}(a''_i)^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b''_i)^{(1)}(G,t) = (r_i)^{(1)}$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

| | |
|---|---|
| $\|(a_i^{''})^{(1)}(T_{14}^{'},t)-(a_i^{''})^{(1)}(T_{14},t)\| \leq (\hat{k}_{13})^{(1)}\|T_{14}-T_{14}^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$<br><br>$\|(b_i^{''})^{(1)}(G^{'},t)-(b_i^{''})^{(1)}(G,t)\| < (\hat{k}_{13})^{(1)}\|G-G^{'}\|e^{-(\hat{M}_{13})^{(1)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(1)}(T_{14}^{'},t)$ and$(a_i^{''})^{(1)}(T_{14},t)$ . $(T_{14}^{'},t)$ and $(T_{14},t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)},(\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i^{''})^{(1)}(T_{14},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)}=1$ then the function $(a_i^{''})^{(1)}(T_{14},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$ :<br><br>$(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)}$, are positive constants<br><br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)},(\hat{Q}_{13})^{(1)}$ :<br><br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$which together With $(\hat{M}_{13})^{(1)},(\hat{k}_{13})^{(1)},(\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)},(a_i^{'})^{(1)},(b_i)^{(1)},(b_i^{'})^{(1)},(p_i)^{(1)}$, $(r_i)^{(1)}, i=13,14,15,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)}+(a_i^{'})^{(1)}+(\hat{A}_{13})^{(1)}+(\hat{P}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$<br><br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)}+(b_i^{'})^{(1)}+(\hat{B}_{13})^{(1)}+(\hat{Q}_{13})^{(1)}(\hat{k}_{13})^{(1)}]<1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)},(a_i^{'})^{(2)},(a_i^{''})^{(2)},(b_i)^{(2)},(b_i^{'})^{(2)},(b_i^{''})^{(2)}>0, \quad i,j=16,17,18$ | |
| The functions $(a_i^{''})^{(2)},(b_i^{''})^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(\mathrm{p_i})^{(2)}$, $(\mathrm{r_i})^{(2)}$: | |
| $(a_i^{''})^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i^{''})^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i^{'})^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(2)}(T_{17},t)=(p_i)^{(2)}$ | 104 |
| $\lim_{G\to\infty}(b_i^{''})^{(2)}((G_{19}),t)=(r_i)^{(2)}$ | 105 |
| **Definition of** $(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)}$ :<br><br>Where $\boxed{(\hat{A}_{16})^{(2)},(\hat{B}_{16})^{(2)},(p_i)^{(2)},(r_i)^{(2)}}$ are positive constants and $\boxed{i=16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |

| | |
|---|---|
| $\|(a_i'')^{(2)}(T_{17}',t)-(a_i'')^{(2)}(T_{17},t)\| \le (\hat{k}_{16})^{(2)}\|T_{17}-T_{17}'\|e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\|(b_i'')^{(2)}((G_{19})',t)-(b_i'')^{(2)}((G_{19}),t)\| < (\hat{k}_{16})^{(2)}\|\|(G_{19})-(G_{19})'\|\|e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and$(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)},(\hat{M}_{16})^{(2)}\right]$. It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)}=1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},$ are positive constants<br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}<1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)},(\hat{Q}_{13})^{(2)}$ :<br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)},(\hat{k}_{16})^{(2)},(\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)},(a_i')^{(2)},(b_i)^{(2)},(b_i')^{(2)},(p_i)^{(2)},\ (r_i)^{(2)},i=16,17,18,$<br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)}+(a_i')^{(2)}+(\hat{A}_{16})^{(2)}+(\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)}+(b_i')^{(2)}+(\hat{B}_{16})^{(2)}+(\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}]<1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)},(a_i')^{(3)},(a_i'')^{(3)},(b_i)^{(3)},(b_i')^{(3)},(b_i'')^{(3)}>0,\quad i,j=20,21,22$<br>The functions $(a_i'')^{(3)},(b_i'')^{(3)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(3)},\ (r_i)^{(3)}$:<br>$(a_i'')^{(3)}(T_{21},t)\le(p_i)^{(3)}\le(\hat{A}_{20})^{(3)}$<br>$(b_i'')^{(3)}(G_{23},t)\le\ (r_i)^{(3)}\le(b_i')^{(3)}\le(\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2\to\infty}(a_i'')^{(3)}(T_{21},t)=(p_i)^{(3)}$<br>$\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t)=\ (r_i)^{(3)}$<br>**Definition of** $(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)}$ :<br>Where $\boxed{(\hat{A}_{20})^{(3)},(\hat{B}_{20})^{(3)},(p_i)^{(3)},\ (r_i)^{(3)}}$ are positive constants and $\boxed{i=20,21,22}$ | 113 |

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|---|---|
| They satisfy Lipschitz condition:<br><br>$\|(a_i'')^{(3)}(T_{21}', t) - (a_i'')^{(3)}(T_{21}, t)\| \leq (\hat{k}_{20})^{(3)}\|T_{21} - T_{21}'\|e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\|(b_i'')^{(3)}(G_{23}', t) - (b_i'')^{(3)}(G_{23}, t)\| < (\hat{k}_{20})^{(3)}\|\|G_{23} - G_{23}'\|\|e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}', t)$ and$(a_i'')^{(3)}(T_{21}, t)$ . $(T_{21}', t)$ And $(T_{21}, t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21}, t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i, j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25}, t) \leq (p_i)^{(4)} \leq (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}((G_{27}), t) \leq (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2 \to \infty}(a_i'')^{(4)}(T_{25}, t) = (p_i)^{(4)}$<br><br>$\lim_{G \to \infty}(b_i'')^{(4)}((G_{27}), t) = (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$ | 118 |
| They satisfy Lipschitz condition: | 119 |

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|---|---|
| $\|(a_i^{''})^{(4)}(T_{25}^{'},t)-(a_i^{''})^{(4)}(T_{25},t)\| \leq (\hat{k}_{24})^{(4)}\|T_{25}-T_{25}^{'}\|e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\|(b_i^{''})^{(4)}((G_{27})^{'},t)-(b_i^{''})^{(4)}\big((G_{27}),t\big)\| < (\hat{k}_{24})^{(4)}\|\|(G_{27})-(G_{27})^{'}\|\|e^{-(\hat{M}_{24})^{(4)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(4)}(T_{25}^{'},t)$ and$(a_i^{''})^{(4)}(T_{25},t)$ . $(T_{25}^{'},t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)},(\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i^{''})^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)}=1$ then the function $(a_i^{''})^{(4)}(T_{25},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}}<1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)},(\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)},(\hat{k}_{24})^{(4)},(\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)},(a_i^{'})^{(4)},(b_i)^{(4)},(b_i^{'})^{(4)},(p_i)^{(4)},\ (r_i)^{(4)}, i=24,25,26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)}+(a_i^{'})^{(4)}+(\hat{A}_{24})^{(4)}+(\hat{P}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(b_i)^{(4)}+(b_i^{'})^{(4)}+(\hat{B}_{24})^{(4)}+(\hat{Q}_{24})^{(4)}(\hat{k}_{24})^{(4)}]<1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)},(a_i^{'})^{(5)},(a_i^{''})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(b_i^{''})^{(5)}>0, \quad i,j=28,29,30$<br><br>The functions $(a_i^{''})^{(5)},(b_i^{''})^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)},\ (r_i)^{(5)}$:<br><br>$(a_i^{''})^{(5)}(T_{29},t)\leq(p_i)^{(5)}\leq(\hat{A}_{28})^{(5)}$<br><br>$(b_i^{''})^{(5)}\big((G_{31}),t\big)\leq\ (r_i)^{(5)}\leq(b_i^{'})^{(5)}\leq(\hat{B}_{28})^{(5)}$ | 122 |
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}',t)-(a_i'')^{(5)}(T_{29},t)| \le (\hat{k}_{28})^{(5)}|T_{29}-T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})',t)-(b_i'')^{(5)}((G_{31}),t)| < (\hat{k}_{28})^{(5)}||(G_{31})-(G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}',t)$ and$(a_i'')^{(5)}(T_{29},t)$ . $(T_{29}',t)$ and $(T_{29},t)$ are points belonging to the interval $\left[(\hat{k}_{28})^{(5)},(\hat{M}_{28})^{(5)}\right]$. It is to be noted that $(a_i'')^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)}=1$ then the function $(a_i'')^{(5)}(T_{29},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ ,\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}<1$$

**Definition of** $(\hat{P}_{28})^{(5)},(\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)},(\hat{A}_{28})^{(5)}and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)},(a_i')^{(5)},(b_i)^{(5)},(b_i')^{(5)},(p_i)^{(5)},\ (r_i)^{(5)},i=28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)}+(a_i')^{(5)}+(\hat{A}_{28})^{(5)}+(\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)}+(b_i')^{(5)}+(\hat{B}_{28})^{(5)}+(\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}]<1$$

Where we suppose

$(a_i)^{(6)},(a_i')^{(6)},(a_i'')^{(6)},(b_i)^{(6)},(b_i')^{(6)},(b_i'')^{(6)}>0,\quad i,j=32,33,34$ 127

The functions $(a_i'')^{(6)},(b_i'')^{(6)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(6)},\ (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33},t)\le(p_i)^{(6)}\le(\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}),t)\le\ (r_i)^{(6)}\le(b_i')^{(6)}\le(\hat{B}_{32})^{(6)}$$

$lim_{T_2\to\infty}(a_i'')^{(6)}(T_{33},t)=(p_i)^{(6)}$ 128

$\lim_{G\to\infty}(b_i'')^{(6)}\left((G_{35}),t\right)=\ (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)},(\hat{B}_{32})^{(6)}$ :

| | |
|---|---|
| Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)\rvert \leq (\hat{k}_{32})^{(6)} \lvert T_{33} - T_{33}'\rvert e^{-(\hat{M}_{32})^{(6)}t}$<br><br>$\lvert(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}\big((G_{35}), t\big)\rvert < (\hat{k}_{32})^{(6)} \lvert\lvert(G_{35}) - (G_{35})'\rvert\rvert e^{-(\hat{M}_{32})^{(6)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and $(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $\big[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\big]$. It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ :<br><br>$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants<br><br>$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ | 129 |
| **Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ :<br><br>There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$<br><br>$\frac{1}{(\hat{M}_{32})^{(6)}}[(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)} (\hat{k}_{32})^{(6)}] < 1$ | 130 |
| Where we suppose | |
| (III) $(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$<br><br>(JJJ) The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:<br><br>$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$<br><br>$(b_i'')^{(7)}(G_{39}, t) \leq \ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$ | 131 |
| *(KKK)* $\lim_{T_2 \to \infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$<br>*(LLL)*<br>$\lim_{G \to \infty} (b_i'')^{(7)}\big((G_{39}), t\big) = \ (r_i)^{(7)}$ | 132 |

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i^{''})^{(7)}(T_{37}^{'}, t) - (a_i^{''})^{(7)}(T_{37}, t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$$

$$|(b_i^{''})^{(7)}((G_{39})^{'}, t) - (b_i^{''})^{(7)}((G_{39}), t)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'}, t)$ and $(a_i^{''})^{(7)}(T_{37}, t)$ . $(T_{37}^{'}, t)$ and $(T_{37}, t)$ are points belonging to the interval $[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}]$ . It is to be noted that $(a_i^{''})^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ : 134

(MMM) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}} \ , \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$$

**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ : 135

(NNN) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and \ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}, i = 36,37,38,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + \ (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$$

$$\frac{1}{(\hat{M}_{36})^{(7)}}[\ (b_i)^{(7)} + (b_i^{'})^{(7)} + \ (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)} \ (\hat{k}_{36})^{(7)}] < 1$$

Where we suppose

$(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i, j = 40,41,42$ 136

The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded

**Definition of** $(p_i)^{(8)}, \ (r_i)^{(8)}$: 137

$(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ 138

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| $(b_i^{''})^{(8)}((G_{43}),t) \leq \ (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2\to\infty}(a_i^{''})^{(8)}(T_{41},t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G\to\infty}(b_i^{''})^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br><br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'},t) - (a_i^{''})^{(8)}(T_{41},t)\rvert \leq (\hat{k}_{40})^{(8)}\lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'},t) - (b_i^{''})^{(8)}\left((G_{43}),t\right)\rvert < (\hat{k}_{40})^{(8)}\lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)}t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'},t)$ and $(a_i^{''})^{(8)}(T_{41},t)$ . $(T_{41}^{'},t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)},$ are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br><br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}, \ (r_i)^{(8)}, i = 40,41,42,$<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i^{'})^{(8)} + \ (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)}\,(\hat{k}_{40})^{(8)}] < 1$ | 145 |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[\,(b_i)^{(8)} + (b_i^{'})^{(8)} + \ (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i^{'})^{(9)}, (a_i^{''})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (b_i^{''})^{(9)} > 0, \quad i,j = 44,45,46$ | 146 A |

The functions $(a_i^{''})^{(9)}, (b_i^{''})^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:

$$(a_i^{''})^{(9)}(T_{45}, t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$$

$$(b_i^{''})^{(9)}(G_{47}, t) \leq (r_i)^{(9)} \leq (b_i^{'})^{(9)} \leq (\hat{B}_{44})^{(9)}$$

$$\lim_{T_2 \to \infty} (a_i^{''})^{(9)}(T_{45}, t) = (p_i)^{(9)}$$

$$\lim_{G \to \infty} (b_i^{''})^{(9)}(G_{47}, t) = (r_i)^{(9)}$$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i^{''})^{(9)}(T_{45}^{'}, t) - (a_i^{''})^{(9)}(T_{45}, t)| \leq (\hat{k}_{44})^{(9)} |T_{45} - T_{45}^{'}| e^{-(\hat{M}_{44})^{(9)} t}$$

$$|(b_i^{''})^{(9)}((G_{47})^{'}, t) - (b_i^{''})^{(9)}((G_{47}), t)| < (\hat{k}_{44})^{(9)} ||(G_{47}) - (G_{47})^{'}|| e^{-(\hat{M}_{44})^{(9)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(9)}(T_{45}^{'}, t)$ and $(a_i^{''})^{(9)}(T_{45}, t)$ . $(T_{45}^{'}, t)$ and $(T_{45}, t)$ are points belonging to the interval $\left[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\right]$. It is to be noted that $(a_i^{''})^{(9)}(T_{45}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i^{''})^{(9)}(T_{45}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}} \ , \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i^{'})^{(9)}, (b_i)^{(9)}, (b_i^{'})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}} [(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}} [(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)} (\hat{k}_{44})^{(9)}] < 1$$

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| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 152 |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ : | 153 |

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| $G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \le (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i$: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{13})^{(1)}$ , $T_i^0 \le (\hat{Q}_{13})^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| Where $s_{(13)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{16})^{(2)}$ , $T_i^0 \leq (\hat{Q}_{16})^{(2)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\bar{M}_{16})^{(2)} t}$ | |
| By<br><br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[ (a_{16})^{(2)} G_{17}(s_{(16)}) - \left( (a_{16}')^{(2)} + a_{16}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{16}(s_{(16)}) \right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[ (a_{17})^{(2)} G_{16}(s_{(16)}) - \left( (a_{17}')^{(2)} + (a_{17}'')^{(2)} (T_{17}(s_{(16)}), s_{(17)}) \right) G_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[ (a_{18})^{(2)} G_{17}(s_{(16)}) - \left( (a_{18}')^{(2)} + (a_{18}'')^{(2)} (T_{17}(s_{(16)}), s_{(16)}) \right) G_{18}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[ (b_{16})^{(2)} T_{17}(s_{(16)}) - \left( (b_{16}')^{(2)} - (b_{16}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{16}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[ (b_{17})^{(2)} T_{16}(s_{(16)}) - \left( (b_{17}')^{(2)} - (b_{17}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{17}(s_{(16)}) \right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[ (b_{18})^{(2)} T_{17}(s_{(16)}) - \left( (b_{18}')^{(2)} - (b_{18}'')^{(2)} (G_{19}(s_{(16)}), s_{(16)}) \right) T_{18}(s_{(16)}) \right] ds_{(16)}$<br><br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0,t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{20})^{(3)}$ , $T_i^0 \leq (\hat{Q}_{20})^{(3)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\bar{M}_{20})^{(3)} t}$ | |
| By<br><br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[ (a_{20})^{(3)} G_{21}(s_{(20)}) - \left( (a_{20}')^{(3)} + a_{20}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{20}(s_{(20)}) \right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[ (a_{21})^{(3)} G_{20}(s_{(20)}) - \left( (a_{21}')^{(3)} + (a_{21}'')^{(3)} (T_{21}(s_{(20)}), s_{(20)}) \right) G_{21}(s_{(20)}) \right] ds_{(20)}$ | |

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| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a_{22}')^{(3)} + (a_{22}'')^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b_{20}')^{(3)} - (b_{20}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b_{21}')^{(3)} - (b_{21}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{22}(t) = T_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b_{22}')^{(3)} - (b_{22}'')^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{24})^{(4)}$, $T_i^0 \le (\hat{Q}_{24})^{(4)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a_{24}')^{(4)} + a_{24}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a_{25}')^{(4)} + (a_{25}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a_{26}')^{(4)} + (a_{26}'')^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b_{24}')^{(4)} - (b_{24}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b_{25}')^{(4)} - (b_{25}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b_{26}')^{(4)} - (b_{26}'')^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{28})^{(5)}$, $T_i^0 \le (\hat{Q}_{28})^{(5)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)} t}$ | |
| By | 163 |

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| $\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a_{28}')^{(5)} + a_{28}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a_{29}')^{(5)} + (a_{29}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a_{30}')^{(5)} + (a_{30}'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b_{28}')^{(5)} - (b_{28}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b_{29}')^{(5)} - (b_{29}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b_{30}')^{(5)} - (b_{30}'')^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{32})^{(6)}$, $T_i^0 \le (\hat{Q}_{32})^{(6)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a_{32}')^{(6)} + a_{32}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a_{33}')^{(6)} + (a_{33}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a_{34}')^{(6)} + (a_{34}'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b_{32}')^{(6)} - (b_{32}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b_{33}')^{(6)} - (b_{33}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{36})^{(7)}$ , $T_i^0 \leq (\hat{Q}_{36})^{(7)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b'_{37})^{(7)} - (b''_{37})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$ | |
| $\bar{T}_{38}(t) = T_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{38})^{(7)} - (b''_{38})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$<br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} G_{41}(s_{(40)}) - \left( (a'_{40})^{(8)} + a''_{40})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{40}(s_{(40)}) \right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[ (a_{41})^{(8)} G_{40}(s_{(40)}) - \left( (a'_{41})^{(8)} + (a''_{41})^{(8)} (T_{41}(s_{(40)}), s_{(40)}) \right) G_{41}(s_{(40)}) \right] ds_{(40)}$ | |

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| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that $G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\bar{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$<br>$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\bar{M}_{13})^{(1)}} \left(e^{(\bar{M}_{13})^{(1)} t} - 1\right)$ | 167 |
| From which it follows that | 168 |

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| $(G_{13}(t)-G_{13}^0)e^{-(\hat{M}_{13})^{(1)}t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{P}_{13})^{(1)}+G_{14}^0)e^{\left(-\frac{(\hat{P}_{13})^{(1)}+G_{14}^0}{G_{14}^0}\right)}+(\hat{P}_{13})^{(1)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 1 | |
| Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$ | |
| The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0+(\hat{P}_{16})^{(6)}e^{(\hat{M}_{16})^{(2)}s_{(16)}}\right)\right] ds_{(16)} =$<br>$\left(1+(a_{16})^{(2)}t\right)G_{17}^0 + \frac{(a_{16})^{(2)}(\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)}t}-1\right)$ | 169 |
| From which it follows that<br>$(G_{16}(t)-G_{16}^0)e^{-(\hat{M}_{16})^{(2)}t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{P}_{16})^{(2)}+G_{17}^0)e^{\left(-\frac{(\hat{P}_{16})^{(2)}+G_{17}^0}{G_{17}^0}\right)}+(\hat{P}_{16})^{(2)}\right]$ | 170 |
| Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$ | |
| The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0+(\hat{P}_{20})^{(3)}e^{(\hat{M}_{20})^{(3)}s_{(20)}}\right)\right] ds_{(20)} =$<br>$\left(1+(a_{20})^{(3)}t\right)G_{21}^0 + \frac{(a_{20})^{(3)}(\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)}t}-1\right)$ | 171 |
| From which it follows that<br>$(G_{20}(t)-G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[((\hat{P}_{20})^{(3)}+G_{21}^0)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)}+(\hat{P}_{20})^{(3)}\right]$ | 172 |
| Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$ | |
| The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that<br>$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0+(\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$<br>$\left(1+(a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t}-1\right)$ | 173 |
| From which it follows that<br>$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 4 | 174 |
| The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious | |

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| that<br>$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} \left(G_{29}^0 + (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} s_{(28)}}\right)\right] ds_{(28)} =$<br>$\left(1 + (a_{28})^{(5)} t\right) G_{29}^0 + \frac{(a_{28})^{(5)} (\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}} \left(e^{(\hat{M}_{28})^{(5)} t} - 1\right)$ | |
| From which it follows that<br>$(G_{28}(t) - G_{28}^0) e^{-(\hat{M}_{28})^{(5)} t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}} \left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right) e^{\left(-\frac{(\hat{P}_{28})^{(5)} + G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 5 | 175 |
| The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} \left(G_{33}^0 + (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} s_{(32)}}\right)\right] ds_{(32)} =$<br>$\left(1 + (a_{32})^{(6)} t\right) G_{33}^0 + \frac{(a_{32})^{(6)} (\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left(e^{(\hat{M}_{32})^{(6)} t} - 1\right)$ | 176 |
| From which it follows that<br>$(G_{32}(t) - G_{32}^0) e^{-(\hat{M}_{32})^{(6)} t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}} \left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right) e^{\left(-\frac{(\hat{P}_{32})^{(6)} + G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 6<br>Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$ | 177 |
| (k) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that<br>$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} \left(G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}}\right)\right] ds_{(36)} =$<br>$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left(e^{(\hat{M}_{36})^{(7)} t} - 1\right)$ | 178 |
| From which it follows that<br>$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[\left((\hat{P}_{36})^{(7)} + G_{37}^0\right) e^{\left(-\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0}\right)} + (\hat{P}_{36})^{(7)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$ | 180 |

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| $\left(1+(a_{40})^{(8)}t\right)G_{41}^{0}+\frac{(a_{40})^{(8)}(\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left(e^{(\hat{M}_{40})^{(8)}t}-1\right)$ | |
| From which it follows that<br>$(G_{40}(t)-G_{40}^{0})e^{-(\hat{M}_{40})^{(8)}t}\leq\frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}}\left[((\hat{P}_{40})^{(8)}+G_{41}^{0})e^{\left(-\frac{(\hat{P}_{40})^{(8)}+G_{41}^{0}}{G_{41}^{0}}\right)}+(\hat{P}_{40})^{(8)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 8<br>Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that<br>$G_{44}(t)\leq G_{44}^{0}+\int_0^t\left[(a_{44})^{(9)}\left(G_{45}^{0}+(\hat{P}_{44})^{(9)}e^{(\hat{M}_{44})^{(9)}s_{(44)}}\right)\right]ds_{(44)}=$<br>$\left(1+(a_{44})^{(9)}t\right)G_{45}^{0}+\frac{(a_{44})^{(9)}(\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left(e^{(\hat{M}_{44})^{(9)}t}-1\right)$ | |
| From which it follows that<br>$(G_{44}(t)-G_{44}^{0})e^{-(\hat{M}_{44})^{(9)}t}\leq\frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}}\left[((\hat{P}_{44})^{(9)}+G_{45}^{0})e^{\left(-\frac{(\hat{P}_{44})^{(9)}+G_{45}^{0}}{G_{45}^{0}}\right)}+(\hat{P}_{44})^{(9)}\right]$<br>$(G_i^0)$ is as defined in the statement of theorem 9<br>Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}<1$ and to choose<br>$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+((\hat{P}_{13})^{(1)}+G_j^0)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[((\hat{Q}_{13})^{(1)}+T_j^0)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric<br>$d\left((G^{(1)},T^{(1)}),(G^{(2)},T^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\hat{M}_{13})^{(1)}t}\}$ | 185 |

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_{i}^{(2)}\right|\leq\int_{0}^{t}(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_{0}^{t}\{(a_{13}^{'})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}\left|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$
$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\,G^{(2)},T^{(2)}\right)\right)$$

(186)

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}\ and\ (\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_{+}$.

If instead of proving the existence of the solution on $\mathbb{R}_{+}$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)},i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0\ and\ T_i\,(t)=0$

From 19 to 24 it results

$$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\geq 0$$

$$T_i\,(t)\geq T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0\quad\text{for t}>0$$

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2\ and\ \left((\widehat{M}_{13})^{(1)}\right)_3$ : (187)

**Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}\ and\ G_{15}$ . indeed if

$G_{13}<(\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt}\leq\left((\widehat{M}_{13})^{(1)}\right)_1-(a_{14}^{'})^{(1)}G_{14}$ and by integrating

$$G_{14}\leq\left((\widehat{M}_{13})^{(1)}\right)_2=G_{14}^0+2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}^{'})^{(1)}$$

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| In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\widehat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |
| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |

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|---|---|
| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{19}}, \widetilde{T_{19}}$ : $\left(\widetilde{G_{19}}, \widetilde{T_{19}}\right) = \mathcal{A}^{(2)}(G_{19}, T_{19})$ | 195 |
| It results<br><br>$\left\|\tilde{G}_{16}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{16})^{(2)} \left\|G_{17}^{(1)} - G_{17}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}}\, ds_{(16)} +$<br><br>$\int_0^t \{(a_{16}^{'})^{(2)} \left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right)\left\|G_{16}^{(1)} - G_{16}^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} +$<br><br>$G_{16}^{(2)} \left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)}, s_{(16)}\right) - (a_{16}^{''})^{(2)}\left(T_{17}^{(2)}, s_{(16)}\right)\right\| \; e^{-(\widehat{M}_{16})^{(2)} s_{(16)}} e^{(\widehat{M}_{16})^{(2)} s_{(16)}} \} ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)} - (G_{19})^{(2)}\right\| e^{-(\widehat{M}_{16})^{(2)} t} \leq$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)} + (a_{16}^{'})^{(2)} + (\widehat{A}_{16})^{(2)} + (\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right) d\left(((G_{19})^{(1)}, (T_{19})^{(1)}; (G_{19})^{(2)}, (T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ and $(\widehat{Q}_{16})^{(2)} e^{(\widehat{M}_{16})^{(2)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i = 16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i$ (t) $= 0$ and $T_i$ (t) $= 0$<br><br>it results<br><br>$G_i$ (t) $\geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(2)} - (a_i^{''})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\} ds_{(16)}\right]} \geq 0$<br><br>$T_i$ (t) $\geq T_i^0 e^{(-(b_i^{'})^{(2)} t)} > 0$ for t $> 0$ | 200 |
| **Definition of** $\left((\widehat{M}_{16})^{(2)}\right)_1$, $\left((\widehat{M}_{16})^{(2)}\right)_2$ and $\left((\widehat{M}_{16})^{(2)}\right)_3$ :<br><br>**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if<br><br>$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a_{17}^{'})^{(2)} G_{17}$ and by integrating<br><br>$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a_{17}^{'})^{(2)}$ | 201 |

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|---|---|
| In the same way , one can obtain<br>$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$<br>If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively. | |
| **Remark 9**: If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. | 202 |
| **Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b_{17}')^{(2)}$ then $T_{17} \to \infty$.<br>**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :<br>Indeed let $t_2$ be so that for $t > t_2$<br>$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$ | 203 |
| Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to<br>$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results | 204 |
| $T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded.<br>The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(t),t\right) = (b_{18}')^{(2)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 205 |
| It is now sufficient to take $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}} < 1$ and to choose<br>$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have | 207 |
| $\frac{(a_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[(\widehat{P}_{20})^{(3)} + \left((\widehat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{20})^{(3)}$ | 208 |
| $\frac{(b_i)^{(3)}}{(\widehat{M}_{20})^{(3)}}\left[\left((\widehat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{20})^{(3)}\right] \leq (\widehat{Q}_{20})^{(3)}$ | 209 |
| In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 210 |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |

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|---|---|
| Indeed if we denote<br><br>**Definition of** $\widetilde{G_{23}}, \widetilde{T_{23}} :\left( \widetilde{(G_{23})}, \widetilde{(T_{23})} \right) = \mathcal{A}^{(3)}\left((G_{23}), (T_{23})\right)$ | 212 |
| It results<br><br>$\left\|\tilde{G}_{20}^{(1)} - \tilde{G}_{i}^{(2)}\right\| \leq \int_0^t (a_{20})^{(3)} \left\|G_{21}^{(1)} - G_{21}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}}\, ds_{(20)} +$<br><br>$\int_0^t \{(a_{20}^{'})^{(3)} \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) \left\|G_{20}^{(1)} - G_{20}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} +$<br><br>$G_{20}^{(2)} \left\|(a_{20}^{''})^{(3)}\left(T_{21}^{(1)}, s_{(20)}\right) - (a_{20}^{''})^{(3)}\left(T_{21}^{(2)}, s_{(20)}\right)\right\| \; e^{-(\widehat{M}_{20})^{(3)} s_{(20)}} e^{(\widehat{M}_{20})^{(3)} s_{(20)}} \} ds_{(20)}$<br><br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0, t]$<br><br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)} - G_{23}{}^{(2)}\right\| e^{-(\widehat{M}_{20})^{(3)} t} \leq$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)} + (a_{20}^{'})^{(3)} + (\widehat{A}_{20})^{(3)} + (\widehat{P}_{20})^{(3)} (\widehat{k}_{20})^{(3)}\right) d\left(\left((G_{23})^{(1)}, (T_{23})^{(1)}; (G_{23})^{(2)}, (T_{23})^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i \; and \; T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t} \; and \; (\widehat{Q}_{20})^{(3)} e^{(\widehat{M}_{20})^{(3)} t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)}, i = 20,21,22$ depend only on $T_{21}$ and respectively on $(G_{23}) (and \; not \; on \; t)$ and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i\,(t) = 0 \; and \; T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(3)} - (a_i^{''})^{(3)} (T_{21}(s_{(20)}), s_{(20)})\} ds_{(20)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)} t\right)} > 0 \;$ for t > 0 | 216 |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2 \; and \; \left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21} \; and \; G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left((\widehat{M}_{20})^{(3)}\right)_1 / (a_{21}^{'})^{(3)}$ | 217 |

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|---|---|
| In the same way , one can obtain<br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}')^{(3)}$<br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)}\left((G_{23})(t),t\right) = (b_{21}')^{(3)}$ then $T_{21} \to \infty$.<br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br>Indeed let $t_3$ be so that for $t > t_3$<br>$(b_{21})^{(3)} - (b_i'')^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}'')^{(3)}\left((G_{23})(t),t\right) = (b_{22}')^{(3)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}} < 1$ and to choose<br>$(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[(\hat{P}_{24})^{(4)} + \left((\hat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\bar{M}_{24})^{(4)}}\left[\left((\hat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{24})^{(4)}\right] \leq (\hat{Q}_{24})^{(4)}$ | 223 |
| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}, \max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\bar{M}_{24})^{(4)}t}\}$ | 225 |

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left(\widetilde{(G_{27})}, \widetilde{(T_{27})}\right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_{i}^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\, ds_{(24)} +$$

$$\int_0^t \{(a_{24}^{'})^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right)\left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} \left|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}^{''})^{(4)}\left(T_{25}^{(2)}, s_{(24)}\right)\right|\ e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}}\} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$
$$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)} + (a_{24}^{'})^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d\left(((G_{27})^{(1)}, (T_{27})^{(1)};\ (G_{27})^{(2)}, (T_{27})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

(226)

**Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i = 24{,}25{,}26$ depend only on $T_{25}$ and respectively on $(G_{27})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

(227)

**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(4)} - (a_i^{''})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(4)} t\right)} > 0 \quad \text{for t} > 0$$

(228)

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ :

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}^{'})^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}^{'})^{(4)}$$

(229)

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| In the same way , one can obtain<br>$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a_{26}')^{(4)}$<br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br>Indeed let $t_4$ be so that for $t > t_4$<br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded.<br>The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}\left((G_{27})(t),t\right) = (b_{26}')^{(4)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br>Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + \left((\widehat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |
| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\widehat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{28})^{(5)}\right] \leq (\widehat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right) =$ | 236 |

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| $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}\left((G_{31}),(T_{31})\right)$<br><br>It results<br><br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$<br><br>$\int_0^t\{(a_{28}^{'})^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br><br>$G_{28}^{(2)}\left\|(a_{28}^{''})^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}^{''})^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br><br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses on it follows | |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\le$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}^{'})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}^{''})^{(5)}$ and $(b_{28}^{''})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(5)}$ and $(b_i^{''})^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br><br>it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(5)}-(a_i^{''})^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(5)}t\right)}>0$ for t > 0 | 239 |
| **Definition of** $\left((\widehat{M}_{28})^{(5)}\right)_1,\left((\widehat{M}_{28})^{(5)}\right)_2$ *and* $\left((\widehat{M}_{28})^{(5)}\right)_3$ :<br><br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if | 240 |

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| $G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq \left((\widehat{M}_{28})^{(5)}\right)_1 - (a'_{29})^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a'_{29})^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a'_{30})^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b''_{30})^{(5)}\left((G_{31})(t),t\right) = (b'_{30})^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |
| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | |

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| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a_{32}^{'})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a_{32}^{''})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a_{32}^{''})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\le$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a_{32}^{'})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a_{32}^{''})^{(6)}$ and $(b_{32}^{''})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(6)}$ and $(b_i^{''})^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on $(G_{35})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 249 |
| **Remark 27:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(6)}-(a_i^{''})^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(6)}t\right)}>0$ for t > 0 | 250 |

**Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a'_{33})^{(6)} G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(6)}((G_{35})(t),t)) = (b_{33}^{'})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i^{''})^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right)$, $t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.
The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}^{''})^{(6)}\left((G_{35})(t),t(t),t\right) = (b_{34}^{'})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}, G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

$$\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\widehat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{36})^{(7)} \quad (256)$$

$$\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\widehat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{36})^{(7)}\right] \leq (\widehat{Q}_{36})^{(7)} \quad (257)$$

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| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right) =$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\,ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\le$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i=36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 260 |
| **Remark 32:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$<br><br>it results<br><br>$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(7)}-(a_i^{''})^{(7)}(T_{37}(s_{(36)}),s_{(36)})\}ds_{(36)}\right]}\ge 0$ | 261 |

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| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(7)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{36})^{(7)})_1, ((\widehat{M}_{36})^{(7)})_2$ *and* $((\widehat{M}_{36})^{(7)})_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq ((\widehat{M}_{36})^{(7)})_1 - (a_{37}')^{(7)}G_{37}$ and by integrating<br><br>$G_{37} \leq ((\widehat{M}_{36})^{(7)})_2 = G_{37}^0 + 2(a_{37})^{(7)}((\widehat{M}_{36})^{(7)})_1/(a_{37}')^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq ((\widehat{M}_{36})^{(7)})_3 = G_{38}^0 + 2(a_{38})^{(7)}((\widehat{M}_{36})^{(7)})_2/(a_{38}')^{(7)}$<br><br>If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b_{37}')^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right)$, $t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b_{38}'')^{(7)}((G_{39})(t),t) = (b_{38}')^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |
| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose<br>$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + ((\widehat{P}_{40})^{(8)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{40})^{(8)}$ | 267 |
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| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{40})^{(8)}\right]\leq(\hat{Q}_{40})^{(8)}$ | |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)},(T_{43})^{(1)}),((G_{43})^{(2)},(T_{43})^{(2)})\right)=$ $\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{43})},\widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})},\widetilde{(T_{43})}\right)=\mathcal{A}^{(8)}((G_{43}),(T_{43}))$ | 270 |
| It results<br><br>$\left\|\tilde{G}_{40}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{40})^{(8)}\left\|G_{41}^{(1)}-G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\,ds_{(40)}+$ $\int_0^t\{(a_{40}')^{(8)}\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)\left\|G_{40}^{(1)}-G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}+$ $G_{40}^{(2)}\left\|(a_{40}'')^{(8)}\left(T_{41}^{(1)},s_{(40)}\right)-(a_{40}'')^{(8)}\left(T_{41}^{(2)},s_{(40)}\right)\right\|\ e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0,t]$<br><br>From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)}-(G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t}\leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)}+(a_{40}')^{(8)}+(\widehat{A}_{40})^{(8)}+(\widehat{P}_{40})^{(8)}(\widehat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)},(T_{43})^{(1)};\ (G_{43})^{(2)},(T_{43})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a_{40}'')^{(8)}$ and $(b_{40}'')^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(8)}$ and $(b_i'')^{(8)}, i=40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})$*(and not on* $t$*)* and hypothesis can replaced by a usual Lipschitz condition. | 274 |
| **Remark 37** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$ | 275 |

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| it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(8)} - (a_i'')^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{\left(-(b_i')^{(8)} t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}')^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}')^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)} \left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}')^{(8)}$<br><br>If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(8)}((G_{43})(t), t)) = (b_{41}')^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i'')^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right)$, $t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded.<br>The same property holds for $T_{42}$ if $\lim_{t\to\infty} (b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$ | 279 |
| It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have | 279 A |

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)}+\left((\hat{P}_{44})^{(9)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{44})^{(9)}\right]\leq(\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)},(T_{47})^{(1)}),\left((G_{47})^{(2)},(T_{47})^{(2)}\right)\right)=$$
$$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})},\widetilde{(T_{47})}:\left(\widetilde{(G_{47})},\widetilde{(T_{47})}\right)=\mathcal{A}^{(9)}\left((G_{47}),(T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{44})^{(9)}\left|G_{45}^{(1)}-G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)}+$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)\left|G_{44}^{(1)}-G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}+$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)},s_{(44)}\right)-(a_{44}^{''})^{(9)}\left(T_{45}^{(2)},s_{(44)}\right)\right|\;e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval [0, t]

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t}\leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)}+(a_{44}^{'})^{(9)}+(\widehat{A}_{44})^{(9)}+(\widehat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)},(T_{47})^{(1)};\;(G_{47})^{(2)},(T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i\ and\ T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}\ and\ (\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i=44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\ (t)=0\ and\ T_i\ (t)=0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(9)} - (a_i'')^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i')^{(9)} t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2$ *and* $\left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}')^{(9)} G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_1 / (a_{45}')^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)} \left((\widehat{M}_{44})^{(9)}\right)_2 / (a_{46}')^{(9)}$$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(9)} ((G_{47})(t), t)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i'')^{(9)}((G_{47})(t), t) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right)$, $t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty} (b_{46}'')^{(9)} ((G_{47})(t), t) = (b_{46}')^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14}, t) + (a_{14}'')^{(1)}(T_{14}, t) \leq -(\sigma_1)^{(1)}$

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| $-(\tau_2)^{(1)} \le -(b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G,t) - (b_{14}^{''})^{(1)}(G,t) \le -(\tau_1)^{(1)}$ | |
| **Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ :<br><br>By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 281 |
| **Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ :<br><br>By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$ | 282 |
| **Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :-<br><br>If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by<br>$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,<br><br>and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$<br><br>$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$ | 283 |
| and analogously<br><br>$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,<br><br>and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$<br><br>$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$<br><br>are defined | 284 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G_{13}^0 e^{(S_1)^{(1)}t}$<br><br>where $(p_i)^{(1)}$ is defined by equation<br><br>$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | 285 |
| $\left(\frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})}\left[e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le\right.$<br><br>$\left.\frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a_{15}^{'})^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}^{'})^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}^{'})^{(1)}t}\right)$ | 286 |

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| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(\mathrm{T}_{17}, t) + (a_{17}'')^{(2)}(\mathrm{T}_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}\big((G_{19}), t\big) - (b_{17}'')^{(2)}\big((G_{19}), t\big) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the | 300 |
| roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |

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| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$, and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$<br>$\left. \frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$<br>$\frac{(a_{18})^{(2)} T_{16}^0}{(\mu_2)^{(2)}((R_1)^{(2)} + (r_{16})^{(2)} + (R_2)^{(2)})}\left[e^{((R_1)^{(2)} + (r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^0 e^{-(R_2)^{(2)}t}$ | 315 |
| **Definition of** $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a_{16}')^{(2)}$ | 317 |

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| $(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$ | |
| $(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)} = (b_{18}^{'})^{(2)} - (r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)} \le -(a_{20}^{'})^{(3)} + (a_{21}^{'})^{(3)} - (a_{20}^{''})^{(3)}(T_{21}, t) + (a_{21}^{''})^{(3)}(T_{21}, t) \le -(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)} \le -(b_{20}^{'})^{(3)} + (b_{21}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t) - (b_{21}^{''})^{(3)}((G_{23}), t) \le -(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ :<br>By $(\nu_1)^{(3)} > 0$, $(\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0$, $(u_2)^{(3)} < 0$ the roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and<br>By $(\bar{\nu}_1)^{(3)} > 0$, $(\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0$, $(\bar{u}_2)^{(3)} < 0$ the<br>roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$<br>and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ | 320 |
| **Definition of** $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by<br>$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$,<br>and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$<br>$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}$, ***if*** $(\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$ | 321 |
| and analogously<br>$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}$, ***if*** $(u_0)^{(3)} < (u_1)^{(3)}$<br>$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}$, ***if*** $(u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)}$, and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$ | 322 |

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| $(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},$ ***if*** $(\bar{u}_1)^{(3)} < (u_0)^{(3)}$<br>Then the solution of global equations satisfies the inequalities<br>$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$<br>$(p_i)^{(3)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ | 323 |
| $\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$<br>$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a'_{22})^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a'_{22})^{(3)} t}\right] + G_{22}^0 e^{-(a'_{22})^{(3)} t}\right)$ | 324 |
| $\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ | 325 |
| $\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ | 326 |
| $\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b'_{22})^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b'_{22})^{(3)} t}\right] + T_{22}^0 e^{-(b'_{22})^{(3)} t} \le T_{22}(t) \le$<br>$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ | 327 |
| **Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:-<br>Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a'_{20})^{(3)}$<br>$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$<br>$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b'_{20})^{(3)}$<br>$(R_2)^{(3)} = (b'_{22})^{(3)} - (r_{22})^{(3)}$ | 328 |
| **Behavior of the solutions of equation**<br>**Theorem:** If we denote and define<br>**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :<br>$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying<br>$-(\sigma_2)^{(4)} \le -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$<br>$-(\tau_2)^{(4)} \le -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}((G_{27}), t) - (b''_{25})^{(4)}((G_{27}), t) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)}, (v_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, v^{(4)}, u^{(4)}$ :<br>By $(v_1)^{(4)} > 0, (v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations | 329 |

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| $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\left(v^{(4)}\right)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$ and $(b_{25})^{(4)}\left(u^{(4)}\right)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(v_0)^{(4)}$ :-<br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br>$(m_2)^{(4)} = (v_0)^{(4)}, (m_1)^{(4)} = (v_1)^{(4)}$, ***if*** $(v_0)^{(4)} < (v_1)^{(4)}$<br>$(m_2)^{(4)} = (v_1)^{(4)}, (m_1)^{(4)} = (\bar{v}_1)^{(4)}$ , ***if*** $(v_4)^{(4)} < (v_0)^{(4)} < (\bar{v}_1)^{(4)}$,<br>and $\boxed{(v_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br>$(m_2)^{(4)} = (v_4)^{(4)}, (m_1)^{(4)} = (v_0)^{(4)}$, ***if*** $(\bar{v}_4)^{(4)} < (v_0)^{(4)}$ | 330 |
| and analogously<br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |
| $\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)} + (r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)} - (b_{26}^{'})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}^{'})^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}^{'})^{(4)}t} \le T_{26}(t) \le$ | 336 |

$\frac{(a_{26})^{(4)}T_{24}^{0}}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t}-e^{-(R_2)^{(4)}t}\right]+T_{26}^{0}e^{-(R_2)^{(4)}t}$

**Definition of** $(S_1)^{(4)},(S_2)^{(4)},(R_1)^{(4)},(R_2)^{(4)}$:- 337

Where $(S_1)^{(4)}=(a_{24})^{(4)}(m_2)^{(4)}-(a_{24}^{'})^{(4)}$

$(S_2)^{(4)}=(a_{26})^{(4)}-(p_{26})^{(4)}$

$(R_1)^{(4)}=(b_{24})^{(4)}(\mu_2)^{(4)}-(b_{24}^{'})^{(4)}$

$(R_2)^{(4)}=(b_{26}^{'})^{(4)}-(r_{26})^{(4)}$

**Behavior of the solutions of equation** 338

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)},(\sigma_2)^{(5)},(\tau_1)^{(5)},(\tau_2)^{(5)}$ four constants satisfying

$-(\sigma_2)^{(5)}\le-(a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}-(a_{28}^{''})^{(5)}(T_{29},t)+(a_{29}^{''})^{(5)}(T_{29},t)\le-(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)}\le-(b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(b_{28}^{''})^{(5)}((G_{31}),t)-(b_{29}^{''})^{(5)}((G_{31}),t)\le-(\tau_1)^{(5)}$

**Definition of** $(v_1)^{(5)},(v_2)^{(5)},(u_1)^{(5)},(u_2)^{(5)},v^{(5)},u^{(5)}$ : 339

By $(v_1)^{(5)}>0$ , $(v_2)^{(5)}<0$ and respectively $(u_1)^{(5)}>0$ , $(u_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_1)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_1)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$ and

**Definition of** $(\bar{v}_1)^{(5)},,(\bar{v}_2)^{(5)},(\bar{u}_1)^{(5)},(\bar{u}_2)^{(5)}$ : 340

By $(\bar{v}_1)^{(5)}>0$ , $(\bar{v}_2)^{(5)}<0$ and respectively $(\bar{u}_1)^{(5)}>0$ , $(\bar{u}_2)^{(5)}<0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2+(\sigma_2)^{(5)}v^{(5)}-(a_{28})^{(5)}=0$
and $(b_{29})^{(5)}\left(u^{(5)}\right)^2+(\tau_2)^{(5)}u^{(5)}-(b_{28})^{(5)}=0$

**Definition of** $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)},(v_0)^{(5)}$ :-

If we define $(m_1)^{(5)},(m_2)^{(5)},(\mu_1)^{(5)},(\mu_2)^{(5)}$ by

$(m_2)^{(5)}=(v_0)^{(5)},(m_1)^{(5)}=(v_1)^{(5)}$, ***if*** $(v_0)^{(5)}<(v_1)^{(5)}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(\bar{v}_1)^{(5)}$ , ***if*** $(v_1)^{(5)}<(v_0)^{(5)}<(\bar{v}_1)^{(5)}$,
and $\boxed{(v_0)^{(5)}=\frac{G_{28}^{0}}{G_{29}^{0}}}$

$(m_2)^{(5)}=(v_1)^{(5)},(m_1)^{(5)}=(v_0)^{(5)}$, ***if*** $(\bar{v}_1)^{(5)}<(v_0)^{(5)}$

and analogously 341

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| $(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)}$, $\boldsymbol{if}\ (u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$, $\boldsymbol{if}\ (u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)}, \boldsymbol{if}\ (\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{28}(t) \le G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \le G_{29}(t) \le \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \le G_{30}(t) \le$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \le T_{28}(t) \le \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b'_{30})^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b'_{30})^{(5)} t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)} t} \le T_{30}(t) \le$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$<br>$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define<br>**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :<br>$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying | 349 |

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| $-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$<br><br>$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \le -(\tau_1)^{(6)}$ | |
| **Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ :<br><br>By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and | 350 |
| **Definition of** $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ :<br><br>By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$<br>and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$<br>**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-<br><br>If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by<br><br>$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,<br>and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$<br><br>$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$ | 351 |
| and analogously<br><br>$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,<br>and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$<br><br>$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$ | 352 |
| Then the solution of global equations satisfies the inequalities<br><br>$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$<br><br>where $(p_i)^{(6)}$ is defined by equation | 353 |
| $\frac{1}{(m_1)^{(6)}} G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G^0_{32}e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)}G^0_{32}}{(m_1)^{(6)}((S_1)^{(6)} - (p_{32})^{(6)} - (S_2)^{(6)})}\left[e^{((S_1)^{(6)} - (p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G^0_{34}e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$<br>*(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |

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|---|---|
| $T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}}T_{32}^{0}e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}}T_{32}^{0}e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)}T_{32}^{0}}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t}-e^{-(b_{34}^{'})^{(6)}t}\right]+T_{34}^{0}e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)}T_{32}^{0}}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t}-e^{-(R_2)^{(6)}t}\right]+T_{34}^{0}e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)},(S_2)^{(6)},(R_1)^{(6)},(R_2)^{(6)}$:- Where $(S_1)^{(6)}=(a_{32})^{(6)}(m_2)^{(6)}-(a_{32}^{'})^{(6)}$ $(S_2)^{(6)}=(a_{34})^{(6)}-(p_{34})^{(6)}$ $(R_1)^{(6)}=(b_{32})^{(6)}(\mu_2)^{(6)}-(b_{32}^{'})^{(6)}$ $(R_2)^{(6)}=(b_{34}^{'})^{(6)}-(r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$: $(\sigma_1)^{(7)},(\sigma_2)^{(7)},(\tau_1)^{(7)},(\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}-(a_{36}^{''})^{(7)}(T_{37},t)+(a_{37}^{''})^{(7)}(T_{37},t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(b_{36}^{''})^{(7)}((G_{39}),t)-(b_{37}^{''})^{(7)}((G_{39}),t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)},(v_2)^{(7)},(u_1)^{(7)},(u_2)^{(7)},v^{(7)},u^{(7)}$: By $(v_1)^{(7)}>0,(v_2)^{(7)}<0$ and respectively $(u_1)^{(7)}>0,(u_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_1)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_1)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)},,(\bar{v}_2)^{(7)},(\bar{u}_1)^{(7)},(\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)}>0,(\bar{v}_2)^{(7)}<0$ and respectively $(\bar{u}_1)^{(7)}>0,(\bar{u}_2)^{(7)}<0$ the roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2+(\sigma_2)^{(7)}v^{(7)}-(a_{36})^{(7)}=0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2+(\tau_2)^{(7)}u^{(7)}-(b_{36})^{(7)}=0$ **Definition of** $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)},(v_0)^{(7)}$:- If we define $(m_1)^{(7)},(m_2)^{(7)},(\mu_1)^{(7)},(\mu_2)^{(7)}$ by | 362 |

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|---|---|
| $(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)},$ ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ , ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)},$<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br><br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)},$ ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br><br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)},$ ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$ , ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)},$<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br><br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)},$ ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{36}(t) \leq G_{36}^0 e^{(S_1)^{(7)} t}$<br><br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \leq G_{37}(t) \leq \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)} t}$ | 365 |
| $(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})} \left[ e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)} t} \right] + G_{38}^0 e^{-(S_2)^{(7)} t} \leq G_{38}(t) \leq$<br>$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a_{38}')^{(7)})} [e^{(S_1)^{(7)} t} - e^{-(a_{38}')^{(7)} t}] + G_{38}^0 e^{-(a_{38}')^{(7)} t})$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)} t} \leq T_{36}(t) \leq \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b_{38}')^{(7)})} \left[ e^{(R_1)^{(7)} t} - e^{-(b_{38}')^{(7)} t} \right] + T_{38}^0 e^{-(b_{38}')^{(7)} t} \leq T_{38}(t) \leq$<br><br>$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)} + (r_{36})^{(7)} + (R_2)^{(7)})} \left[ e^{((R_1)^{(7)} + (r_{36})^{(7)})t} - e^{-(R_2)^{(7)} t} \right] + T_{38}^0 e^{-(R_2)^{(7)} t}$ | 369 |
| **Definition of** $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:-<br><br>Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a_{36}')^{(7)}$ | 370 |

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

**Behavior of the solutions of equation** 371

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \le -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \le -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \le -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \le -(\tau_1)^{(8)}$

**Definition of** $(v_1)^{(8)}, (v_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, v^{(8)}, u^{(8)}$ : 372

By $(v_1)^{(8)} > 0, (v_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

**Definition of** $(\bar{v}_1)^{(8)}, , (\bar{v}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{v}_1)^{(8)} > 0, (\bar{v}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0, (\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\big(v^{(8)}\big)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\big(u^{(8)}\big)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

**Definition of** $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}, (v_0)^{(8)}$ :-

If we define $(m_1)^{(8)}, (m_2)^{(8)}, (\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (v_0)^{(8)}, (m_1)^{(8)} = (v_1)^{(8)}$, ***if*** $(v_0)^{(8)} < (v_1)^{(8)}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (\bar{v}_1)^{(8)}$, ***if*** $(v_1)^{(8)} < (v_0)^{(8)} < (\bar{v}_1)^{(8)}$,
and $\boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$

$(m_2)^{(8)} = (v_1)^{(8)}, (m_1)^{(8)} = (v_0)^{(8)}$, ***if*** $(\bar{v}_1)^{(8)} < (v_0)^{(8)}$

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| and analogously<br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},\ \boldsymbol{if}\ (u_0)^{(8)} < (u_1)^{(8)}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}, \boldsymbol{if}\ (u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)},$<br>and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}\ (\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{40}(t) \le G_{40}^0 e^{(S_1)^{(8)} t}$<br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \le G_{41}(t) \le \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \le G_{42}(t) \le \right.$<br>$\left.\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a'_{42})^{(8)})}\left[e^{(S_1)^{(8)} t} - e^{-(a'_{42})^{(8)} t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)} t}\right)$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \le T_{40}(t) \le \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b'_{42})^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b'_{42})^{(8)} t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)} t} \le T_{42}(t) \le$<br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$<br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$<br>$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$<br>$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$ | 381 |
| **Behavior of the solutions of equation 37 to 92**<br>**Theorem 2:** If we denote and define | 382 |

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$$

$$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$
and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,
and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,
and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{44}(t) \le G^0_{44} e^{(S_1)^{(9)} t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G^0_{44} e^{((S_1)^{(9)} - (p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G^0_{44} e^{(S_1)^{(9)} t}$$

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| ( $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})}\left[e^{((S_1)^{(9)}-(p_{44})^{(9)})t}-e^{-(S_2)^{(9)}t}\right]+G_{46}^{0}e^{-(S_2)^{(9)}t}\le G_{46}(t)\le$ $\frac{(a_{46})^{(9)}G_{44}^{0}}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})}[e^{(S_1)^{(9)}t}-e^{-(a'_{46})^{(9)}t}]+G_{46}^{0}e^{-(a'_{46})^{(9)}t})$ | |
| $\boxed{T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$ | |
| $\frac{1}{(\mu_1)^{(9)}}T_{44}^{0}e^{(R_1)^{(9)}t}\le T_{44}(t)\le\frac{1}{(\mu_2)^{(9)}}T_{44}^{0}e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$ | |
| $\frac{(b_{46})^{(9)}T_{44}^{0}}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})}\left[e^{(R_1)^{(9)}t}-e^{-(b'_{46})^{(9)}t}\right]+T_{46}^{0}e^{-(b'_{46})^{(9)}t}\le T_{46}(t)\le$ $\frac{(a_{46})^{(9)}T_{44}^{0}}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})}\left[e^{((R_1)^{(9)}+(r_{44})^{(9)})t}-e^{-(R_2)^{(9)}t}\right]+T_{46}^{0}e^{-(R_2)^{(9)}t}$ | |
| **Definition of** $(S_1)^{(9)},(S_2)^{(9)},(R_1)^{(9)},(R_2)^{(9)}$:- Where $(S_1)^{(9)}=(a_{44})^{(9)}(m_2)^{(9)}-(a'_{44})^{(9)}$ $(S_2)^{(9)}=(a_{46})^{(9)}-(p_{46})^{(9)}$ $(R_1)^{(9)}=(b_{44})^{(9)}(\mu_2)^{(9)}-(b'_{44})^{(9)}$ $(R_2)^{(9)}=(b'_{46})^{(9)}-(r_{46})^{(9)}$ | |

**Proof :** From global equations we obtain

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left((a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right) - (a''_{14})^{(1)}(T_{14},t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)}\right) \le \frac{dv^{(1)}}{dt} \le -\left((a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}} < (v_1)^{(1)} < (\bar{v}_1)^{(1)}$

$$v^{(1)}(t) \ge \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(v_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (v_2)^{(1)}}}$$

it follows $(v_0)^{(1)} \le v^{(1)}(t) \le (v_1)^{(1)}$

In the same manner , we get

$$v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{v}_1)^{(1)} - (v_0)^{(1)}}{(v_0)^{(1)} - (\bar{v}_2)^{(1)}}}$$

From which we deduce $(v_0)^{(1)} \le v^{(1)}(t) \le (\bar{v}_1)^{(1)}$

If $0 < (v_1)^{(1)} < (v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}} < (\bar{v}_1)^{(1)}$ we find like in the previous case,

$$(v_1)^{(1)} \le \frac{(v_1)^{(1)} + (C)^{(1)}(v_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((v_1)^{(1)} - (v_2)^{(1)}\right)t\right]}} \le v^{(1)}(t) \le$$

$$\frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (\bar{v}_1)^{(1)}$$

If $0 < (v_1)^{(1)} \le (\bar{v}_1)^{(1)} \le \boxed{(v_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$ , we obtain

$$(v_1)^{(1)} \le v^{(1)}(t) \le \frac{(\bar{v}_1)^{(1)} + (\bar{C})^{(1)}(\bar{v}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{v}_1)^{(1)} - (\bar{v}_2)^{(1)}\right)t\right]}} \le (v_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

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**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

$$\text{For } 0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

| | |
|---|---|
| $\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}$ , $\boxed{(\bar{C})^{(2)}=\frac{(\bar{\nu}_1)^{(2)}-(\nu_0)^{(2)}}{(\nu_0)^{(2)}-(\bar{\nu}_2)^{(2)}}}$ | |
| From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ | 392 |
| If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case,<br><br>$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)}+(C)^{(2)}(\nu_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}}{1+(C)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)}-(\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$<br><br>$\frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$ | 393 |
| If $0 < (\nu_1)^{(2)} \leq (\bar{\nu}_1)^{(2)} \leq (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $\nu^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t)=\frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t)=\frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)}G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)}T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain | 398 |

$$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)v^{(3)} - (a_{21})^{(3)}v^{(3)}$$

**Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)}v^{(3)} - (a_{20})^{(3)}\right) \le \frac{dv^{(3)}}{dt} \le -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)}v^{(3)} - (a_{20})^{(3)}\right)$$

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From which one obtains

For $0 < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$

$$v^{(3)}(t) \ge \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(v_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (v_2)^{(3)}}}$$

it follows $(v_0)^{(3)} \le v^{(3)}(t) \le (v_1)^{(3)}$

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In the same manner , we get

$$v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{v}_1)^{(3)} - (v_0)^{(3)}}{(v_0)^{(3)} - (\bar{v}_2)^{(3)}}}$$

**Definition of** $(\bar{v}_1)^{(3)}$ :-

From which we deduce $(v_0)^{(3)} \le v^{(3)}(t) \le (\bar{v}_1)^{(3)}$

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If $0 < (v_1)^{(3)} < (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}} < (\bar{v}_1)^{(3)}$ we find like in the previous case,

$$(v_1)^{(3)} \le \frac{(v_1)^{(3)} + (C)^{(3)}(v_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}}{1 + (C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((v_1)^{(3)} - (v_2)^{(3)}\right)t\right]}} \le v^{(3)}(t) \le$$

$$\frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (\bar{v}_1)^{(3)}$$

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If $0 < (v_1)^{(3)} \le (\bar{v}_1)^{(3)} \le (v_0)^{(3)} = \frac{G^0_{20}}{G^0_{21}}$ , we obtain

$$(v_1)^{(3)} \le v^{(3)}(t) \le \frac{(\bar{v}_1)^{(3)} + (\bar{C})^{(3)}(\bar{v}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}}{1 + (\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{v}_1)^{(3)} - (\bar{v}_2)^{(3)}\right)t\right]}} \le (v_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(3)}(t)$ :-

$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}$, $\boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}$, $\boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}^{'})^{(4)} - (a_{25}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right) - (a_{25}^{''})^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get

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$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)}+(C)^{(4)}(\nu_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}}{1+(C)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)}-(\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$, we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)}+(\bar{C})^{(4)}(\bar{\nu}_2)^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}}{1+(\bar{C})^{(4)}e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)}-(\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(4)}(t)$ :-

$$(m_2)^{(4)} \leq \nu^{(4)}(t) \leq (m_1)^{(4)}, \quad \boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}, \quad \boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)}G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)}T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29},t)\right) - (a_{29}^{''})^{(5)}(T_{29},t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{5 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}}{1 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)} + (\bar{C})^{(5)}(\bar{\nu}_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}}{1 + (\bar{C})^{(5)} e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)} - (\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}'')^{(5)} = (a_{29}'')^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} =$

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$(\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)} G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)} T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}, t)\right) - (a_{33}^{''})^{(6)}(T_{33}, t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \leq \frac{d\nu^{(6)}}{dt} \leq -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \geq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\nu_1)^{(6)}$

In the same manner , we get

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$$\nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \leq \nu^{(6)}(t) \leq (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case,

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$$(\nu_1)^{(6)} \leq \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \leq \nu^{(6)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \leq (\bar{\nu}_1)^{(6)} \leq \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain

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$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$(m_2)^{(6)} \leq v^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{v^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $v^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .

Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then $(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)v^{(7)} - (a_{37})^{(7)} v^{(7)}$$

**Definition of** $v^{(7)}$ :- $\boxed{v^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)} v^{(7)} - (a_{36})^{(7)}\right) \leq \frac{dv^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$v^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq v^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get

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$$\nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \le \nu^{(7)}(t) \le (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \le \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \le \nu^{(7)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\bar{\nu}_1)^{(7)}$$

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If $0 < (\nu_1)^{(7)} \le (\bar{\nu}_1)^{(7)} \le \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

$$(\nu_1)^{(7)} \le \nu^{(7)}(t) \le \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \le (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$$(m_2)^{(7)} \le \nu^{(7)}(t) \le (m_1)^{(7)}, \quad \boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$$

In a completely analogous way, we obtain

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**Definition of** $u^{(7)}(t)$ :-

$$(\mu_2)^{(7)} \le u^{(7)}(t) \le (\mu_1)^{(7)}, \quad \boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}'')^{(7)} = (a_{37}'')^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}'')^{(7)} = (b_{37}'')^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

$$\frac{dv^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right) - (a''_{41})^{(8)}(T_{41},t)v^{(8)} - (a_{41})^{(8)}v^{(8)}$$

**Definition of** $v^{(8)}$ :- $\boxed{v^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_2)^{(8)}v^{(8)} - (a_{40})^{(8)}\right) \le \frac{dv^{(8)}}{dt} \le -\left((a_{41})^{(8)}\left(v^{(8)}\right)^2 + (\sigma_1)^{(8)}v^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(8)}, (v_0)^{(8)}$ :-

For $0 < \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}} < (v_1)^{(8)} < (\bar{v}_1)^{(8)}$

$$v^{(8)}(t) \ge \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(v_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (v_2)^{(8)}}}$$

it follows $(v_0)^{(8)} \le v^{(8)}(t) \le (v_1)^{(8)}$

In the same manner , we get

$$v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{v}_1)^{(8)} - (v_0)^{(8)}}{(v_0)^{(8)} - (\bar{v}_2)^{(8)}}}$$

From which we deduce $(v_0)^{(8)} \le v^{(8)}(t) \le (\bar{v}_8)^{(8)}$

If $0 < (v_1)^{(8)} < (v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}} < (\bar{v}_1)^{(8)}$ we find like in the previous case,

$$(v_1)^{(8)} \le \frac{(v_1)^{(8)} + (C)^{(8)}(v_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)}e^{\left[-(a_{41})^{(8)}\left((v_1)^{(8)} - (v_2)^{(8)}\right)t\right]}} \le v^{(8)}(t) \le$$

$$\frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (\bar{v}_1)^{(8)}$$

If $0 < (v_1)^{(8)} \le (\bar{v}_1)^{(8)} \le \boxed{(v_0)^{(8)} = \frac{G^0_{40}}{G^0_{41}}}$ , we obtain

$$(v_1)^{(8)} \le v^{(8)}(t) \le \frac{(\bar{v}_1)^{(8)} + (\bar{C})^{(8)}(\bar{v}_2)^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)}e^{\left[-(a_{41})^{(8)}\left((\bar{v}_1)^{(8)} - (\bar{v}_2)^{(8)}\right)t\right]}} \le (v_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(8)}(t)$ :-

$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then
$(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$, we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)}G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)}T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$$

$$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0\ ,$$

$$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$$

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| $with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations | 426 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ | 427 |
| $(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ | 428 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , | 429 |
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |

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| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$<br>$with\ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | 434 |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$ | 436 A |

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| $with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |
| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\ ]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\ ]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\ ]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\ ]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\ ]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\ ]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\ ]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\ ]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |

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| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27}))\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}((G_{27}))\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27}))\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |

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| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |
| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(k) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$F(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) +$ | 486 |

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| $(a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | 488 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |

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| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{13}=\frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}=\frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | 494 |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations<br><br>$G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :-<br><br>After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations | 499 |

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| $G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a'_{32})^{(6)}+(a''_{32})^{(6)}(T^*_{33})]}$ , $G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a'_{34})^{(6)}+(a''_{34})^{(6)}(T^*_{33})]}$ | |
| **Definition and uniqueness of** $T^*_{37}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T^*_{37}$ for which $f(T^*_{37}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{36} = \frac{(a_{36})^{(7)}G_{37}}{[(a'_{36})^{(7)}+(a''_{36})^{(7)}(T^*_{37})]}$ , $G_{38} = \frac{(a_{38})^{(7)}G_{37}}{[(a'_{38})^{(7)}+(a''_{38})^{(7)}(T^*_{37})]}$ | 500 |
| **Definition and uniqueness of** $T^*_{41}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T^*_{41}$ for which $f(T^*_{41}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)}G_{41}}{[(a'_{40})^{(8)}+(a''_{40})^{(8)}(T^*_{41})]}$ , $G_{42} = \frac{(a_{42})^{(8)}G_{41}}{[(a'_{42})^{(8)}+(a''_{42})^{(8)}(T^*_{41})]}$ | 501 |
| **Definition and uniqueness of** $T^*_{45}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a''_i)^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T^*_{45}$ for which $f(T^*_{45}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)}G_{45}}{[(a'_{44})^{(9)}+(a''_{44})^{(9)}(T^*_{45})]}$ , $G_{46} = \frac{(a_{46})^{(9)}G_{45}}{[(a'_{46})^{(9)}+(a''_{46})^{(9)}(T^*_{45})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b'_{13})^{(1)}(b'_{14})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br><br>$[(b'_{13})^{(1)}(b''_{14})^{(1)}(G) + (b'_{14})^{(1)}(b''_{13})^{(1)}(G)]+(b''_{13})^{(1)}(G)(b''_{14})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G^*_{14}$ such that $\varphi(G^*) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b'_{16})^{(2)}(b'_{17})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br><br>$[(b'_{16})^{(2)}(b''_{17})^{(2)}(G_{19}) + (b'_{17})^{(2)}(b''_{16})^{(2)}(G_{19})]+(b''_{16})^{(2)}(G_{19})(b''_{17})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that | 504 |

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| there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br><br>$\left[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})\right] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |
| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right] + (b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right] + (b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right] + (b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right] + (b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if | 510 |

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| $\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$ $\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right] + (b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi(G_{43}{}^{*}) = 0$ | |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right] + (b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$<br><br>Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^{*}$ such that $\varphi((G_{47})^{*}) = 0$ | |
| Finally we obtain the unique solution<br><br>$G_{14}^{*}$ given by $\varphi(G^{*}) = 0$ , $T_{14}^{*}$ given by $f(T_{14}^{*}) = 0$ and<br><br>$G_{13}^{*} = \frac{(a_{13})^{(1)}G_{14}^{*}}{[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15}^{*} = \frac{(a_{15})^{(1)}G_{14}^{*}}{[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}^{*})]}$<br><br>$T_{13}^{*} = \frac{(b_{13})^{(1)}T_{14}^{*}}{[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G^{*})]}$ , $T_{15}^{*} = \frac{(b_{15})^{(1)}T_{14}^{*}}{[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G^{*})]}$<br><br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^{*}$ given by $\varphi((G_{19})^{*}) = 0$ , $T_{17}^{*}$ given by $f(T_{17}^{*}) = 0$ and | 512 |
| $G_{16}^{*} = \frac{(a_{16})^{(2)}G_{17}^{*}}{[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}^{*} = \frac{(a_{18})^{(2)}G_{17}^{*}}{[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 513 |
| $T_{16}^{*} = \frac{(b_{16})^{(2)}T_{17}^{*}}{[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}((G_{19})^{*})]}$ , $T_{18}^{*} = \frac{(b_{18})^{(2)}T_{17}^{*}}{[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}((G_{19})^{*})]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br><br>$G_{21}^{*}$ given by $\varphi((G_{23})^{*}) = 0$ , $T_{21}^{*}$ given by $f(T_{21}^{*}) = 0$ and<br><br>$G_{20}^{*} = \frac{(a_{20})^{(3)}G_{21}^{*}}{[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}^{*} = \frac{(a_{22})^{(3)}G_{21}^{*}}{[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}^{*})]}$<br><br>$T_{20}^{*} = \frac{(b_{20})^{(3)}T_{21}^{*}}{[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}{}^{*})]}$ , $T_{22}^{*} = \frac{(b_{22})^{(3)}T_{21}^{*}}{[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}{}^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 515 |

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| Finally we obtain the unique solution<br><br>$G_{25}^{*}$ given by $\varphi(G_{27}) = 0$ , $T_{25}^{*}$ given by $f(T_{25}^{*}) = 0$ and<br><br>$G_{24}^{*} = \frac{(a_{24})^{(4)} G_{25}^{*}}{[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}^{*} = \frac{(a_{26})^{(4)} G_{25}^{*}}{[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 516 |
| $T_{24}^{*} = \frac{(b_{24})^{(4)} T_{25}^{*}}{[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}((G_{27})^{*})]}$ , $T_{26}^{*} = \frac{(b_{26})^{(4)} T_{25}^{*}}{[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}((G_{27})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution<br><br>$G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br><br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 518 |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br><br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br><br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br><br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br><br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br><br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br><br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br><br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br><br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 523 |

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| $T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$<br>$T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | 523<br>A |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_{+})$ then the above equilibrium point is asymptotically stable.<br>**Proof:**_Denote<br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br>$G_i = G_i^{*} + \mathbb{G}_i$ , $T_i = T_i^{*} + \mathbb{T}_i$<br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}}(T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j}(G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*}\mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*}\mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*}\mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*}\mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*}\mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*}\mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions | 531 |

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| $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i^{''})^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^*\mathbb{T}_{17}$ | 534 |
| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a_{17}^{'})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^*\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a_{18}^{'})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^*\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b_{16}^{'})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^*\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b_{17}^{'})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^*\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b_{18}^{'})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^*\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(3)}$ *and* $(b_i^{''})^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{21}^{''})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b_i^{''})^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a_{20}^{'})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^*\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a_{21}^{'})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^*\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a_{22}^{'})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^*\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b_{20}^{'})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^*\mathbb{G}_j\right)$ | 544 |

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| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b_{21}^{'})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b_{22}^{'})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(4)}\ and\ (b_i^{''})^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^{*}) = (q_{25})^{(4)}\ ,\ \frac{\partial(b_i^{''})^{(4)}}{\partial G_j}((G_{27})^{*}\ ) = s_{ij}$ | 548 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^{*}\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^{*}\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^{*}\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^{*}\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^{*}\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^{*}\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}\ and\ (b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial(a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^{*}) = (q_{29})^{(5)}\ ,\ \frac{\partial(b_i^{''})^{(5)}}{\partial G_j}(\ (G_{31})^{*}\ ) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |

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| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a'_{28})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G^*_{28}\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a'_{29})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G^*_{29}\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a'_{30})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G^*_{30}\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b'_{28})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T^*_{28}\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b'_{29})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T^*_{29}\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b'_{30})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T^*_{30}\mathbb{G}_j\right)$ | 562 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(6)}$ *and* $(b''_i)^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G^*_i + \mathbb{G}_i \quad , T_i = T^*_i + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{33})^{(6)}}{\partial T_{33}}(T^*_{33}) = (q_{33})^{(6)}$ , $\frac{\partial(b''_i)^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a'_{32})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G^*_{32}\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a'_{33})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G^*_{33}\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a'_{34})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G^*_{34}\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b'_{32})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T^*_{32}\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b'_{33})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T^*_{33}\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b'_{34})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T^*_{34}\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(7)}$ *and* $(b''_i)^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |

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| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \ , \frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left( (G_{39})^{**} \right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |
| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a_{38}^{'})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G_{38}^*\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b_{36}^{'})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T_{36}^*\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b_{37}^{'})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T_{37}^*\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b_{38}^{'})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T_{38}^*\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{41}^{''})^{(8)}}{\partial T_{41}}(T_{41}^*) = (q_{41})^{(8)} \ , \frac{\partial (b_i^{''})^{(8)}}{\partial G_j}\left( (G_{43})^{*} \right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a_{40}^{'})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G_{40}^*\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a_{41}^{'})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G_{41}^*\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a_{42}^{'})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G_{42}^*\mathbb{T}_{41}$ | 583 |

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| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br>**Proof: Denote** | 586 A |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{45}^{''})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \;\; , \frac{\partial(b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$<br><br>$\left[\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\Big]$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\Big)$<br><br>$+\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\Big)$<br><br>$\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\Big)$ | |

$$\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$\left((\lambda)^{(1)}\right)^2+\left((b_{13}')^{(1)}+(b_{14}')^{(1)}-(r_{13})^{(1)}+(r_{14})^{(1)}\right)(\lambda)^{(1)}\Big)$$

$$+\left((\lambda)^{(1)}\right)^2+\left((a_{13}')^{(1)}+(a_{14}')^{(1)}+(p_{13})^{(1)}+(p_{14})^{(1)}\right)(\lambda)^{(1)}\Big)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)}+(a_{13}')^{(1)}+(p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^{*}+(a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^{*}\right)$$

$$\left((\lambda)^{(1)}+(b_{13}')^{(1)}-(r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^{*}+(b_{14})^{(1)}s_{(13),(15)}T_{13}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\}\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$$

$$\left[\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\Big]$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^2+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^2+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\}\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$$

$$\left[\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\Big]$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^2+\left((b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left(\left((\lambda)^{(3)}\right)^2+\left((a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)}+(a_{20}^{'})^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)}+(b_{20}^{'})^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\Big)\right]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^2+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left(\left((\lambda)^{(4)}\right)^2+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\Big)\right]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^2+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^2+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(6)}+(b_{34}^{'})^{(6)}-(r_{34})^{(6)}\{\left((\lambda)^{(6)}+(a_{34}^{'})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(33)}T_{33}^{*}\Big)$$

$$+\left((\lambda)^{(6)}+(a_{33}^{'})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G_{32}^{*}+(a_{32})^{(6)}(q_{33})^{(6)}G_{33}^{*}\Big)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(32)}T_{32}^{*}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^2+\left((b_{32}^{'})^{(6)}+(b_{33}^{'})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^2+\left((a_{32}^{'})^{(6)}+(a_{33}^{'})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a_{32}^{'})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G_{33}^{*}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G_{32}^{*}\right)$$

$$\left((\lambda)^{(6)}+(b_{32}^{'})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T_{33}^{*}+(b_{33})^{(6)}s_{(32),(34)}T_{32}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(7)}+(b_{38}^{'})^{(7)}-(r_{38})^{(7)}\{\left((\lambda)^{(7)}+(a_{38}^{'})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(37)}T_{37}^{*}\Big)$$

$$+\left((\lambda)^{(7)}+(a_{37}^{'})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G_{36}^{*}+(a_{36})^{(7)}(q_{37})^{(7)}G_{37}^{*}\Big)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(36)}T_{36}^{*}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^2+\left((b_{36}^{'})^{(7)}+(b_{37}^{'})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^2+\left((a_{36}^{'})^{(7)}+(a_{37}^{'})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a_{36}^{'})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G_{37}^{*}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G_{36}^{*}\right)$$

$$\left((\lambda)^{(7)}+(b_{36}^{'})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T_{37}^{*}+(b_{37})^{(7)}s_{(36),(38)}T_{36}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(8)}+(b_{42}^{'})^{(8)}-(r_{42})^{(8)}\}\{\left((\lambda)^{(8)}+(a_{42}^{'})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(q_{40})^{(8)}G_{40}^{*}\Big)\right]$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(41)}T_{41}^{*}\Big)$$

$$+\left((\lambda)^{(8)}+(a_{41}^{'})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G_{40}^{*}+(a_{40})^{(8)}(q_{41})^{(8)}G_{41}^{*}\Big)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(40)}T_{40}^{*}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^2+\left((b_{40}^{'})^{(8)}+(b_{41}^{'})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^2+\left((a_{40}^{'})^{(8)}+(a_{41}^{'})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a_{40}^{'})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G_{41}^{*}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G_{40}^{*}\right)$$

$$\left((\lambda)^{(8)}+(b_{40}^{'})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T_{41}^{*}+(b_{41})^{(8)}s_{(40),(42)}T_{40}^{*}\Big)\}=0$$

$$+$$

$$(\lambda)^{(9)}+(b_{46}^{'})^{(9)}-(r_{46})^{(9)}\}\{\left((\lambda)^{(9)}+(a_{46}^{'})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a_{44}^{'})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G_{45}^{*}+(a_{45})^{(9)}(q_{44})^{(9)}G_{44}^{*}\Big)\right]$$

$$\left((\lambda)^{(9)}+(b_{44}^{'})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T_{45}^{*}+(b_{45})^{(9)}s_{(44),(45)}T_{45}^{*}\Big)$$

$$+\left((\lambda)^{(9)}+(a_{45}^{'})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G_{44}^{*}+(a_{44})^{(9)}(q_{45})^{(9)}G_{45}^{*}\Big)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45} + (b_{45})^{(9)}s_{(44),(44)}T^*_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^2 + \left((b'_{44})^{(9)} + (b'_{45})^{(9)} - (r_{44})^{(9)} + (r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^2 + \left((a'_{44})^{(9)} + (a'_{45})^{(9)} + (p_{44})^{(9)} + (p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)} + (a'_{44})^{(9)} + (p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45} + (a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left((\lambda)^{(9)} + (b'_{44})^{(9)} - (r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45} + (b_{45})^{(9)}s_{(44),(46)}T^*_{44}\Big)\} = 0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future.** It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al**

**Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal**

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i(x)\, xi$ where $x_i \in R^n_{i,}$ $x = (x_1,\ldots,x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i(x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i(\mu)$ of the $A_i(x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i Xi(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010)
It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$(11) \qquad \frac{dxi}{dt} = x_i f_i(x) \qquad i = 1,\ldots,k$$

Where $x_i$ is the abundance of population $i$, $x = (x_1,\ldots,x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output, creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and

review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 11. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,...x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \; i=1,...,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided

there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \in C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i(x)$ is continuous

**A2:** $A_i(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For $K \subset C$ and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N_\delta(K) = \{x \in C : \|x - y\| < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N_\delta(\Gamma(A))$, *and* $\|A(x) - \tilde{A}(x)\| \leq \delta$ *for all* $x \in N_\delta(\Gamma(A))$. (2) is *robustly permanent* if there exist $\delta > 0$ and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C \backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A)\times R^n_i$ defined by $(x,y).t = (x.t,B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)|=|v_i(x)|=1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ and $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ and $t \geq 0$.**Interaction rates.** Given any x $\epsilon$ $\Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right)$ *in* $||B_i(t,x)||$ when $x_i$ =0, $\lambda_i(x)$ provides an upper bound to the rate of growth of population *i* when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population *i* persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population *i* is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) In\ |B_i\ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x) = limsup_{t\rightarrow\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = \underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni} = E_{i(x)}\ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \leq a|\ B_i(t,x)u_i(x)|+ |B_i(t,x)w| \leq |\ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)|\ w|)$. Hence, $\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$\underset{t\rightarrow\infty}{limsup}\frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \epsilon R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty} \left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\epsilon$ $C_i^+$. Then, we can write $v = au_i(x) + w$ with $a > 0$ and $w \epsilon F_i$.

Equation (3) implies

$$|B_i(t,x)v| \geq a|B_i(t,x)u_i(x)| - |B_i(t,x)w|$$

$$\geq |B_i(t,x)u_i(x)|(a - \alpha + exp(-\beta t)|w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \geq \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\ u_i(x)| = In\ |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\ u_i(x.t)||\ B_i(t,x)\ u_i(x)|$$

$$= \text{In } b_i(s,t.x)\ b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x) = 0$ implies

$$\frac{d}{dt} b_i(t,x) = \lim_{s\to 0} \frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0} \frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\ b_i(s,x.t)$$

$$=$$

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x) = \int_0^t (A_i(x.s)\ u_i(x.s),\ u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \leq 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma>0$ such that $|x.t| \leq \gamma$ for all $t \geq 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty} \frac{1}{t} \text{In } |(x.t)_i|$$

$$\leq \limsup_{t\to\infty} \frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0$. $If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\geq 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty} \frac{1}{t} In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp($\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B*.

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle A_i(x)u_i(x), u_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:* $\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle A_i(x.s)u_i(x.s), u_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(x) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu(\Pi_{i\in I} \boldsymbol{C}_i^+)$=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $h_i(x) = \langle A_i(x)u_i(x), u_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ($\Pi_{i\in I} \boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I} \boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I} \boldsymbol{C}_i^+$ such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following equivalent conditions hold For all invariant measures $\mu$ supported by *K* (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $\mathrm{p}_i > 0$ such that (5) $\sum_{1\le i\le k} \mathrm{p}_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by *K*.

then *K* is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and *K* is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by *K*.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $\mathrm{A_i}(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $\mathrm{A}_i(x)$ can be decomposed into a finite number, say $\mathrm{m}_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle \mathrm{A}_i^j(x)\mathrm{u}_i^j(x), \mathrm{u}_i^j(x)\rangle \, \mathrm{d}\mu(x)$where $\mathrm{A}_i^j(x)$ is the submatrix of $\mathrm{A_i}(x)$ corresponding to the *j*-th irreducible component of $\mathrm{A_i}(x)$ *and* $\mathrm{u}_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $\mathrm{A}_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le \mathrm{m}_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \leq 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\rightarrow \infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$ . Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \rightarrow R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large so that $|\int_c h(x)\, d\mu(x) - \int_c h(x) d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\, d\mu(x) - \int_c h(x.t) d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x) d\mu(x)|$ is

$\leq \quad |\int_c h(x.t)\, d\mu(x) - \int_c h(x.t)\, d\mu_n(x)| + |\int_c h(x.t)\, d\mu_n(x) - \int_c h(x)\, d\mu(x)|$

$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\, d\mu_n(x)| + |\int_c h(x^n.t)|\, d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\, d\mu_n(x)| + |\int_c h(x) d\mu_n(x) - \int_c h(x) d\mu(x)|$

$\leq \quad 3 \in$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and t > 0 are arbitrary, we have $\int_c h(x.t) d\mu(x) = \int_c h(x) d\mu(x)$ for all t >0 and all bounded continuous functions $h: C \rightarrow R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x) u_i^n(x), u_i^n(x) \rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop. 3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \rightarrow A_i(x)$ uniformly for $x \in N_1((A))$ as $n \rightarrow \infty$, $[26, Prop. 3.2]$ implies that $u_i^n(x) \rightarrow u_i(x)$ converges uniformly for $x \in$

$N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)| d\mu(x) + |\int_c h_i^n(x)\, d\mu(x) - \int_c h_i^n(x)\, d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\, d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.\ 2.1]$ implies that either there exists $\mathcal{Y} \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $\mathcal{Y} \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t} \int_0^t \delta_{y.s}\, ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $\mathcal{Y}.s$. Dissipativeness of (2) and weak* compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{l}{t} \int_0^t h_i(y.s) ds$$

$$= \limsup_{t\to\infty} \int_c h_i\, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i\, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then

there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i\ (\mu = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

$$(6)\ \min \max \lambda i(\mu) = \min_{\mu} \max_{p\epsilon\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_i P_i\ \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\in$ $iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is

saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of $\Gamma$(A) such that V.t c int $V$ for all t > 0.Let $\eta > 0$ be such that $W$=:$N\,\eta$($\Gamma$(A))c $V$ Let p:c→ $[0,1]$ $be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_{1,}\ldots,\tilde{A}_{k})$ by

$$\tilde{A}_i(x) = \{\frac{A(x)}{A(x)-\frac{\delta}{2Idp\,(x)}} \quad \frac{if\ i\epsilon I}{if\ i \notin I}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})$c $V$ c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1,\ldots,k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t-$invariant families of $C^1$ discs $W^s_x$,$W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). $Since\ E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K$\subset \partial$ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K$\subset II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set *K* if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$ and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent.* Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume* $x \mapsto A(x)$ *is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists* $j \in \{1,\ldots,k\}$ *such that* $M_i$ ----- $\{ x \in C : x_l \gg 0 \text{ for } l \neq j\}$ *Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by* $U_i M_i$.

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup ($\mu$) to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures $\mu$ with support in* $\partial$C, $\lambda_i(\mu) > 0$ *for all* $I \in \{1,\ldots,k\} \setminus supp(\mu)$ *(2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j$ i = 1,..,k j=1,…,m

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1, \ldots, x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin $\lambda(0)$, which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species,* e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i(0) > 0$ for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2(E_1)$ and $\lambda_2(E_2)$ are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0 < e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \; f_i^j(x^j_i) = 0$ i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product $(E_1^1, \ldots, E_1^m)$, and the matrix $A^2(E_1)$ is close to the diagonal matrix $\text{diag}(f^i_2(E_1^j))^m_{j=1.}$ Hence the invasion rate $\lambda_2(E_1)$ is close to $\max_j f_2^j(E^j_1) > 0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j(E^j_1)>0$ and $\max_j f_2^j(E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial C$ is given by two equilibria $\{0,E_1\}$, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2(E_1) > 0$. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$ for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j(E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$$\frac{dx_1^j}{dt} = x_1^j(1-x_1^j - \beta^j x_2^j - \alpha^j x_3^j) + \sum d_1^{jk} x_1^k - e_1^j x_1^j$$

$$\frac{dx_2^j}{dt} = x_2^j(1-\alpha^j x_1^j - x_2^j - \beta^j x_3^j) + \sum d_2^{jk} x_2^k - e_2^j x_2^j$$

$$\frac{dx_3^j}{dt} = x_i^j(1-\beta^j x_1^j - \alpha^j x_2^j - \beta^j x_3^j) + \sum d_3^{jk} x_3^k - e_3^j x_3^j$$

Where $a^j \in (0,1)$ and $\beta^j > {}^1$ for all *j*. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all *i, j*) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0< e^j_i \ll 1$ for all *i,* j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2(E_1)$and $\lambda_3(E_1)$are close to $\max_j 1-a^j$ and $\max_j 1-\beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1-a^j$ and $\max_j 1-\beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i(0) = 1> 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation

reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1-a^j) > -\max_j(1-\beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x+a)^n =$

(9) $$\frac{dx_i^j}{dt} = x_i^j\left(r_i - \sum_s a_{is}x_s^j\right) + \sum_l d_i^{jl}x_i^l - e_i^j x_i^j, \qquad i = 1,\dots,k;\ j = 1,\dots,m$$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is}x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\Gamma$. We further assume that for each patch $j$ and each species $i$ the immigration rate equals the emigration rate , i. e,

(10) $$e_i^j = \sum_l d_i^{jl}$$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^1 \forall i,j,l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $$\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is}x_s), \qquad i = 1, \dots, k$$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is}x_s), \quad i = 1,\dots,k$

It has been shown [12,ch.16.1,16.2] That for $k$ =3,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R^3_+$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with\; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1 \to E_2 \to E_3 \to E_1}$ then the following inequality holds

12) $$\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial$C which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This

implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However,

permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma\, I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & 0 \\ \gamma & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \left( \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1 - y) - b(N)\ y \right)$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon+x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} - y_i)$$

With

$$(16)\quad F(x)_i = x_i g\,(x_i, x_i - 1) = \frac{x_i}{1+x_i+p x_{i-1+Kp\, x_i x_i - 1}}$$

Here $x_i$ and $y_i$ are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\quad \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag\,(x_i, x_i) & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

## Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons

and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have

the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1\ (t)$ and $P_2\ (t)$ denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t\geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1\ (t)$ and $P_2\ (t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1\ (a, t)\ da$ and $p_2\ (a, t)\ da$ individuals with ages between $a$ and $a + da\ (a\geq 0)$ at time $t$ so that we have

(1.31) $P_1\ (t) = \int_0^\infty \rho_1(a, t)\ da,\ P_2\ (t) = \int_0^\infty \rho_2(a, t)\ da,\ t\geq 0.$

The rates of change of the two species densities are by definition $D\rho_1\ (a, t)$ and $D\rho_2\ (a, t)$ where

(1.32) $D\rho_i\ (a, t) = \limsup_{h\to 0} \frac{\rho_i(a+h,t+h)-\rho_i(a,t)}{h},\ i = 1, 2.$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.33) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a,t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a,t) = 0, & \text{'} \\ \rho_1(0,t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a,t)da, & t>0, \\ \rho_2(0,t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a,t)da, \end{cases}$$ (1.4)

(1.5) $\rho_i(a,s) = \varphi_i(a,s),\ a.0,\ s\epsilon[-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a,s)\,da,\ s\in[-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon[-\tau, 0]$. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic (*as*

$t \to \infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such "harmless" natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 11. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |,\ s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1 / \partial P_1, \partial f_1 / \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2 / \partial P_1, \partial f_2 / \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\,(R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

$(\beta_1^*\ , \beta_2^*$ *are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for i= 1, 2,*

$|b_i\,(a, P_1^{'}, P_2^{'})-b_i\,(a, P_1^{"}, P_2^{"})| \leq \beta_i\,[||P_1^{'}- P_1^{"}||+||P_2^{'}-P_2^{"}]$

$B_{\ 1}$ , $B_{\ 2}$ Are positive constants),

$b_i(°, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2,$

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 1} \text{x}\ k_{\ 2}, \text{R}^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ \text{x}\ k_{\ 2} \text{x}\ k_{\ 1}, \text{R}^-),$$

$A_3$. $\varphi_{\ i}\ (\cdot\ , s) \in L^{\ 2}\ (R^{\ +}, R^{\ +})\ \cap C(R^{\ +}, R^{\ +})$ for each $s \in\ [-\tau, 0]$,

$i, = 1,\ 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i$: $R^{\ +} \times [-\tau, T] \to R^{\ +}$ $(i, = 1,\ 2,)$ With the following properties:

$S_1$. $P_i(\cdot\ , t) \in L^{\ 1}\ (R^{\ +}, R^{\ +})$ for each $t\ \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot\ ) \in C\ ([-\tau, T], R^{\ +})$ for each $a\ \in R^{\ +}$ ;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^{\ +} \times \{R^{\ +} \cap\ [0, T]\}$ And is continuous $(i, = 1,\ 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,)\ \in R^{\ +} \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = \quad P_i(x), (i, = 1,\ 2,)\ ,$$

Then (1.3) considered at $(1.3)$ considered at $(a + x,\ t + x)$ becomes

$$\frac{dp_{\ i}(x)}{dx} + f_i(a + x,\ P(t + x))\ \ P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^{\infty} f_i(a+s, P(t+S))ds \right]$$

And hence

(2.1) $p_1(a+x, t+x) = p_1(a,t)\, exp\left[-\int_o^x f_1(a+s, p(t+s))ds\right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a,t) = \varphi_1(a-t, 0)\exp\left[-\int_o^t f_1\left(a-t+s, p_1(s)\right) ds\right]$

$$O \leq t \leq a, i = 1,2,$$

With

$$P_1(s)=(p_1(s),p_2(s-t)),$$

$$P_2(s)=(p_1(s-t),p_2(s)),$$

For $0 \leq a < t$ we choose *(a,t)=(0,t-a)in (1.3)and x=a in (2.1)* so that

(2.3) $p_i(a,t)= B_i(t-a)exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$$t > a \geq, i = 1,2,$$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i and\ L_i\ (i = 1,2)$ as follows:

$$M_i(a,t, p_i(t)) = exp\left[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds\right]$$

(2.5) $L_i(a,t,\ p_i(t)) = \exp\left[-\int_o^t f_1(a+s,\ p_i(s))ds\right]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a)\, M_1(a, t, p_i(t)da + \int_o^{\infty} \varphi_i(a, 0)\ L_i(a,t,p_i(t)da,$

$$t > 0, i = 1,2,$$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^{\infty} \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.\left(a,t,P_i(t)\right)da$

$$+\int_0^{\infty} b_i\left(a+t, P_i(t)\right)\varphi_i(a,0)L_i\left(a,t,P_i(t)\right)da$$

$$i = 1, 2, \; t > 0$$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \; \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^{\infty} \varphi_i\,(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i\,(0)\, exp(\beta_i^* t)\left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \; t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2\ ),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in \mathrm{C}([0,T], R^+), \quad B_i \in \mathrm{C}([0,T], R^+)\ , i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in \mathrm{C}([-\tau, T], R\ ) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $\mathrm{C}^+[-\tau, T\ ]$. Then for any

$(x, y) \in \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$\mathrm{C}_T = \mathrm{C}^+[-\tau, T\ ], \quad \mathrm{C}_T^2 = \mathrm{C}^+[-\tau, T\ ] \times \mathrm{C}^+[-\tau, T\ ].$$

Members of $\mathrm{C}_T$ will be denoted by $Z_t$ with the meaning that

$Z\ (t+s) = z_t(s)$ *for* $s \in [-\tau, T\ ]$ *and* $t \in [0\ , T\ ]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in \mathrm{C}_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2\ )$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$\mathrm{B}\ (\mathrm{t}) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \ t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}),\ \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for *i =1,2* .

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \{P_t \in (P_{1t}, P_{2t}) \in C_{t*}^2 \mid \|P_t - \Phi_t\|_{t*} \le r^*\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on [-$\tau$, $t^*$]

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a, s)da \text{ for } s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a, 0)da \text{ for } s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left\{\sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \quad t^*\right\}$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then □ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \leq T$. we then will have to show that for small $t^* > 0$ , □ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|□(P_t^{(1)}) - □(P_t^{(2)})|$ for t ≤ $t^*$.

Corresponding to the chosen $P_t^{(1)}$ , $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$ . Then from

(2.6),

$$|□(P^{(1)}(t)) - □(P^{(2)}(t)| \leq \left| \int_0^t B_1^{(1)}(a)\left\{M_1\left(a,t,P_t^{(1)}\right) - M_1\left(a,t,P_t^{(2)}\right)\right\}da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)]\, M_1(a,t,P_t^{(2)})da \right|$$

$$+ \left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x - y| \; for \; x, y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1\, \beta_1^*\, \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left|J_3\right| \leq \int_o^\infty \varphi(a,o)[\int_o^t \left|f_1(a+s,P_t^{(1)}(s) - f_1(a+s,P_t^{(2)}(s))\right| ds]da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left|B_1^{(1)}(a) - B_1^{(2)}(a)\right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a [B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a [B_1^{(2)}(\sigma)$$

$$\text{X}[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o)\left[b_1\left(\sigma + a, P_t^{(1)}(a)\right) - b_1\left(\sigma + a P_t^{(2)}(a)\right)\right], L_1\left(\sigma, a, P_t^{(1)}(a)\right) d\sigma$$

$$+\int_o^\infty \varphi(\sigma, o) b_1\left(\sigma + a, P_t^{(2)}(a)\right)\left[L_1\left(\sigma, a, P_t^{(1)}(a)\right) - L_1\left(\sigma, a, P_t^{(2)}(a)\right)\right] d\sigma$$

$$= R_1 + R_2 + R_3 + R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma, P_t^{(1)}(a)\right) - b_1\left(\sigma + a, P_t^{(2)}(a)\right)\right| \leq B_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \text{ for } a \in [o, t^*]$$

And hence we have

$$|R_3(a) + R_4(a)| \leq \beta_1 \Phi_1(0) ||P_t^{(1)} - P_t^{(2)}||_t^* + \Phi_1(0) \beta_1 d_1 ||P_t^{(1)} - P_t^{(2)}||_t^{*a},$$

$$|R_2(a)| = [\int_0^a [B_1^{(2)}(\sigma)\{b_1(a-\sigma, P_t^{(1)}(a)) - b_1(a-\sigma, P_t^{(2)}(a))\} M_1(\sigma, a, P_t^{(1)}(a))$$

$$+ b_1(a, -\sigma, P_t^{(2)}(a))\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$+ b_1\left(a-0, P_t^{(2)}(a)\right)\{M_1(\sigma, a, P_t^{(1)}(a)) - M_1(a-\sigma, P_t^{(2)}(a))\}] d\sigma]$$

$$\leq \int_0^a [\beta_1^* \Phi_1(0) e^{\delta 1^a} \beta_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} + \beta_1 d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} (a-o)] do$$

$$\leq \beta_1^* \Phi_1(0) e^{\delta 1^a} /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} \{\beta_1 a + d_1 a^2/2\}$$

Thus

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right|$$

$$\leq \int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)-\mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{B}_1^{*\mathrm{d}}\sigma$$

$$+ \Phi_1(0)\|P_t^{(1)}-P_t^{(2)}\|_{t*}\left\{B_1 + B_2\, d_1\, a + B_1^{*}\ \left(B_2 a + d_1 a^2/2\right) e^{\delta 1^a}\right\}$$

$$\leq \beta_1^{*}\int_o^a \left| \mathrm{B}_1^{(1)}(\sigma)- \mathrm{B}_1^{(2)}(\sigma) \right| \mathrm{do}+\left(\mathrm{r}^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\, g_1^{(a)}$$

Where $g_1(a)\rightarrow \beta_1$ as $a\rightarrow 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*}\{g_1(a)+0(a)\}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)|\leq \left| \int_0^a [B_1^{(2)}(\sigma)\, B_1^{(2)}(\sigma),]\, b_1(a))-b_1(a-\sigma,\, P_t^{(1)}(a))\, M_1(\sigma,\, a,\, P_t^{(1)}(a))do \right|$$

$$\left| B_1^{*}\left(r^{*} + \|\Phi_t\|_{t*}\right)\| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a\rightarrow 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(1) as \quad a \rightarrow o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_t o(t) as \quad t \rightarrow o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\,(P_t^{(1)} - \pi\left(P_t^{(2)}\right)\| \leq (r^{*} + \|\Phi_t\|_{t*})\| P_t^{(1)} - P_t^{(2)}\|_{t*} o(t^{*})$$

As $t^{*}\rightarrow o$ showing that if $t^{*}$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^{*})$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o\leq t \leq t^{*} for\ t^{*}$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^{+}$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^{*}$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^{*}]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

(3.1) $\frac{dp\,P_2^*(a)}{da}+f_i(\mathrm{a}, P_1^*, P_2^*)\,P_i^*(a) = 0,\ \ i = 1, 2, a > 0,$

(3.2) $P_i^* = \int_o^\infty\ P_i^*(a)da, i = 1,\ 2,$

(3,3) $P_i^*=\int_o^\infty\ b_i(a,\ P_1^*,\ P_2^*)P_i^*(a)da,\ i = 1.\ \ 2$

All solutions of (3.1)are of the from

(3.4) $P_i^*(a) = P_i^*(o)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1^*, P_2^*)ds\right], i = 1,\quad 2, a \geq 0,$

Which together with (3.3) lead to

$1=\int_o^\infty\ b_1(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_1(s, P_1^*, P_2^*)ds\right]da,$

(3.5) $1=\int_o^\infty\ b_2(a, P_1^*, P_2^*)\ \exp\left[-\int_o^a\ f_2(s, P_1^*, P_2^*)ds\right]da$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $F_{i.}\ (P_1, P_2) =\ \int_o^\infty\ b_i(a, P_1, P_2)\ \exp\left[-\int_o^\infty\ f_i\,(s, P_1, P_2)ds\right]da,$

$i$ =1, 2, $\quad P_1 \geq 0, \quad P_2 \geq 0,$

**And suppose the following hold:**

(3.7) $A_{4.}\quad F_{i.}\ (0,\ 0) > 1,\ i = 1, 2$:

$A_{5.}$ *there exists a positive constant c for which*

$F_1$(C,0)< $F_{2.}$(C,0),

(3.8) $\quad F_1$(0.C)< $F_{2.}$(0,C),

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o,$

$P_2^* > 0$ such that

(3,9) $F_1(P_1^*, P_2^*) = 1 =\ F_2(P_1^*, P_2^*)$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $P_2^*(a) = [P_2^* \exp[-f_i(s, P_1^*, P_2^*)ds])/(\int_o^\infty \exp[-\int_o^a f_i(s, P_1^*, P_2^*)ds]da)$,

$i = 1, 2, \quad a \geq 0,$

Proof Consider the elements of the surfaces defined by

$Z = F_1(P_1, P_2), z = F_2(P_1, P_2), \ P_1 \geq 0, P_2 \geq 0,$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i / \partial P_j < 0$ *and* , by ($A_2$), $\partial b_i / \partial P_j \leq 0$ for all

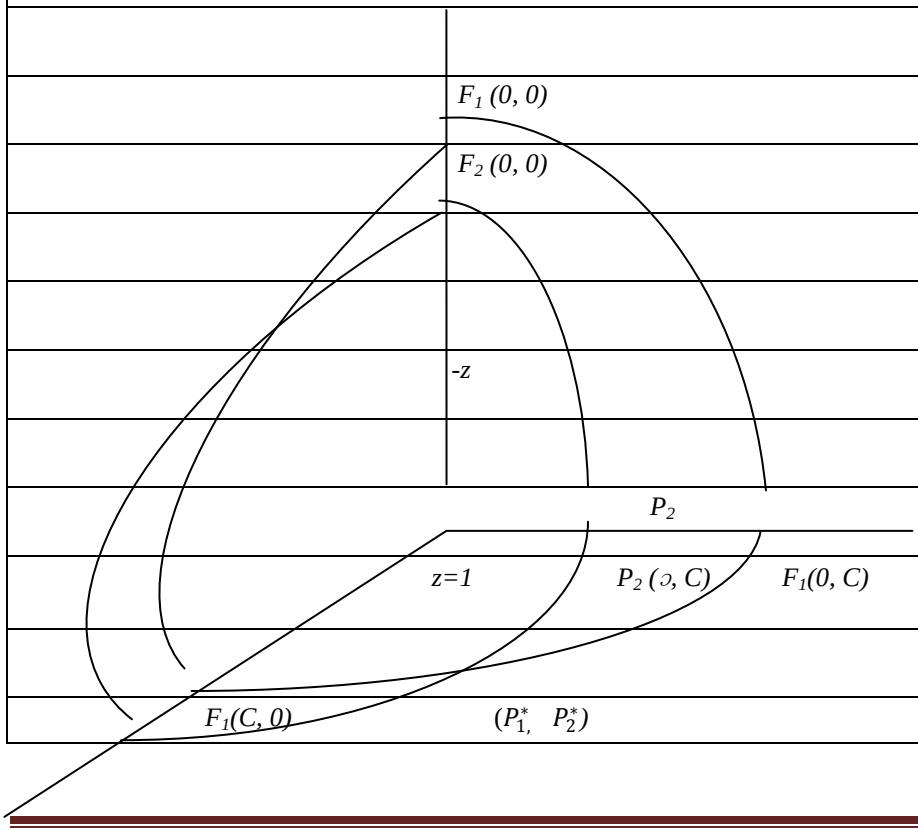
$P_1 \geq 0, P_2 \geq 0, \quad i = 1, 2,$

$j = 1, 2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= $1, P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$F_1(P_1, P_2 = 1$ *and* $F_2(P_1, P_2) = 1, \ F_1(P_1, P_2 \geq 0,$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.



$F_2(C, 0)$

(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p^*_1, p^*_2)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p^*_1, p^*_2$, we let

$$(3.11)\quad \begin{cases} p_i(a,t) = p^*_i(a) + u_i(a,t) \\ p_i(t) = p^*_i + p_i(t), \quad i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

$$(3.12)\quad \begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_i}\, p_i(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial b_i}{\partial p^*_i}\, p_i(t)\right\} da \end{cases}$$

Where $i = 1,2$ and (3.13)

$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

$$(3.14)\quad \begin{cases} u_i(a,t) = u_i(a) \exp[\lambda t], \quad u_i(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty u_i(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{du_i}{da} + \lambda u_i = -f_i\left(a, p^*_1, p^*_2\right) u_i(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_i\right\},$$

(3.15)

$$\dot{q}_i(0) = \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) q_i(a)da + \int_0^\infty p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p_j^*}\, p_i^*\right\} da\,,$$

Where $i = 1,\ 2,$ and

$$p_{11}^* = p_1^*, \quad p_{12}^* = p_2^* \exp\{-\lambda\tau\},$$

$$p_{21}^* = p_1^*,\ \exp\{-\lambda\tau\},\ p_{22}^* = p_2^*.$$

If we integrate the first of (3.15) and use the fact that $q_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right] da$$

$$= \int_0^\infty \left[b_1\left(a, p_1^*, p_2^*\right) - f_1\left(a, p_1^*, p_2^*\right)\right] q_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right] da$$

$$= \int_0^\infty \left[b_2\left(a, p_1^*, p_2^*\right) - f_2\left(a, p_1^*, p_2^*\right)\right] q_2(a)da,$$

From the first of (3.15) we get

(3.18) $$q_i(a) = \left\{q_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\right\}$$

$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\} e^{\lambda s} ds\,, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$q_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\} da\right.$$

$$\left. - \int_0^\infty b_i\left(a, p_1^*, p_2^*\right) p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a \left(\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i^*\right) e^{\lambda s} ds\right\} da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a \left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p_j^*}\, p_i^*\right\} e^{\lambda s} ds\,, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)} \int_0^\infty b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} da, \quad i = 1, 2,$$

Now supplying $\varphi_1(a)$ and $\varphi_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11}) p_1^* - A_{12} p_2^* \exp(-\lambda\tau) = g_{11}(\lambda) p_1^* + g_{12}(\lambda) p_2^* \exp(-\lambda\tau),$$

(3.21)

$$-A_{21} p_1^* \exp(-\lambda\tau) + (\lambda - A_{22}) p_2^* = g_{21}(\lambda) p_1^* \exp(-\lambda\tau) + g_{22}(\lambda) p_2^*$$

Where

(3.22) $$A_{ij} = \int_0^\infty p_i^*(a) \left[\frac{\partial b_i}{\partial p_j^*} - \frac{\partial f_i}{\partial p_j^*}\right] da,$$

(3.23) $$g_{ij}(\lambda) = [p_i^*(0) c_i]^{-1} \left\{\int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} da\right\} \left\{\int_0^\infty \left[p_i^*(a) \frac{\partial b_i}{\partial p_j^*} - b_i(a, p_1^*, p_2^*) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right)\right] da\right\}$$

$$- \int_0^\infty \sigma_i(a) p_i^*(a) e^{-\lambda a} \left(\int_0^a \frac{\partial f_i}{\partial p_j^*} e^{\lambda s} ds\right) da, \quad i, j = 1, 2,$$

(3.24) $$\sigma_i(a) = b_i(a, p_1^*, p_2^*) - f_i(a, p_1^*, p_2^*), \quad i = 1, 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $(p_1^*, p_2^*)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et \begin{bmatrix} \lambda - (A_{11} + g_{11}(\lambda)) & -(A_{12} + g_{12}) e^{-\lambda\tau} \\ -(A_{21} + g_{21}) e^{-\lambda\tau} & \lambda - (A_{22} + g_{22}(\lambda)) \end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp[-2\lambda\tau] = \lambda s_1(\lambda) + s_2(\lambda, \tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda, \tau) = [A_{21} g_{12}(\lambda) + A_{12} g_{21}(\lambda) + g_{12}(\lambda) g_{21}(\lambda)] \exp[-2\lambda\tau]$

$$-A_{11} g_{22}(\lambda) - A_{22} g_{11}(\lambda) - g_{11}(\lambda) g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ =1,\ 2.)\ \ (see(A_1)\ and\ (A_2))$;

$A_7.\ \ \{A_{11} + A_{22} + \| S_1(0) \|\}^2 < 4\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\}$,

$$\| S_1(0) \| = | g_{11}(0)| + | g_{22}(0)|\ ,$$

$$\| S_2(0) \| = | A_{11}\, g_{12}(0)| + | A_{12}\, g_{21}(0)| + | g_{12}(0)\, g_{21}(0)|$$

$$+ | A_{11}\, g_{22}(0)| + | A_{22}\, g_{11}(0)| + | g_{11}(0)\, g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i\ (a,\ t) = \phi(a) \exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda, \tau)$ and $F_2(\lambda, \tau)$ as follows:

$$F_1(\lambda, \tau) = \lambda^2 - \lambda\,(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21} \exp\ [-2\lambda\tau];$$

$$F_2(\lambda, \tau) = \lambda S_1(\lambda) + S_2(\lambda, \tau)\ ,$$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$ , $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda\ \ \mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda, \tau) - P_2(\lambda, \tau) \right|$$

$$\geq \left| F_1(\lambda, 0) \right| - F_2(\lambda, \tau) \Big|$$

$$\geq |\lambda|^2 - |\lambda| \{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0, \tau) \|\}$$

$$\geq \left[ (|\lambda| - P)^2 + (q) \right]^2$$

Where

$$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2\ ,$$

$$q = \left\{ 4\,|\,(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0, T) \| \,| - |A_{11} + A_{22} + \| S_1(0) \| \,|^2 \right\}/2,$$

It will now follow, from (A) and (3.26), $F_1\ (\lambda, \tau) = F_2(\lambda, \tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles,

interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the "Grand design "is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational. It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their

insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection. Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance

upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

## Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this

connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 21. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \left\{ \begin{array}{l} \textit{an individual's contribution} \\ \textit{to population growth} \end{array} \right\} N(t) \qquad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the

population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \qquad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \right\} \left\{ 1 - \frac{N(t-\tau)}{K} \right\}. \qquad (1.3)$$

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s)\, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\left\{ \frac{dN(t)}{dt} = N(t) = \right\} \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s))\, ds \right\} \qquad (1.4)$$

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^t f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that "time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)(dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate}.$$

(1.5)

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\left(N(t-\tau)\right) - m(N(t))$$

(1.6)

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \quad (1.7)$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is

almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates; $b_1$, $m_1$ $(i = 1, 2)$ are continuous with continuous partial derivatives for all $x_1 \geq 0$ $(I = 1, 2)$; also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ $(i, j = 1, 2)$, (2.2)

(xxxi) $b_1(0) = 0$, $m_1(0, x_2) \equiv 0, b_2(0) = 0$, $m_2(x_1, 0) \equiv 0$,

(2.3)For some $x_1^* > 0, x_2^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xxxii) There exist positive constants $\delta_1$, $\delta_2$ such that

$$b_1(\delta_1) - m_1(\delta_1, x_2) < 0,$$

$$b_2(\delta_2) - m_2(x_1, \delta_2) < 0,$$

for $x_1 \geq 0$, $x_2 \geq 0$, (2.5)

(xxxiii) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha,\beta) = 0 = b_2(\beta) - m_2(\alpha,\beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the $(x_1, x_2)$- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\left.\begin{aligned}\frac{dx_1(t)}{dt} &= r_1x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),\\ \frac{dx_2(t)}{dt} &= r_2x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),\end{aligned}\right\}$$

(2.7)

Where $r$, $a_{ij}$, $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i(0) > 0$, $x_2(0) > 0$ does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \neq 0$, $x_2(t) \neq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\left.\begin{aligned} \frac{dX_1(t)}{dt} &= \beta_{11} X_1(t) + \beta_{12} X_2(t), \\ \frac{dX_2(t)}{dt} &= \beta_{21} X_1(t) + \beta_{22} X_2(t), \end{aligned}\right\} \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$a_{11}\alpha > a_{21}\,\beta \; and \; a_{22}\,\beta > a_{12}\,\alpha,$

Which together imply$a_{11}\; a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\; a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1 \; (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1 \; (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J \; (i, \; j = 1, 2; i \neq j \;)$.

## 22. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i, j = 1, 2)$ be a set of nonnegative constants with $\tau = \max \tau_{ij}$ , $i, j = 1, 2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t - \tau_{11})\big) - m_1\big(x_1(t), x_2(t - \tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t - \tau_{22})\big) - m\big(x_1(t - \tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i = 1, 2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s) \; \text{0}, \qquad s \in [-\tau, 0], \qquad i = 1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\left\{\psi_1^{(n)}(s)\right\}, s \in [-\tau, 0], i = 1,2; (n = 1,2, \ldots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s \in [-\tau, 0], \qquad i = 1,2 \tag{3.3}$$

Let $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ be the solution of (3.1) corresponding to the initial condition

$$\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\} = \left\{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\right\}, \quad s \in [-\tau, 0],. \tag{3.4}$$

Consider the solution $\left\{x_1^{(n)}(t), x_2^{(n)}\ (t)\right\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0, \tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i, j = 1, 2))$ for which

$$x_1^{(n)}\ (t^*) = 0\ \ and\ x_1^{(n)}(t)(\ t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \tag{3.5}$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0\ on\ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [\ 2\tau^*, 3\tau^*]\ etc$. Thus it will follow that so long as $(x_1^{(n)}(t),\ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t)\ \geq 0\ x_2^{(n)}(t)\ \geq 0.$$

If we now consider the limits as $n\ \to\ \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t),\ x_2^{(n)}(t)\ \right\} = \{x_1(t), x_2(t)\} \tag{3.2}$$

With $x_1(t)$ , $x_2(t)$ where $\left(x_1(t), x_2(t)\right)$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1\ or\ x_2$ we have

$$\lim_{t\to t_1-} x_i\ (t) = \infty. \tag{3.7}$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i\ (t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1\big(x_1\ (t_2 - \tau_{11})\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1\big(x_1\ (t_2)\big) - m_1\big(x_1(t_2), x_2\ (t_2 - \tau_{12})\big)$$

$$< b_1(\delta_1) - m_1\big(\delta_1, x\ (t_2 - \tau_{12})\big) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ ∎.

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1\ (t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1\ (t - \tau_{11}) - m_{12}Y_2\ (t - \tau_{12}),$$

$$\frac{dY_2\ (t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1\ (t - \tau_{21}) - m_{22}Y_2\ (t - \tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i}\ at\ (\alpha, \beta). \qquad (3.10)$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ ∎ $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\frac{dy_1(t)}{dt} = f_1\big(y_1(t)\big) - g_1\big(y_1(t), y_2(t)\big),$$

$$\frac{dy_2(t)}{dt} = f_2\big(y_1(t)\big) - y_2(t) - g_2(y_2(t)), \qquad (4.1)$$

Where $f_i$ $and$ $g_i$ $i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i$ $and$ $g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i$ $and$ $g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0$ $(i = 1,2)$; $also\ we\ require$

$$\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \text{ 0}; \quad \frac{\partial g_1}{\partial y_2} \text{ 0},$$

$$\frac{\partial f_2}{\partial y_1} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}; \quad \frac{\partial g_2}{\partial y_2} \text{ 0}, \quad \text{Where } y_1 \text{ 0}, y_1 \text{ 0}, \tag{4.2}$$

(xli)

$$f_1(0) = 0, \quad g_1(0, y_2 \equiv 0,$$

$$f_2(0, y_2) = 0 = f_2(y_1, 0); \quad g_2(0) = 0, \tag{4.3}$$

(xlii) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xliii) There exists a point $(\alpha^*\beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$f_1(\alpha^*) - g_1(\alpha^*, \beta^*) = 0,$$

$$f_2(\alpha^*, \beta^*) - g_2(\beta^*) = 0, \tag{4.5}$$

(xliv) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \text{ 0},$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \tag{4.6}$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_{l,}$ $a_{ij}$ $(i, j = 1,2)$are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t$ 0 for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all t let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l\ (0)$ satisfy $0< y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0)$ $\eta_l and\ y_2(0)$ $\eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t$ ). Suppose $y_l(t)$ does not exist for all $t$ ; then for some $t_l < \infty\ lim_{t\to t_l-} y_l(t) = \infty$and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t$ ; if $y_2(t)$ is not defined for all $t$ , then for some $t_3$ we will have $lim_{t-t_3-} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2)\ f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t$ .

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_1} U_1(t) - \frac{\partial g_1}{\partial y_2} U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1} U_1(t) - \frac{\partial f_2}{\partial y_2} U_2(t) - \frac{\partial g_2}{\partial y_2} U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{vmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{vmatrix} = 0,$$

$$\text{Where} \quad \left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\} \text{ at } (\alpha^*, \beta^*) \quad (4.8)$$

*( i,j =1, 2 )*

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\lambda_1 + \lambda_2 = (f_{11} - g_{11}) + (f_{22} - g_{22}),$$
$$\lambda_1\lambda_2 = (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\frac{\partial g_1}{\partial y_1} > \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1},$$
$$\frac{\partial g_2}{\partial y_2} > \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \quad \text{Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 15. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$
$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

(5.2)

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds],$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_2(t)N_1(t), \end{aligned}\right\} \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t), \\ \frac{dN_2(t)}{dt} &= -bN_2(t) + \beta N_1(t)N_2(t). \end{aligned}\right\} \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\left.\begin{aligned} \frac{dN_1(t)}{dt} &= a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t), \\ \frac{dN_2(t)}{dt} &= a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \end{aligned}\right\} \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the

case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$

$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^\infty \delta(a)N_2(t-a)e^{-d*a}\,da, \quad (5.6)$$

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$

$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)), \quad (5.7)$$

Where $\tau_{ij}$ $(i,j = 1,2)$ are nonnegative constants with $\tau = \max \tau_{ij}, i,j = 1,2$; the birth and death rate functions $f_1, f_2, g_1, g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s) = \phi_1(s) > 0, \quad s \in [-\tau, 0].$$

$$N_1(s) = \phi_1(s) > 0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1} > 0, \frac{\partial f_2}{\partial N_2} > 0, f_2(N_2, N_2) > 0 \; whenever \; N_1 > 0, \; N_2 > 0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s) > 0$ and $N_2(s) > 0$ for $s \in [-\tau, 0]$, solution of (5.7) exist for all $t \geq 0$ and remain nonnegative for all $t \geq 0$. also $(\alpha^*, \beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*, \beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange

confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state* $(\alpha^*, \beta^*)$ *is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\left.\begin{aligned} \frac{dY_1(t)}{dt} &= -m_{11}\,Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}), \\ \frac{dY_2(t)}{dt} &= -m_{22}\,Y_1(t) + m_{21}Y_1(t-\tau_{21}) - b_{22}Y_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.1)$$

Where $m_{1J}$ , $b_{i1}$ $(i, j = 1,2)$ are positive constants while $\tau_{1J}(i, j = 1,2)$ are non negative with $\tau = \max(\tau_{1J})(i, j = 1,2$.as is shown in the appendix, the following system dominates (6.1*)*

$$\left.\begin{aligned} \frac{dZ_1(t)}{dt} &= -m_{11}\,Z_1(t) + b_{11}Z_1(t-\tau_{11}) - m_{12}Z_2(t-\tau_{12}), \\ \frac{dZ_2(t)}{dt} &= -m_{22}\,Z_1(t) + m_{21}Z_1(t-\tau_{21}) - b_{22}Z_2(t-\tau_{22}), \end{aligned}\right\} \quad t > 0, \quad (6.2)$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \;\; for\; s \le 0 \;\; for\; which\; Y_i(s)\; is\; defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$\left.\begin{aligned} Z &= (Z_1, Z_2)^{\tau}, \qquad \|Z\| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t) \\ B &= \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad \|B\| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \\ A &= \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}. \end{aligned}\right\} \quad (6.3)$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_J [b_{jJ} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jJ} | ]$ $B=(b_{1J})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \leq \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm $\|B\|$ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \; or \; -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix} \frac{\lambda+m_{11}}{0} & \frac{0}{\lambda+22} \end{bmatrix} - \begin{matrix} b_{11}^{e-\lambda T_{11}} & -m_{12}^{e-\lambda T_{12}} \\ -m_{21}^{e-\lambda T_{21}} & b_{22}^{e-\lambda T_{22}} \end{matrix}$$

$=\lambda I - A - B(\lambda,T_{iJ})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\alpha < 0$.

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\alpha > 0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\alpha$ in (6.8)and define

$G(z-\alpha,T_{IJ})=M(z)=(z-a)I-A-B(z-\alpha,T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$\mathrm{Re}(\lambda) < -a$

is the following :

$-\mu(aI + A) > \|B_1(a,T_{ij})\|$,

Where

$B_1(a,T_{ij}) = \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{12}} \\ -m_{21}\, e^{-\lambda T_{21}} & b_{22}\, e^{-\lambda T_{22}} \end{pmatrix}$

Since $\mu(\mathfrak{a}\, I + A) = \mathfrak{a} + \boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\mathfrak{a} + \mu(A) + \|B_I(\mathfrak{a}, T_{IJ})\| < 0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a} f(\mathfrak{a}, T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as\ \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$f(0, T_{Ij}) < 0 \;\; for\ \mathfrak{a} \in [0, \mathfrak{a}^{*})$

And hence

$f(\mathfrak{a}, T_{Ij}) = 0$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root of$\max \begin{pmatrix} b_{11}\, e^{-\lambda T_{11}} & -m_{12}\, e^{-\lambda T_{21}} \\ b_{22}\, e^{-\lambda T_{22}} & -m_{21}\, e^{-\lambda T_{12}} \end{pmatrix} + \mathfrak{a} - \min \begin{pmatrix} m_{11} & \\ & m_{22} \end{pmatrix} = 0$

Where the maximum in (6.18) is taken over$b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij}\,(i, = j\ 1{,}2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}.$ (6.19)

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^*$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^*$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that

delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement “all animal populations have the reproductive capacity to increase without limit, yet none di” and in his concluding remarks writes “ this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated” More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution “Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist.” In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render

intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self –regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_I(x) - m_I(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_2(x,y) - m_2(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^* y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \succ \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^* y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau$ = max $\tau_{ij}$, $i,j$ = 1,2. Then

$$\left.\begin{aligned} \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{11}\left|Y_1(\mathrm{t})\right| + b_{11}\left|Y_1(\mathrm{t}-\tau_{11})\right| + m_{12}\ Y_2(\mathrm{t}-\tau_{12})\Big|, \\ \frac{d\left|Y_1(\mathrm{t})\right|}{dt} &\le -m_{22}\left|Y_2(\mathrm{t})\right| + m_{21}\left|Y_1(\mathrm{t}-\tau_{21})\right| + b_{22}\ \left|Y_2(\mathrm{t}-\tau_{21})\right|, \end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\ \left|\ Y_1(t)\ \right| = \left\{\begin{array}{l} \frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) \ge 0, \\ \frac{dY_1(\mathrm{t})}{dt}\ if\ \ Y_1(\mathrm{t}) < 0, \end{array}\right\}\ i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned} \frac{dZ_1(\mathrm{t})}{dt} &= -m_{11}\ Z_1(\mathrm{t}) + b_{11}Z_1(\mathrm{t}-\tau_{11}) + m_{12}\ Z_2(\mathrm{t}-\tau_{12}), \\ \frac{dZ_2(\mathrm{t})}{dt} &= -m_{22}Z_2(\mathrm{t}) + m_{21}Z_1(\mathrm{t}-\tau_{21}) + b_{22}\ Z_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$$Z_1(\mathrm{s}) > \left|Y_1(s)\right|,\ \ s \in [-\tau, 0],\ i = 1,2. \qquad (A.5)$$

Then

$$\left|Y_1(t)\right| < Z_1(\mathrm{t})\ \text{for t} \ge 0 \quad \text{and} \quad i = 1,2. \qquad (A.6)$$

PROOF. Let $U_1(\mathrm{t}) = Z_i(\mathrm{t}) - \left|Y_1(s)\right|$, $i = 1,2.$ $\mathrm{t} \ge -\tau.$ Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(\mathrm{t})}{dt} &\ge -m_{11}\ U_1(\mathrm{t}) + b_{11}U_1(\mathrm{t}-\tau_{11}) + m_{12}\ U_2(\mathrm{t}-\tau_{12}), \\ \frac{dU_2(\mathrm{t})}{dt} &\ge -m_{22}U_2(\mathrm{t}) + m_{21}U_1(\mathrm{t}-\tau_{21}) + b_{22}\ U_2(\mathrm{t}-\tau_{22}), \end{aligned}\right\} t > 0, \quad (A.7)$$

With

$$U_1(s) = Z_i(\mathrm{s}) - \left|Y_i(s)\right| > 0, \quad i = 1,2, s \in [-\tau, 0]. \qquad (A.8)$$

Consider (A.7)-(A.8) first on the interval $[\ \tau, 0]$ : It will follow from (A.7) that $U_1(\mathrm{t}) \ge U_1(0)\ \exp\ [-m_{11}\mathrm{t}]$

$$+\int_0^t \left[b_{11}U_1(\mathrm{s}-\tau_{11}) + m_{12}\ U_2(\mathrm{s}-\tau_{12})\right] \exp\left[-m_{11}(\mathrm{t}-\mathrm{s})\mathrm{ds} > 0,\right] \qquad (A.9)$$

For all $\mathrm{t} \in [0, \tau^*]$ since $U_i(\mathrm{s}) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(\mathrm{t}) > 0$ for $\mathrm{t} \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(\mathrm{t}) > 0, U_2(\mathrm{t}) > 0$ for $\mathrm{t} \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$$H(Z_1,Z_2)(t)=Z_1(t)+Z_2(t)$$

$$+\int_{1-T}^{t} b_{11}\, Z_1(s)ds+\int_{1-T}^{t} m_{12}\, Z_2(s)ds$$

$$+\int_{1-T}^{t} m_{21}\, Z_1(s)ds+\int_{1-T}^{t} b_{22}\, Z_{22}(s)ds$$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$$0 \leq |\, Y_{1,}(t)\,| \leq Z_1(t) \to 0 \text{ as } t \to \infty,$$

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}+\frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$=-[\, Z_1(t)\{ m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{ m_{22} - (\, b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t),Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t) \to 0$ as $t \to \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t) \to 0, Z_2(t) \to 0$ as $t \to \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s \in [0,\infty)$, $i, j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$$\int_0^\infty sk_{ij}(s)ds < \infty;\ \int_0^\infty k_{ij}(s)ds=1,\ I,j=1,2.$$

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where $m_{ij}, b_{ij}$ $(i, j = 1, 2)$ are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \leq Z_i(t) \to \infty \text{ as } t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\begin{cases} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{cases}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

| |
|---|
| $+m_{21}\int_0^{\infty} k_{12}(u)(\int_{t-u}^{t} Z_1(s)ds)du$ |
| $+b_{22}\int_0^{\infty} k_{22}(u)(\int_{t-u}^{t} Z_2(s)ds)du$ |
| Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12)) |
| $(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]$ |
| $< 0$(using (A.18)) for $Z_1(t) > 0$ $and$ $Z_2(t) > 0$. |
| By Lemma 6 below the result will follow |
| Lemma 6. Let H(t)=H$(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose |
| $m_{11} > b_{11} + m_{21}$ and $m_{22} > b_{22}$ $\|m_{12}$ |
| Then |
| $Z_1(t) + Z_2(t) \to 0$ast $\to \infty$. |
| PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence |
| $\{t_n\} \to \infty$ as n $\to \infty$ such that |
| $[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon$ $(A.22)$ |
| For some sufficiently small positive number $\varepsilon$ and hence |
| $\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, ....$ |
| By the continuity and differentiability of H with respect to t, |
| $H(t_n - \varepsilon) - H(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$ |
| |
| If $\varepsilon$ is small we will have |
| $= \frac{dH(t^*)}{dt}\varepsilon/2$ |
| So that |
| $H(t_n) \le H(t_n - \varepsilon) - \varepsilon^2/2$ |
| $\le H(t_{n-1}) - \varepsilon^2/2$ |

| |
|---|
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$ |
| $\leq H(t_{n-2}) - 2(\varepsilon^2/2)$: |
| $\leq H(t_0) - n(\varepsilon^2/2)$: $t_0=0$. |
| (A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$) |
| We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$ |
| Where A and B are defined as in (6.3). |
| We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems |

**Gopalsamy**

## Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of

foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 31. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ $and$ $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single

species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 32. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0,$$

**(2.1)**

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

$(H_2)\ f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

$(H_3)$ There exist positive constants $b_{il},\ b_i^u,\ a_{il},\ a_i^u\ (l = 1, 2)$ such that

$$b_{il} - a_{il}x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u/a_i^u, \quad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} |h(t)|, h \in B_\omega^\tau\right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| > b_l = \min(b_{1l}/a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| > b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1$ *in* $\bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{cases} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{cases} \quad \text{for t> } 0, \quad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t - s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+\int_0^1 G_{12}\,(m,0,t-s)g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds,$$

$$N_2(t,x_1,x_2) = \int_0^{\omega} G_{22}\,(m,\omega,t-s)g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds \qquad \textbf{(2.6)}$$

$$+\int_0^1 G_{21}(m,0,t-s)g_2\,\big(m,s,x_1(s-\tau_{21}),x_2(s)\big)ds,$$

For $t > 0$, with

$$g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big) = mx_1(s) + x_1(s)f_1\big(s,x_1(s),x_2(s-\tau_{12})\big),$$

$$g_2\big(m,s,x_1(s-\tau_{21}),x_2(s)\big) = mx_2(s) + x_2\,(s)f_2\big(s,x_1(s-\tau_{21})x_2,(s)\big), \qquad \textbf{(2.7)}$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}\,(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

$$G_{ij}\,(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2, \qquad \textbf{(2.8)}$$

$$N(x_1\,,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1\,, x_2) \\ N_2(n\omega + t, x_1\,, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

$m$ being a suitably selected positive constant (more will be said below) while $n$ is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ ∈ and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ ∈ then $N\,(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ ∈ $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t),x_2(t))$ , $t$ ∈ $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = \big(x_1(t),x_2(t)\big) \quad and\ (x_1,x_2)\ \in K\ \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1, x_2)$ assuming that $N(x_1, x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1, x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1, x_2)(t)$ . It will follow from (2.9) and the assumed periodicity of $N(x_1, x_2)$ $that$ $N(x_1, x_2)$ is continuous

For $t$ $\in -\tau$. Now we will verify the periodicity of $N(x_1, x_2)$ in $t$. A simple computation leads to

$$N_1(t + w, x_1, x_2) - N_1(t, x_1, x_2) = e^{-mt} F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m, \omega, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{\omega} G_{12}(m, 0, t - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_{\omega}^{\omega+t} G_{12}(m, 0, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$- \int_0^{\omega} G_{11}(m, \omega - s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

$$+ \int_0^{t} G_{12}(m, 0, -s) g_1\big(m, s, x_1(s), x_2(s - \tau_{12})\big) ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B$, thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m, t, x_1(t), x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m, t, x_1(t-\tau_{21}), x_2(t)\right),$$

For any arbitrary real constant *m* and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

(xxxi) Either $x_1, x_2$ or both $x_1$ *and* $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ *and* $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ *and* $x_2$ cannot be periodic.

(xxxii) While $(x_1, x_2) \in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ *or* $dx_2(t^*)/dt < 0$, which will imply that the values of $x_1(t^*)$ *or* $x_2(t^*)$ cannot be repeated from below.

(xxxiii) Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ *or* $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above.

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1)$ *and* $\lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}$. *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq x_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq x_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda x_2(t) f_2(t, x_1(t - \tau_{21}), x_2(t)) + x_2(t) m(\lambda - 1)$$

For all $t \geq 0$ such that $x_1(t) \geq b_1^u/a_1^u$ *or* $x_2(t) \geq b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ *or* $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \leq x_1^* < b_1^u/a_1^u$, $x_2(t) \leq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) \geq \int_0^\omega G_{11}(m, \omega, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}(m, 0, t - s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds.$$

**(2.14)**

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for $m$ for the case $N_1(t, x_1, x_2)$ 0 and then choose $m = \max(m_1, m_2)$. Thus we have

$$N: K \cap (G_2 / G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \to B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} for\ t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\ t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let $N: K \cap (H_2 / H_1) \to K$ be completely continuous and which satisfies the following conditions:*

***(u)*** *There exists $(z_1, z_2) \in K \subset B$, $\|(z_1, z_2)\| = 1$ and a $\mu$ satisfying*

$$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(v) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that $f_1, f_2$, of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N\ (x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 33. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq x1,x2 \leq C*$ *where* $C*$*some positive constant is. Assume the following:*

(ee) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(ff) N is monotonic nondecreasing on $K^*$.

(gg) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda (1 + \eta)\ N(x_1, x_2)$. Then N has at most one fixed point different from (0,0) in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \leq \|x_1\| \leq b_1^u/a_1^u,\ b_{2l}/a_{2l} \leq \|x_2\| \leq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2/G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2/G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \qquad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$x\{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \qquad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2\ (t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \text{ and } 0 \not\equiv x_2(s) - y_2(s);$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2)\ - N_2\ (t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$$N\,(x_1,\ x_2)\ - N\ (y_1, y_2) \text{ for } (x_1 - y_1,\ x_2 - y_2) \in\ K^* - \{(0,0)\}.$$

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2)\ \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0< \lambda <1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N\,(\lambda x_1,\ \lambda x_2) \geq \lambda(1 + \eta)\ N\,(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1\ (t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1\ (t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m, \omega, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m, 0, t - s) g_1\big(s, x_1(s),\ x_2(s - \tau_{12})\big) H_1(s, \lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s, \lambda,\ x_1,\ x_2) = \frac{m\ x_1(s) + x_1(s) f_1\big(s, \lambda\ x_1(s), \lambda\ x_2(s - \tau_{12})\big)}{m\ x_1(s) + x_1(s) f_1\big(s, x_1(s), x_2(s - \tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$H_1(s, \lambda, x_1, x_2) \leq \eta_1$ for $0 < \lambda < 1, s > 0, (x_1, x_2) \in K^*$, **(3.8)**

And hence

$N_1(t, \lambda x_1, \lambda x_2) \leq \lambda(1 + \eta_1) N_1(t, x_1, x_2)$.

A similar argument will show that there exists an $\eta_2 > 0$ such that

$N_2(t, \lambda x_1, \lambda x_2) \leq \lambda(1 + \eta_2) N_2(t, x_1, x_2)$,

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique $\omega$ - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ –}\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ –}\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ –}\tau, \qquad \textbf{(4.1)}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t) X_1(t) - a_{12}(t) x_2^*(t - \tau_{12})\, X_2(t - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t) x_1^*(t - \tau_{21}) x_1(t - \tau_{21}) - a_{22}(t) x_2^*(t) x_2(t), \qquad \textbf{(4.2)}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t}-\tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t-\tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(\mathrm{t}))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \qquad \textbf{(4.4)}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(\mathrm{t}-\tau_{12})|X_2(\mathrm{t}-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(\mathrm{t}-\tau_{21})|X_1(\mathrm{t}-\tau_{12})| - b_{22}x_2^*(t)|X_2(\mathrm{t})|, \end{aligned} \qquad \textbf{(4.5)}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \qquad \textbf{(4.6)}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x^*_1(t), x^*_2(t), t \in -\tau\}$ of $(2.1)-(2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12},$$

**(4.7)**

*Where the constants* $b_{ij}(i,j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x^*_2(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x^*_1(u)|X_{1,}(u)|du. \quad \textbf{(4.8)}$$

Since $x^*_1, x^*_2$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21}) x^*_1(t)|X_{1,}(t)| + (|b_{22}| - b_{22}) x^*_2(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}$. $Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(|b_{11}| - b_{21}) x^*_1(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x^*_2(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq -\int_{t_m-\varepsilon}^{t_m} \frac{\mathrm{dv}(\mathrm{T})}{\mathrm{dT}}\,\mathrm{d}\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$$v(t_m - \varepsilon) - \mathrm{v}(t_m) \geq \varepsilon^2/2 \text{ and hence}$$

$$v(t_m) \geq v(t_m - \varepsilon) - (\varepsilon^2/2)$$

$$\geq v(t_m - 1)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_m - 2)\text{-}(\varepsilon^2/2)$$

$$\geq v(t_0)\text{-m}\,(\varepsilon^2/2)$$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t \geq 0$.

this contradiction shows that $|X_{1,}(\mathrm{t})| + |X_2(\mathrm{t})| \to 0$ as $\mathrm{t} \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1\big(\mathrm{t}, x_1(\mathrm{t}), x_2(\mathrm{t} - T_{12})\big)1 \quad \left\{ \frac{x_1(\mathrm{t}) + a x_2(\mathrm{t} - T_{12})}{K_1(\mathrm{t})} \right\}$$

$$f_2(\mathrm{t}, x_1(\mathrm{t} - {}_{21})x_2(\mathrm{t})) = 1 - \left\{ \frac{\beta x_1(\mathrm{t} - T_{21}) + x_2(\mathrm{t})}{K_2(\mathrm{t})} \right\}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\operatorname{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(t)\right)\Big| = \min_{t}, \beta/K_2(t)$$

$$\max_{t>T}\{-a_{22}(t)\}\Big| = \max_{t>T}(1/K_2(t)) > \min_{t>T}\left(-a_{12}(t)\right)\Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality $0< \alpha,\beta<1$, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_1\left(t,x_1(t),x_2(t-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(t)}\frac{dx_1(t)}{dt}=f_2(t,x_1(t-T_{21}),x_2(t))$$

Since $\left(x_1(t),x_2(t)\right)\rightarrow\left(x_1^*(t),x_2^*(t)\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient

condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(\mathrm{t})}{dt} x_1(\mathrm{t})[1 - x_1(\mathrm{t}) + 0.4x_2(\mathrm{t} - \mathcal{T})/k_1(\mathrm{t})]$$

$$\frac{dx_2(\mathrm{t})}{dt} x_2(\mathrm{t})[1 - \{0.8\, x_1(\mathrm{t} - \mathcal{T}) + x_2(\mathrm{t})\}/k_2(\mathrm{t})]$$

Where $k_1(\mathrm{t}) = 20 + \sin\ (\pi\mathrm{t})$ and $k_2(\mathrm{t})$=2.5+sin ($\pi$t)

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There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations

may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of “differance” (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the

bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

**Abstract**

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent

oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 21. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

$t > t_0;$

$t_0 \in (-\infty, \infty); i = 1, 2, \ldots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \ldots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \ldots, n)$ by the following:

$$\inf_{t \in R} b_l(t) = b_i^l; \qquad \sup_{t \in R} b_l(t) = b_l^u, \qquad i, j = 1, 2, \ldots, n.$$

$$\inf_{t \in R} a_{lj}(t) = a_{lj}^l; \qquad \sup_{t \in R} a_{lj}(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u (b_j^u / a_{ij}^l). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l \{b_l^u - a_{ll}^l x_i\}; \qquad t > t_0, i = 1, 2, \ldots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \ldots, n; t \geq t_0. \tag{1.5}$$

(1.11) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \ldots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \; b_i^l - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right)\Big)(1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \; x_i^l \; for \; t \; t_0, \; i = 1, 2, \ldots, n. \qquad (1.7)$$

22. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}_n | x_* \; x_l \; x^*; i = 1, 2, \ldots, n\}$$

(1.8)

*Where*

$$x_* = \min_{1} x_l^l; \qquad x_*$$

$$= \min_{1} x_i^u \qquad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(221) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(222) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(223) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(224) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(225) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(226) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of Southampton, UK. "This is a cool, new idea, and I'm up for trying to see it in experiments," he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(227) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team's work. Einstein's theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth's surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(228) Because of gravity's effect on (e&eb) space-time, Pikovski's team realised that variance in a molecule's position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(229) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(230) "In most situations decoherence is due to (e) something external; here it's as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself," adds Pikovski.

**A practical limit**

(231) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(232) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(233) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

| |
|---|
| (234) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below). |
| (235) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints) |
| (236) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain. |
| (237) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy. |
| (238) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**. |
| (239) If gravity's effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy. |
| (240) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (241) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (242) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like

Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$\left||f\ (t+\ \tau,x) - \ f(t,x)|\right| = \sum_{l=1}^{m} |f_l\ (t+\ \tau,x) = \ f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1} \int_0^T f(t,x)\exp(-i\,\lambda_j\,t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j\ \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or$\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times$*

$S, \{g(t+,x)\}$ *is also uniformly convergent on* $\boldsymbol{R} \times S$ *then the module of* $g(t+,x)$ *is contained in the module of* $f(t+,x)$.

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters* $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ *of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number* $\mu$ *such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J \neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i = 1,2,\dots,n. \tag{2.3}$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say* $w(t) = w_1(t), \dots, w_n(t), t \in \boldsymbol{R}$, *whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l \left\{ b_l(t) - \sum_{j=1}^{n} a_{lJ}(t) x_J \right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

Uniformly in $(x_1, \dots, x_n) \in S$ for $t \in (-\infty, \infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0), \dots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta, \infty), i = 1, 2, \dots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let $U$ be any compact subset of $\beta, \infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \tag{2.7}$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*)$$

(2.8)

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1, (s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \geq t_0, \tag{2.9}$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1, (t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \geq v(s)$$

$$\leq (1/x^*)\sum_{i=1}^{n}|x_l(s)-x_1(s+\tau_m+\tau_k)|. \qquad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s)=\sum_{i=1}^{n}\{sign[x_i(s)-x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds}-\frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\leq\sum_{l=1}^{n}|b_l(s)-b_l(s+\tau_m-\tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s)-x_i(s+\tau_m-\tau_k)|+\sum_{\substack{j=1\\ j\neq l}}^{n}a_{ij}(s)|x_j(s)-x_j(s+\tau_m-\tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(s)|a_{ij}(s+\tau_m-\tau_k)|\,x_j(s+\tau_m-\tau_k) \qquad (2.11)$$

$$\leq-\mu x_*v(x(s),x(s+\tau_m-\tau_k))+\sum_{l=1}^{n}|b_l(s)-b_i(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|. \qquad (2.12)$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n}|b_l(s)-b_1(s+\tau_m-\tau_k)|+x^*\sum_{i=1}^{n}\sum_{J=1}^{n}|a_{iJ}(s)-a_{iJ}(s+\tau_m-\tau_k)|.$$

$$=\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_i(\sigma+\tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{ij}\ (\sigma+\tau_k) - a_{ij}\,(\sigma+\tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k), x(t+\tau_m)\big) \le v\big(x(t_0), x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$$\le [(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] \ + (\varepsilon/2x^*)$$

$$< (\varepsilon/x^*) \text{ On using (2.8), (2.10)} \qquad (2.14)$$

For $m \ge k \ge n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $v$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \le x_i(t) \le x^*, i = 1,2,\dots,n; t \ge t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \ge k \ge n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t), \dots, w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*, x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*, x^*], i = 1,2,\dots,n;$ for $t \ge t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \dots, x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt} x_i(t+\tau_k) - \frac{d}{dt} x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n\left(t+\tau_k\right)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_m), \dots, x_n(t+\tau_m)\right)|$$

Where $f_l\ (i = 1, 2, \dots, n)$ are as in (2.4). Since $\mathrm{x}(\mathrm{t}+\tau_k) = x_1(t+\tau_k), \dots, x_n(t+\tau_k) \in S$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ [illegible] $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)$$

$$-f_1\left(t+\tau_m, x_1(t+\tau_k), \dots, x_n(t+\tau_k)\right)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t+\tau_k) \in S$ and $x_l(t+\tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ [illegible] $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t+\tau_m, x_1(t+\ \tau_k), \dots, x_n(t+\ \tau_k)\right)$$

$$-f_1\left(t+\tau_m, x(t+\tau_m), \dots, x_n(t+\tau_m)\right)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case

concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \neq k$ $n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} |\frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ((t + \tau_m)| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty}\frac{d}{dt}x_1(t+\tau_k) \quad exists\ for\ i=1,2,\dots,n$$

Uniformly on all compact subsets of $(-\infty,\infty)$. Thus we have

$$\lim_{k\to\infty}\lim_{k\to\infty}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h}$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{x_l(t+\tau_k+h)-x_l(t+\tau_k)}{h} \qquad (2.22)$$

$$=\lim_{h\to 0}\frac{w_1(t+h)-w_1(t)}{h}, \quad i=1,2,\dots,n,$$

Showing that (since the left side of (2.22) exists) $w_i(i=1,2,\dots,n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt}=\lim_{k\to\infty}[\,f_i\left(t+\tau_k,x_1(t+\tau_k),\dots,x_n\,(t+\tau_k)\right)$$

$$-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big)$$

$$+-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big)]$$

$$=-f_l\big(t+\tau_k,w_1(t),\dots,w_n(t)\big); \quad i=1,2,\dots,n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1,f_2,\dots,f_n$ on $R\times S$ and hence $w(t)=\big(w_1(t),\dots,w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t)=\big(w_1(t),\dots,w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t,x_1,\dots,x_n),f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$ for $(x_1,x_2\dots,x_n)\in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k,x_1,\dots,x_n),f_2(t,+\tau_k,x_1,\dots,x_n),\dots,f_n(t,+\tau_k,x_1,\dots,x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1(t, +\tau_k) \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l(t, +\tau_k) \to w_i(t + \tau), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ [illegible] $n0(\varepsilon)$ then

$$\sum_{l=1}^{n} |b_l(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t + \tau_k) - a_{ij}(t + \tau_m)| < \left(\frac{\varepsilon}{4x^*}\right) \mu x_* \qquad (2.25)$$

$$n(2x^*/x_*) \exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s + \tau_m - \tau_k)\big)$ for $s \in [t, t + \tau_k]$ defined by

$$v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s + \tau_m - \tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s + \tau_m - \tau_k)\big) = \sum_{i=1}^{n} \{\text{sign}[w_l(s) - w_i(s + \tau_m - \tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds} \cdot \frac{1}{w_l(s)} - \frac{dw_l(s + \tau_m - \tau_k)}{ds} \frac{1}{w_l(s + \tau_m - \tau_k)}\right\}$$

$$-\mu \sum_{l=1}^{n} |w_l(s) - w_l(s + \tau_m - \tau_k)| + \sum_{l=1}^{n} |b_i(s) - b_l(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{l=1}^{n} \sum_{J=1}^{n} \left|a_{lJ}(s) - a_{lJ}(s + \tau_m - \tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s)-\ b_l(s+\ \tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s)-a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k=\sigma$ then we have from (2.29),

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right| \qquad (2.30)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_*)/(4x^*)]\ +[(\varepsilon\mu x_*)/(4x^*)]$$

(2.31)

Integrating (2.31) over $(t, t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called "shrinking operations"), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang's model or to Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during $0 \le t < 1$. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$$\square\ e^{-\mu x * T_k}\, v\big(w(t), w(t + T_m - T_k)\big) + (\varepsilon/2x^{*})$$

$\leq e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t + T_m - T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$\leq n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \leq \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m)| < \varepsilon\ for\ m \geq k \geq n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty\ as\ k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\ as\ k \to \infty\ on\ \mathbf{R}$ follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T > 0$. *Then if* (1.2), (1.3)*and* (2.3)*hold the system* (1.1)*will have a periodic solution* $w(t) = \{w,(t), \ldots\ldots, w_n(t)\}$*of period* $T$ *such that*

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots)$*and note that*

$$x_1(t+KT) - w_1(t) \to 0\ as\ k\ \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) -\ w_1(t) \to 0\ \ as\ \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0\ \ as\ k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \dots, w_n(t)\} of (1.1) is\ globally\ asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \dots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \dots, n$ is defined for all t□0 and satisfies

$$\lim_{t\to\infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \dots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu\ Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= { g|g:**R**→ $\boldsymbol{R}\ and\ g(t) \lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}\ (t)y_1(t)\right\},$$

$i=1,2,\ldots,n;\ t>t_0 \in R,$

Where

$$B_i \in H(b_i);\left(A_{ij}\right), \qquad \mathrm{i,j} = 1,2,\ldots,\mathrm{n},$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{iJ}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\ldots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\ldots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\ldots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \quad i = 1,2,\ldots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1(1) - B_1(t)| < \delta \qquad = \sum_{i=1}^{n} |x_1(t) - y_i(t)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{aligned} &\beta_1 \leq x_i(t) \leq \beta_2, \\ &\beta_1 \leq y_i(t) \leq \beta_2, \end{aligned} \quad i = 1,2,\dots,n; t \geq t_0 \qquad (3.7)$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \qquad (3.8)$$

It is easy to see that

$$\frac{1}{\beta_2}\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1}\sum_{i=1}^{n}|x_i(t)-y_i(t)|,\ t\geq t_0\in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\left(x(t),y(t)\right)\leq-\mu\sum_{i=1}^{n}|x_i(t)-y_i(t)|+\sum_{i=1}^{n}|b_i(t)-B_i(t)|$$

$$+\beta_2\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}(t)-A_{ij}(t)|;\quad t\geq t_0,$$

$$\leq-\mu\,\beta_1\,V\left(x(t),y(t)\right)+\sum_{i=1}^{n}|b_i(t)-B_i(t)|$$

$$+\beta_2\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}(t)-A_{ij}(t)|;\quad t\geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0,t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2}\sum_{i=1}^{n}|x_i(t)-y_i(t)|\leq V(x(t),y(t))\leq V(x(t_0),y(t_0))$$

$$+\left[\sum_{i=1}^{n}\sup_{t\in R}|b_i(t)-B_i(t)|+\beta_2\sum_{i=1}^{n}\sum_{i=1}^{n}\sup_{t\in R}|a_{ij}(t)-A_{ij}(t)|\right]\frac{1}{\mu\,\beta_1},$$

$$t\geq t_0,$$

And hence

$$\sum_{i=1}^{n} | x_1(t) - y_1(t) | \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} | x_1(t_0) - y_1(t_0) |$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[ \sum_{i=1}^{n} \sup_{t \in R} | b_i(t) - B_i(t) |$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying "The cause is by cause," which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} | a_{ij}(t) - A_{ij}(t) | \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Prescientiial Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is**

**to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ ?0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ ?\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ ?\infty,$$

$$\rho_1(t,0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1\left[\frac{dQ_2}{dt}\right]_1 - x_2\left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_1 - x_4\left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1 P_1) + (f_2)_1(Q_1))(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \leq \rho_1(t, a_1) \leq \rho_1^*(t, a_1),$$

$$0 \leq Q_1(t) \leq Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\,\rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\,\rho_1^*(t, a_1)da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt}$$

$$= -b_1\, Q_1^* + g_1(P_1^*)Q_1^*, \qquad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*\ (t, a_1)da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \qquad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \qquad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\, da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\, da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1} \rho_1(t,a_1)\, da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1 (P_0) a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t,a_1) = \rho(t,a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1(t,a_1)da_1 = \int_0^\infty \rho_1(t,a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))u_1 - (\rho_0)_1(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0^{'}Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1 + (h_{a_1}^{'})(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^{\infty} u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^{\infty} \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t, a_1) = \tilde{u}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_1}\, da_1$. Then,

A solution of the first differential equation for $\tilde{u}(a_1)$ gives,

$$\tilde{u}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{u}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})}\tau_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^\infty \beta_1(a_1)e^{-\gamma_1 a_1 - (f_{10})_1 a_1 - \int_0^a \mu_1(\tau_1)d\tau}\int_0^a (\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1 + \int_0^\tau \mu_1(s_1)ds}d\tau_1\, da_1$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1 - \int_0^\infty e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}\int_0^a e^{f_{10}\tau + \int_0^\tau \mu(s)ds + y\tau}\alpha_1(\tau)d\tau\, da\right)w = 0. \quad (23)$$

Define $Y(\gamma, a) = e^{-ya - f_{10}a - \int_0^a \mu(\tau)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta\,(a)Y(\gamma, a)da\right)u(0)$$

$$+w\int_0^\infty \beta(a)Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da = 0, (24)$$

$$\left(\int_0^\infty Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^\infty Y(\gamma,a)\int_0^a \alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w = 0.$$

Equation (20) has a solution of the form of $u(t,a) = u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I(y,a) = \int_0^a \rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^l\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^l\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right) = 0. \quad (25)$$

Since

$$\rho_0(a) = \rho_0(0)e^{-\int_0^a \mu(\tau)d\tau - f_{10}a}$$

$$I(\gamma,a) = \int_0^a \rho_0(\tau)e^{\gamma\tau + f_{10}\tau + \int_0^\tau \mu(s)ds}d\tau$$

$$= \frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a) = e^{-f_{10}a - \int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1, a_1) = E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma}\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1, a_1)I(\gamma_1, a_1)da = \frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^1]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\ (\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1)$$

$$+\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1 + \frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(\,a_1)da_1 - \int_0^\infty E_1(\,a_1)e^{-\gamma_1 a_1}da_1\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1 + \frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(\,a_1)da_1) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1 = 1 - \int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1 \neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\ \int_0^a \alpha_1\,(\tau_1)Y^{-1}(\gamma_1, \tau_1)d\tau_1 da_1 > 0$. Therefore, the characteristic solution is

$$\gamma_1 = -(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(\,a_1)da_1.$$

Since $f_1'(P_0)_1 > 0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1), 0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*) \int_0^\infty \beta_1(a_1) p_1^*(a_J) p_1^*(a_1) da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0) e^{\int_0^a \mu_1(\tau_1) d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)(\,[f_2^{'}]_1(z_1)\,)\int_0^\infty a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1}da_{\square}$$

$-h^{'}(z_1)\int_0^\infty \beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$H_1(0) = 1 - \int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1}da_1 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^-(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y}1^a da_1}$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R\big(f_1(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider

perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$M_1(t) \equiv \int_0^\infty m_1(t, a_1)da_1 = P_1(t) - P_1^*,$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - \left[p^*\right]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1)m_1(t,a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1}\, p_1^*(0)W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $\left[f_{1*}^{'}\right]_1$ denoting $\left[f^{'}{}_1(P^*)\right]_1 \left[f_{2*}^{'}\right]_1$ denoting

$[f_2]_1$ $(Q_1^*,$ $h_1^*$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $\left[g_*^{'}\right]_1$

Denoting $\left[g^{'}\right]_1 (P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - \left[f^{'}{}_{1*}\right]_1 [p_1^*](a_1) M_1 - \left[f_{2*}^{'}\right]_1 p_1^*(a_1) W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1) m_1(t,a_1) da_1 + \frac{[h_*^1]_1}{[h^*]_1} [p^*]_1(0)\, W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1 [Q_1^*] M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1) da_1$$

$$= -\int_0^\infty [m_a]_1 (t,a_1) da_1 - \int_0^\infty \mu(a_1) m(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f^{'}{}_{1*}\right]_1 \int_0^\infty p_1^*(a_1) da_1\right) M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1) da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\, m_1(t,a_1) da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 - \left[f_{2*}^{'}\right]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^{\infty} ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + \left[f_{1*}^{'}\right]_1 P_1^*\right) M_1 + \left(\frac{\left[h_*^{'}\right]_1}{[h_*]^1}\, P_1^*(0) - f_{2*}^{'} P_1^*\right) W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_*\right]_1 Q_1^*\, M_{\cdot 1}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{\cdot *}]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1 = ([h_*]_1\beta_1(\theta_1) - \mu_1(\theta_1))M_{\cdot 1}$$

Where $0 < \theta_1 = \theta_1(t) < \infty$.

And if $a_1^2\,[a_1^2]_1 < 4\,[a_2]_1 < 4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1 = \text{Re}\lambda_2\left[\lambda_2\right]_1 = -\frac{1}{2}a_1[a2\ ]_1 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1) - \mu_1(a_1)\right|$. Then $|n_1(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_1^2 < 4\,a_2 e$ and $[a_1] > 2\Delta$,or $[a_1^2] \geq 4[a_2]e$ and $(a_1 - \sqrt{(a_1^2)_1 - 4(a_2)_1 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+1}\right)$ and $\beta_1(a_1) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1)= K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1)= [0x_1]/(1 + x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1 + x_1)$and $h_1(x_1)=(1+\omega_1 x_1)/(1 + x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_1$i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*}$ $[h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2}[\ g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2}$ $[h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

# AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

**Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model**

**Gravitational quantum wave equation:**

**Equation Time-dependent equation (Wikipedia)**

**(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]**

**(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.**

**(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).**

**(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2**

**(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.**

**(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.**

| |
|---|
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |

| |
|---|
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |
| **(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]** |
| **Time-dependent Schrödinger equation** |
| **(Single non-relativistic particle)** |
| **(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)** |
| **(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").** |
| **(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.** |
| **(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.** |
| **(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.** |
| **(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).** |
| **(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)** |

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⊠ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would**

**take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38)** $$\frac{dr}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{(m_1 m_2)(m_1+m_2)}{r^3},$$

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

$$t = \frac{5}{256}\frac{c^5}{G^3}\frac{r^4}{(m_1 m_2)(m_1+m_2)}.$$

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1 + [\mu_0]_1 + 1 + \frac{K_1 b_1}{\delta_1 - b_1} + (m_1 + [\mu_0]_1 + \frac{K_1 b_1}{\delta_1 - b_1} + \sigma_1)\, Q_1^*)^3$$

$$= m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1 + Q^*)(1 + \omega_1 Q_1^*$$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_1 a e_1^{mu}\ [\mu_0]_1 - \frac{a_1}{a_1 + 1}$$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}]\,P^*_1$$

$$[a_2]_1 = \left[f^1_{2*}\right]P^*_1 - \frac{h^1_*}{h_*}P^*(0)$$

$$e=[\,g^1_*]_1\,Q^*_1$$

$$\eta_1(t) = \left[h_*\right]_1 \beta_1\left(\theta_1(t)\right) - \mu_1(\theta(t)),$$

The system(31)can be written as the following system:

$$[M(t)]\,[a_1]M_1 - [a_2]_1 W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_1 - [a_2]_1 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_1 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_1(0) + \int_0 \mathrm{e}^{At}$$

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## (Module Three)

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to**

**be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society **Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995**

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 1, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ 0 $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of

creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_3\ (Q_3) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3 P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \tag{4}$$

$$\rho_3(t_3, 0) = h_3(Q_3)\int_0^\infty \beta_3\,(a_3)\rho_3(t, a_3)da, \tag{5}$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \tag{6}$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \ \rho_3(t, a_3) \ \rho_3^*(t, a_3),$$

$$0 \ Q_3(t) \ Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \tag{7}$$

$$\rho_3^*\,(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*\,(t, a_3)da_3 \tag{8}$$

$$\frac{dQ_3^*}{dt} = -b_3\,Q_3^* + g_3(P_3^*)Q_3^*, \tag{9}$$

With $P_3^*(t) = \int_0^\infty \rho_3^*\ (t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_3(a)e^{-\int_0^a \mu_3(\tau)dt}e^{-ra}\ da_3 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(t)dt}\ da_3 > 1, \quad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^{\infty} [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t,a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^{\infty} \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t,a_3) = \rho(t,a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3(t,a_3)da_3 = \int_0^\infty \rho_3(t,a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0^{'}Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}^{'})(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho\mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^{\infty} u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^{\infty} \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t, a_3) = \tilde{u}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\, da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3) Y_3(\gamma_3, a_3) da_3\right) u_3(0)$$

$$+w \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3 = 0,$$

(24)

$$\left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right) \tilde{u}_3(0)$$

$$+\left(-1 - \int_0^\infty Y_3(\gamma_3, a_3) \int_0^a (a_3)_3(\tau)_3 (Y^{-3})_3(\gamma_3, \tau_3) d\tau_3\, da_3\right) w = 0.$$

Equqtion (20) has a solution of the form of $u_3(t, a_3) = u_3(a_3) e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0) \not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3) Y^{-3}\,(\gamma_3, a_3) d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^l \left(\int_0^\infty Y_3(\gamma_3, a_3) da_3\right)\left(\int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3) Y_3(\gamma_3, a_3) da\right)\left(1 + (f_{30}^l)_3 \int_0^\infty Y_3(\gamma_3, a_3) I_3(\gamma_3, a_3) da_3\right)$$

$$= 0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3) = (\rho_0)_3(0)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - (f_{30})_3 a_3}$$

$$I_3(\gamma_3, a_3) = \int_0^a (\rho_0)_3(\tau_3)e^{\gamma_3\tau_3 + (f_{30})_3\tau + \int_0^\tau \mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(\,a_3)da_3 - \int_0^\infty E_3(\,a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1+\frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^{\infty}E_3(a_3)da_3)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\ (\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(P_3^*) + [f_2]_3(Q_3^*))P_3^* = 0, \quad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^\infty \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \quad (27)$$

$$g_3(P_3^*) - b_3 = 0, \quad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3 a_3}\,da_3 < 1. \quad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3))\int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3)\int_0^{\infty}\beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\quad da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty}\beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3}da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty}\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}}da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - \left[p^*\right]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3)M - [f_{2*}^3]_3 p_3^*(\mathrm{a})W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $\left[f_{3*}^{'}\right]_3$ denoting $\left[f^{'}{}_3(P^*)\right]_3 \left[f_{2*}^{'}\right]_3$ denoting $[f_2]_3$ $(Q_3^*,$ $h_{*}$ denoting $h_*^{'}$ denoting $h_3^{'}$ $(Q_3^*)$, and $\left[g_*^{'}\right]_3$

Denoting $\left[g^{'}\right]_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - \left[f^{'}{}_{3*}\right]_3 [p_3^*](a_3)M_3 - \left[f_{2*}^{'}\right]_3 p_3^*(a_3)W_3$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t,a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3) da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3) da_3 - \int_0^\infty \mu(a_3) m(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f^{'}{}_{3*}]_3 \int_0^\infty p_3^*(a_3) da_3\right) M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3) da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3) da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 - [f_{2*}^{'}]_3 P_3^* W_{.3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3 \beta_3(a_3) - \mu_3(a_3)) m_3(t,a_3) da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [f_{3*}^{'}]_3 P_3^*\right) M_3 + \left(\frac{[h_*^{'}]_3}{[h_*]^3}\, P_3^*(0) - f_{2*}^{'} P_3^*\right) W_3, \qquad (33)$$

$$\frac{dW_3(t)}{dt}[g'_*]_3 Q_3^* M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2[a_3^2]_3 < 4[a_2]_3 < 4a_2 e$,

$\text{Re}\,\lambda_3[\lambda_3]_3 = \text{Re}\lambda_2[\lambda_2]_3 = -\frac{3}{2}a_3[a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty}\left|h_*[h_*]_3\beta_3(a_3) - \mu_3(a_3)\right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_3^2 < 4a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0]$ 0.5. Assume$[f_3](x_3)= K_3x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3)= [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3)=(1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])\,{}_3^3\ [\mathrm{K}]_{\,3}[P_0]\,{}_3^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)\ [P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}\,{}_{2*}^3] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}\,{}_*^3] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}\,{}_*^3] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_3 + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\, Q_3^*)^3$$

$$= m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_3\, ae_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3 + 3}$$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}]\,P^*_3$$

$$[a_2]_3 = \left[f^3_{2*}\right]P^*_3 - \frac{h^3_*}{h_*}P^*(0)$$

$$e=[g^3_*]_3\,Q^*_3$$

$$\eta_3(t) = \left[h_*\right]_3\beta_3\left(\theta_3(t)\right) - \mu_3\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_3\right]M_3 - [a_2]_3W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_3 - [a_2]_3 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_3 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_3(0) + \int_0 \mathrm{e}^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins χ1,2∈[−0.99,0.99] aligned (e&eb) with the orbital angular momentum, component masses m1,2∈[2,48]M⊙, and (e&eb) total mass Mtotal≤50 M⊙. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise**

**ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large**

**uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely**

**the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=)**

**62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (FΛF), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e)**

**self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground**

**and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an**

**observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as: arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate**

**the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of**

**both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent**

**rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb)**

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Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors**

**provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any $\ell$-local Hamiltonian, where $\ell \geq 2$, can be achieved using decoupling schemes of length at most $O(n\ell - 1 \log n)$. Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of**

**infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any**

**external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 91. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \text{ 0}, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \text{ ∞}. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \text{ ∞},$$

$$\rho_4(t,0) = \int_0^\infty n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4\left[\frac{dQ_2}{dt}\right]_4 - x_2\left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_4 - x_4\left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_4 - x_6\left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ 0 $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ∞; $\beta$ is the age

dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4 P_4) + (f_2)_4(Q_4))(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \ \rho_4(t, a_4) \ \rho_4^*(t, a_4),$$

$$0 \ Q_4(t) \ Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \quad (10)$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a) e^{-\int_0^a \mu_4(\tau)dt} e^{-ra}\, da_4 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(t)dt}\, da_4 > 1, \quad (H1)$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \quad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4} \rho_4(t,a_4)\, da_4$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + c_4 P_4 Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4 P_4 - \lambda_4 P_4 Q_4$$

$$\frac{dQ_4}{dt} = b_4 Q_4 - c_4 P_4 Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 92. THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^{\infty} \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + \ (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0^{'}Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^{\infty} \beta_4\,(a_4)u_4(t,a_4)da_4 + (h_{a_4}^{'})(\rho_0)_4Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^{\infty} \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\ o\ )_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t, a_4)da_4$$

$$u_4(t, 0) = \int_0^\infty \beta_4\,(a_4)u_4(t, a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t, a_4) = \tilde{u}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_4}\,da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})}\tau_4\,(\alpha_4)_4(\tau_4)e^{\gamma_4\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4)e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4}e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds}d\tau_4\,da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau} da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4-\int_0^{\infty} e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4+\int_0^\tau \mu_4(s_4)ds+y\tau}\alpha_4(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_4 a_4-(f_{40})_4 a_4-\int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^a(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\, da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40_4}^{l}\left(\int_0^{\infty} Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^a \mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^a(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$=\frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4a_4}-1)$$

Let

$$E_4(a_4)=e^{-(f_{40})_4a_4-\int_0^a\mu_4(\tau_4)d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4,a_4)=E_4(a_4)e^{-\gamma_4a_4}$. by a simple calculation,

$$\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma}\left(1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4a_4}\,da\right);$$

$$\int_0^{\infty}Y_4(\gamma_4,a_4)I(\gamma_4,a_4)da=\frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^{\infty}E_4(a_4)e^{-\gamma_4a_4}\,da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\,(\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1+\frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4-\int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1+\frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)da_4=1-\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\neq 0,$$

Since $\int_0^\infty \beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^a \alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4>0$. Therefore, the characteristic solution is

$$\gamma_4=-(f_{40}^{'})_4(\rho_0)_4\int_0^\infty E_4(\,a_4)da_4.$$

Since $f_4^{'}(P_0)_4>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^\infty \beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^* = \int_0^\infty p_4^*(a_4)\, da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} da_{,4}$$

And assume that

$$[k_3]_4 \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4 a_4}\, da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)(\,[f_2^{'}]_4(z_4)\,)\int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_{\square}$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}\ da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} d a_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r$, $R\big(f_4(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^{\infty} m_4(t, a_4) da_4 = P_4(t) - P_4^*,$

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - \left[p^*\right]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^{\infty} \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0)W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $\left[f_{4*}^{'}\right]_4$ denoting $\left[f^{'}{}_4(P^*)\right]_4\left[f_{2*}^{'}\right]_4$ denoting $[f_2]_4$ $(Q_4^*,\quad h_4^*$ denoting $h_*^{'}$ denoting $h_4^{'}$ $(Q_4^*)$, and $\left[g_*^{'}\right]_4$

Denoting $\left[g^{'}\right]_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_4]_4+[m_a]_4=-(\mu_4(a_4)+[f_{4*}]_4+[f_{2*}]_4)m_4-\left[f^{'}{}_{4*}\right]_4[p_4^*](a_4)M_4-\left[f_{2*}^{'}\right]_4p_4^*(a_4)W_4$$

$$m_4(t,0)=[h_*]_4\int_0^\infty \beta_4(a_4)m_4(t,a_4)da_4+\frac{[h_*^4]_4}{[h^*]_4}[p^*]_4(0)\,W_{,4} \qquad (30)$$

$$\frac{dW_4}{dt}=[g_*^4]_4[Q_4^*]M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed

correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is

shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way'' point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in

(eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to

(e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's "Extreme Cosmic Censorship" principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair

of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed, authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry

of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular

importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro** **STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L)

mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum

of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=) the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many

gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as: arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded

from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS**

**Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the

critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th] (or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic

Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th] (or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number:

MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly,

by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in (e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its**

**state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~ (1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the

environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement:**

**Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f'_{4*}]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 - [f_{2*}']_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4\beta_4(a_4) - \mu_4(a_4))m_4(t, a_4)da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + [f_{4*}']_4 P_4^*\right)M_4 + \left(\frac{[h_*']_4}{[h_*]^4}\, P_4^*(0) - f_{2*}' P_4^*\right)W_4,$$

(34)

$$\frac{dW_4(t)}{dt}[g'_*]_4\, Q_4^*\, M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{r*}]_4\beta_4(a_4) - \mu_4(a_4))m_4(t,a_4)da_4 = ([h_*]_4\beta_4(\theta_4) - \mu_4(\theta_4))M_{,4}$$

Where $0< \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2[a_4^2]_4<4[a_2]_4<1\, a_2 e$,

$\mathrm{Re}\,\lambda_4[\lambda_4]_4 = \mathrm{Re}\lambda_2[\lambda_2]_4 = -\frac{4}{2}a_4[a2\ ]_4 < 0.$

Let $\Delta=\max_{2<u<\infty}|h_*[h_*]_4\beta_4(a_4) - \mu_4(a_4)|$. Then$|n_4(t)|$ ⊠$\Delta\forall t$ ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 <4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2]$ ⊠$4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4) + d\tau_4}da = \frac{4}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ ⊠0.5. Assume$[f_4](x_4)= K_4x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1+x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1+x_4)$and $h_4(x_4)=(1+\omega_4x_4)/(1+x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [\mathrm{K}][P_0])}{[m_4]_4 + (\mu_0)_4 + [\mathrm{K}][P_0])_4^3\ [\mathrm{K}]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$ i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\ \mathrm{P}^*\ [f_{2*}] = \frac{\sigma \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*} \quad [h_*] = \frac{4+\omega_4 \mathrm{Q}_4^*}{4+\mathrm{Q}_4^*}$

$[f_{4*}] = K_4\ [\mathrm{f}_{2*}^4] = \frac{\sigma_4}{(4+\mathrm{Q}_4^*)^2}[\ \mathrm{g}_*^4] = \frac{\sigma_4}{(4+\mathrm{P}_4^*)^2} \quad [\mathrm{h}_*^4] = \frac{\omega_4 - 4}{(4+\mathrm{Q}_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_4}{(m + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma)^2} + \frac{2\omega_4}{([m_4]_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 + b_4} + \sigma_4)^3} < 1$$

And

$$\frac{1}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^2} + \frac{2\omega_4}{(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the following equation:

$(m_4 + [\mu_0]_4 + 1 + \frac{K_4 b_4}{\delta_4 - b_4} + (m_4 + [\mu_0]_4 + \frac{K_4 b_4}{\delta_4 - b_4} + \sigma_4)\ \mathrm{Q}_4^*)^3$

$=m_4 + [\mu_0]_4 + 3 + \frac{K_4 b_4}{\delta_4 - b_4}(4 + Q^*)(1 + \omega_4 Q_4^*$

Now

$$\left[a_4^2\right] - 1\left[a_2\right]e = \left(\frac{2K_4 b_4}{\delta_4 - b_4}\right)^2 + \frac{1K_4 b_4 \sigma_4 Q_4^*}{(\delta_4 - b_4)(1 + Q_4^*)} + \frac{\sigma_4^2 Q_4^{*2}}{(1 + Q_4^*)^2}$$

$$\frac{1\, b_4 \sigma_4 (\delta_4 - b) Q^*}{\delta_4 (1 + Q_4^*)^2} - \frac{1\, (\delta_4 - b_4)^2 (1 - \omega_4) Q_4^*}{\delta_4 (1 + \omega_4 Q_4^*)(1 + Q_4^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 a e_4^{mu} [\mu_0]_4 - \frac{a_4}{a_4 + 4}$

Hence if $[a_4^2 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q_4^*}{1 + Q_4^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P_4^*$$

$$[a_2]_4 = \left[f_{2*}^4\right] P_4^* - \frac{h_*^4}{h_*} P^*(0)$$

$$e=[\,g_{*}^{4}]_{\,4}\,Q_{4}^{*}$$

$$\eta_4(t) = \left[h_{*}\right]_{4}\beta_{4}\left(\theta_{4}(t)\right) - \mu_{4}\left(\theta(t)\right),$$

The system(34)can be written as the following system:

$$[M(t)]\,[a_{4}]M_4 - [a_2]_{4}W_4 - [n(t)]M_4$$

$$w_4(t) = eM_4$$

Write

$$A=\begin{pmatrix} -[a_4]_{4} - [a_2]_{4} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{4} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$$

The solution of (32)is

$$X_4(t) = e^{A,t}\,[X]_{4}(0) + \int_0 e^{At}$$

**Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The**

**term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the**

**implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December**

**1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with**

**real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-**

**energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105 DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar**

**potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-**

**branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-**

**Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–**

**334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order**

**transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor**

**solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum**

**Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics. Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc]**

**for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures, especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated**

**effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and**

**DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3 [astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak**

**effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero.**

**Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon," Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the**

**rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the**

**position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at**

**ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob**

**Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 93. MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \quad P_6 \; 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \; \infty,$$

$$\rho_6(t,0) = \int_0^\infty n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6 P_6) + (f_2)_6(Q_6))(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6) \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \leq \rho_6(t, a_6) \leq \rho_6^*(t, a_6),$$

$$0 \leq Q_6(t) \leq Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\, \rho_6^*, \quad (7)$$

$$\rho_6^*(t, 0) = \int_0^\infty \beta_6(a_6)\, \rho_6^*(t, a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6 Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t, a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt} e^{-ra}\, da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\, da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6\ h_6(Q_6) = 1, \qquad and\ \ g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t,a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 94. THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t, 0) = \int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^{\infty} u_6(t, a_6) da_6 = \int_0^{\infty} \rho_6(t, a_6) da_6 - \int_0^{\infty} (\rho_0)_6(a_6) da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6 N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6^{'}(P_0) = f_{60}^{'}$, and $f_2^{'}(0) = f_{20}^{'}$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t, 0) = \rho_6(t, 0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^{\infty} \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + \ (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0^{'}Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^{\infty} \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}^{'})(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^{\infty} \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\ o\ )_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho\mathrm{o}]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t, a_6)da_6$$

$$u_6(t, 0) = \int_0^\infty \beta_6\,(a_6)u_6(t, a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t, a_6) = \tilde{u}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_6}\,da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})\tau_6}\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau} da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6-\int_0^{\infty} e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6+\int_0^\tau \mu_6(s_6)ds+y\tau}\alpha_6(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_6 a_6-(f_{60})_6 a_6-\int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^a(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\, da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60_6}^{l}\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$=\frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6a_6}-1)$$

Let

$$E_6(a_6)=e^{-(f_{60})_6a_6-\int_0^a\mu_6(\tau_6)d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(5211) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(5212) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(5213) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(5214) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(5215) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the**

**mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

## The Planck scale

(5216) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(5217) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(5218) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(5219) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(5220) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

### The microscopic structure of spacetime

(5221) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(5222) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(5223) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(5224) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

### Early cosmology

(5225) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(5226) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(5227) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(5228) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(5229) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

## Black holes

(5230) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(5231) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(5232) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(5233) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(5234) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(5235) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(5236) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(5237) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(5238) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have**

**experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(5239) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(5240) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

**String theory**

(5241) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(5242) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(5243) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

(5244) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.**

(5245) **However, when summed the entire series appears to be (=) divergent.**

(5246) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.**

(5247) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.**

(5248) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.**

(5249) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.**

**Loop quantum gravity and spin foams**

(5250) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.**

(5251) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.**

(5252) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.**

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| (5253) **The granular properties of space can be (=) explicitly computed.** |
| (5254) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (5255) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (5256) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (5257) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (5258) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (5259) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

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| (5260) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (5261) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (5262) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (5263) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (5264) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (5265) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

**Others**

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| (5266) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (5267) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (5268) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning** |

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| **of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (5269) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (5270) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |
| (5271) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (5272) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (5273) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (5274) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (5275) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (5276) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (5277) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

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| **Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016** |

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| (5278) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (5279) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (5280) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (5281) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |

(5282) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.**

(5283) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.**

(5284) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.**

(5285) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016**
**DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society**

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(5286) **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.**

(5287) **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.**

(5288) **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.**

(5289) **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.**

(5290) **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).**

(5291) **Bringing them to thermal equilibrium requires (e) an external bath.**

(5292) **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.**

(5293) **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).**

(5294) **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.**

(5295) **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.**

(5296) **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—**

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| **containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (5297) **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (5298) **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (5299) **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (5300) **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (5301) **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (5302) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (5303) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (5304) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (5305) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (5306) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (5307) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (5308) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (5309) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (5310) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (5311) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the** |

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| **eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (5312) **This is a realistic scenario for (e) all known systems.** |
| (5313) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (5314) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (5315) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (5316) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (5317) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (5318) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |
| (5319) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (5320) **So in those systems, the quantum phase transition does indeed exist.** |
| (5321) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (5322) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (5323) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (5324) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (5325) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (5326) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (5327) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |

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| (5328) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (5329) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (5330) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (5331) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (5332) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (5333) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |
| (5334) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (5335) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (5336) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (5337) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (5338) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (5339) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (5340) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (5341) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (5342) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (5343) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (5344) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (5345) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University** |

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| **in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (5346) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (5347) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (5348) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (5349) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (5350) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (5351) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (~104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (5352) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (5353) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (5354) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (5355) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (5356) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |

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| (5357) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (5358) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (5359) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (5360) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (5361) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (5362) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |
| (5363) **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (5364) **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (5365) **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (5366) **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (5367) **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (5368) **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (5369) **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (5370) **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (5371) **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (5372) **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (5373) **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (5374) **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (5375) **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |

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| (5376) **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (5377) **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (5378) **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (5379) **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (5380) **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (5381) **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |
| (5382) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (5383) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (5384) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (5385) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (5386) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (5387) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (5388) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (5389) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (5390) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (5391) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (5392) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (5393) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (5394) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (5395) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (5396) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (5397) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (5398) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (5399) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (5400) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (5401) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (5402) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (5403) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (5404) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (5405) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (5406) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (5407) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (5408) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (5409) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (5410) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |

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| (5411) | **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (5412) | **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (5413) | **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (5414) | **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (5415) | **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |
| (5416) | **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.** |

**Doomed satellite missions**

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| (5417) | **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.** |
| (5418) | **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.** |
| (5419) | **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.** |
| (5420) | **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.** |
| (5421) | **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.** |
| (5422) | **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people** |
| (5423) | **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."** |
| (5424) | **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.** |

(5425) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(5426) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(5427) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

(5428) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.**

(5429) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.**

(5430) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.**

(5431) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.**

(5432) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.**

(5433) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.**

(5434) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.**

(5435) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.**

**Dark matter guides growth of supermassive black holes February 18, 2015**

(5436) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.**

(5437) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.**

(5438) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.**

(5439) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.**

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| (5440) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (5441) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (5442) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (5443) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |
| (5444) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (5445) **We know dark matter exists only from (e) its gravitational effects.** |
| (5446) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (5447) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (5448) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (5449) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (5450) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (5451) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (5452) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (5453) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (5454) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (5455) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (5456) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |

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| (5457) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (5458) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (5459) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (5460) **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (5461) **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (5462) **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (5463) **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (5464) **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (5465) **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (5466) **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (5467) **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (5468) **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (5469) **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |

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| (5470) **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (5471) **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (5472) **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |
| (5473) **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (5474) **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (5475) **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (5476) **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (5477) **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (5478) **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (5479) **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (5480) **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (5481) **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (5482) **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |

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| (5483) **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (5484) **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (5485) **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (5486) **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |
| (5487) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (5488) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (5489) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (5490) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (5491) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (5492) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (5493) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (5494) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (5495) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |

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| (5496) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (5497) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (5498) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (5499) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (5500) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e) electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.** |

**Spheres on trees**

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| (5501) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.** |
| (5502) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.** |
| (5503) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.** |
| (5504) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.** |
| (5505) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.** |
| (5506) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.** |
| (5507) **As a result, it provides (eb) an important clue for finding the global minimum.** |
| (5508) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.** |
| (5509) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.** |

**Thompson's problem 2**

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| (5510) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society** |

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| (5511) | **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.** |
| (5512) | **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.** |

**Implications for Thomson's and Smale's problems**

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| (5513) | **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.** |
| (5514) | **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (5515) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (5516) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (5517) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (5518) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (5519) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (5520) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (5521) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (5522) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (5523) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (5524) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(961) Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(962) The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |
| **(963) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(964) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(965) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(966) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(967) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(968) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(969) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(970) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(971) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(972) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(973) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(974) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(975) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(976) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(977) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |

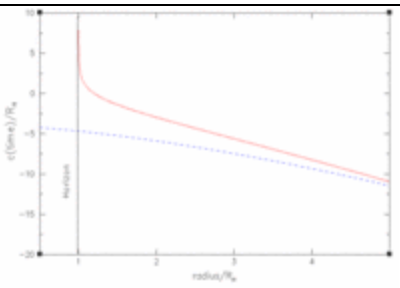
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| **(978) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(979) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(980) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(981) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |
| **(982) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(983) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(984) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(985) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(986) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(987) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(988) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(989) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(990) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(991) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(992) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(993) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(994) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(995) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(996) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |

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| **(997) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(998) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(999) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(1000) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(1001) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |
| **(1002) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(1003) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(1004) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(1005) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(1006) One of these coordinates is (=) called the "coordinate time."** |
| **(1007) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(1008) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(1009) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(1010) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(1011) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |

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| **(1012) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(1013) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(1014) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(1015) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(1016) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |
| **(1017) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(1018) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(1019) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(1020) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(1021) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(1022) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(1023) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(1024) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(1025) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(1026) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(1027) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(1028) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(1029) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |

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| **(1030) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(1031) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(1032) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(1033) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(1034) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |
| **(1035) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(1036) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(1037) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(1038) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(1039) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(1040) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(1041) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(1042) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(1043) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(1044) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(1045) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(1046) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(1047) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |

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| **(1048) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(1049) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(1050) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(1051) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(1052) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |
| **(1053) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(1054) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(1055) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(1056) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

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| Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation, |
| $$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$ |
| $$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$ |
| Substituting in to (25), we get |
| $$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$ |

$$+\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1+\frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^{\infty}E_6(a_6)da_6-\int_0^{\infty}E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6=1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6>0$. Therefore, the characteristic solution is

$$\gamma_6=-(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6'(P_0)_6>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_6 + (\mu_6(a_6) + [f_6]_6(P_6^*) + [f_2]_6(Q_6^*))P_6^* = 0, \quad (26)$$

$$p_6^*(0) = h_6(Q_6^*)\int_0^\infty \beta_6(a_6)p_6^*(a_J)p_6^*(a_6)da_6 \quad (27)$$

$$g_6(P_6^*) - b_6 = 0, \quad (28)$$

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b)$,

From (26),

$$p_6^*(a) = p_6^*(0)e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*)\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}da_6\right)p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\, da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,)\int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6}\, da_{\square}$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\quad da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0,$ (29)

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-\gamma_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0)W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6\left[f_{2*}^{'}\right]_6$ denoting

$[f_2]_6$ $(Q_6^*,\quad h_{6}^{*}$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - \left[f^{'}{}_{6*}\right]_6[p_6^*](a_6)M_6 - \left[f_{2*}^{'}\right]_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\,W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f^{'}{}_{6*}\right]_6\int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6\int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\,m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + \left[f_{6*}^{'}\right]_6 P_6^*\right)M_6 - \left[f_{2*}^{'}\right]_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^{\infty} ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f'_{6*}]_6 P_6^*\right) M_6 + \left(\frac{[h'_*]_6}{[h_*]^6}\, P_6^*(0) - f'_{2*}P_6^*\right) W_6,$$

(36)

$$\frac{dW_6(t)}{dt}\left[g'_*\right]_6 Q_6^*\, M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{"*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6$<4 $[a_2]_6$<4 $a_2 e$,

Re $\lambda_6\left[\lambda_6\right]_6 = \mathrm{Re}\lambda_2\left[\lambda_2\right]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0$.

Let $\Delta$=max$_{2<u<\infty}\left|h_*\,[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then$|n_6(t)|$ ⊠⊿∀t ⊠0. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_6^2$ <4 $a_2 e$ and $[a_6] > 2\Delta$,or $[a_6^2]$ ⊠4$[a_2]e$ and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6 x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)=(1+\omega_6 x_6)/(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [K][P_0])}{[m_6]_6 + (\mu_0)_6 + [K][P_0])_6^3\ [K]_6 [P_0]_6^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_6(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ P^*\ [f_{2*}] = \frac{\sigma Q_6^*}{6+Q_6^*}$ $[h_*] = \frac{6+\omega_6 Q_6^*}{6+Q_6^*}$

$[f_{6*}] = K_6\ [f_{2*}^6] = \frac{\sigma_6}{(6+Q_6^*)^2}[\ g_*^6] = \frac{\sigma_6}{(6+P_6^*)^2}$ $[h_*^6] = \frac{\omega_6 - 6}{(6+Q_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^2} + \frac{2\omega_6}{([m_6]_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^3} < 1$$

And

$$\frac{1}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^2}+\frac{2\omega_6}{(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6+[\mu_0]_6+1+\frac{K_6b_6}{\delta_6-b_6}+(m_6+[\mu_0]_6+\frac{K_6b_6}{\delta_6-b_6}+\sigma_6)\,Q_6^*)^3$$

$$=m_6+[\mu_0]_6+3+\frac{K_6b_6}{\delta_6-b_6}(1+Q^*)(1+\omega_6Q_6^*$$

Now

$$\left[a_6^2\right]-4\left[a_2\right]e=\left(\frac{2K_6b_6}{\delta_6-b_6}\right)^2+\frac{4K_6b_6\sigma_6\,Q_6^*}{(\delta_6-b_6)(1+Q_6^*)}+\frac{\sigma_6^2Q_6^{*2}}{(1+Q_6^*)^2}$$

$$\frac{4\,b_6\sigma_6\,(\delta_6-b)Q^*}{\delta_6(1+Q_6^*)^2}-\frac{4\,(\delta_6-b_6)^2(1-\omega_6)Q_6^*}{\delta_6(1+\omega_6Q_6^*)1+Q_6^*)}P^*(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_6\,ae_6^{mu}\;[\mu_0]_6-\frac{a_6}{a_6+6}$$

Hence if $[a_6^2>4(a_2)_6e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 \, Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_{\,6} = [f_{6*}]_{\,6} + [f_{2*}]_{\,6} + [f_{6*}] \, P_6^*$$

$$[a_2]_{\,6} = \left[f_{2*}^{6}\right] P_6^* - \frac{h_*^6}{h_*} P^*(0)$$

$$e = [\, g_*^6]_{\,6} \, Q_6^*$$

$$\eta_6(t) = \left[h_*\right]_{\,6} \beta_{\,6} \left(\theta_{\,6}(t)\right) - \mu_{\,6}\left(\theta(t)\right),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_{\,6}\right] M_6 - [a_2]_{\,6} W_6 - [n(t)] M_6$$

$$w_6(\mathrm{t}) = \mathrm{e}M_6$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_6(0) + \int_0 \mathrm{e}^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case β→∞ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite

strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes (eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both

numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen**

**Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case $\omega_q=-2/3$ authors derive analytic expressions of (e) critical physical quantities, while (e) for cases $\omega_q\neq-2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits ($e>0.96$) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and

Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the

Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence, authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006,**

**JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R∼1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the

previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11

(2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct

relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2 [hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the

recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time

evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de

Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality

of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short

distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors

clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top. Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium**

**J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-**

**García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity. **Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across

(e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions. The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at $\lambda=\lambda c$, the correlation length diverges as $\xi\sim|\lambda-\lambda c|-\nu$, while (e) the gap between the ground state and first excited state vanishes as (e) $\Delta E\sim\xi-z\sim|\lambda-\lambda c|z\nu$. Exactly at the critical point $\lambda=\lambda c$, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as $\omega\sim|k-k0|z$. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl,**

**Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan**

also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is**

**described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin**

**qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A, 75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of**

**one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation**

**functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the**

**deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb)**

**Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles**

**on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g**

**handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb)**

**the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct ("extreme") types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, 'the field'. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and**

**(e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb) that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ∗This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and**

**Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time**

**dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an**

**electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches**

**x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x. Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being**

**considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states**

**are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics (math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The**

**Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

## 95. MODEL FORMULATION

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^{\infty} \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \; P_7 \; \text{0}, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \text{?}\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \text{?}\infty,$$

$$\rho_7(t,0) = \int_0^{\infty} n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3\left[\frac{dQ_3}{dt}\right]_7 - x_4\left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_7}{dt}\right]_7 - x_6\left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(5525) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(5526) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(5527) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(5528) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(5529) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(5530) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(5531) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(5532) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(5533) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(5534) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(5535) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(5536) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(5537) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(5538) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(5539) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(5540) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(5541) The electronic and (e&eb) structural variability among them is extraordinary.

(5542) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(5543) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(5544) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(5545) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to

Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(5546) Nudge your bath tub just right and (eb) you might end up reversing time.

(5547) Don't worry – you won't end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a "time mirror", which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

(5548) To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse

(5549) Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses.

(5550) But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort.

(5551) The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort.

(5552) Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower

(5553) Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower

**Déjà vu**

(5554) To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape.

(5555) The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says.

(5556) Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says.

(5557) "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique

could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

(5558) Wave control is usually performed by (e) spatially engineering the properties of a medium.

(5559) Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach.

(5560) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors.

(5561) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source.

(5562) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation.

(5563) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption.

(5564) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control.

**Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)**

(5565) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice.

(5566) Quantum mechanics rules the dynamics of (e) light and matter.

(5567) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved.

(5568) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1].

(5569) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest.

(5570) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called

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| scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (5571) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (5572) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (5573) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (5574) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |
| (5575) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (5576) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (5577) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (5578) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (5579) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (5580) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (5581) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (5582) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (5583) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (5584) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (5585) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (5586) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (5587) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (5588) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (5589) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |

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| (5590) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (5591) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (5592) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (5593) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |
| (5594) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (5595) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (5596) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (5597) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (5598) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (5599) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (5600) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (5601) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (5602) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (5603) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (5604) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (5605) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (5606) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (5607) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (5608) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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|---|
| (5609) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (5610) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (5611) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (5612) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |
| (5613) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum. |
| (5614) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big((f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \le \rho_7(t, a_7) \le \rho_7^*(t, a_7),$$

$$0 \le Q_7(t) \le Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\,\rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\,\rho_7^*(t, a_7)da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7 Q_7^* + g_7(P_7^*)Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7)da_7$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \tag{10}$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \tag{$H1$}$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7Q_7 + c_7P_7Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7P_7 - \lambda_7P_7Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 96. THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7 e^{r_7 t - \int_0^t (f_7)_7 (P_7(\tau)) d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau) d\tau}\, da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{(k_7)_7 a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau) d\tau} e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{k_7 a_7} da_7 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7) da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau) d\tau} e^{-(f_7)_7 (P_0) a_7} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7(t, a_7) da_7 = \int_0^\infty \rho_7(t, a_7) da_7 - \int_0^\infty (\rho_0)_7(a_7) da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7N_7 + o_7(N_7), \qquad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \qquad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7\,(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7\,(0) \qquad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7Q_7$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t, a_7) = \tilde{u}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{u}(a_7)$ gives,

$$\tilde{u}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{u}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^\infty \beta_7(a_7) e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a (\alpha_7)_7(\tau_7) e^{-\gamma_7\tau_7} e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds} d\tau_7\, da_7$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\, da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7 + \int_0^\tau \mu_7(s_7)ds + y\tau}\,\alpha_7(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_7 a_7 - (f_{70})_7 a_7 - \int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty}Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^{a}(\rho_0)_7(\tau_7)Y^{-7}\,(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^{a}\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7, a_7) = \int_0^a (\rho_0)_7(\tau_7) e^{\gamma_7\tau_7 + (f_{70})_7\tau + \int_0^\tau \mu_7(s_7)ds_7} d\tau$$

$$= \frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7 a_7} - 1)$$

Let

$$E_7(a_7) = e^{-(f_{70})_7 a_7 - \int_0^a \mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(a_7)da_7 - \int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(a_7)da_7) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) da_7 = 1 - \int_0^\infty \beta_7(a_7) E_7(a_7) e^{-\gamma_7 a_7} da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7) Y_7(\gamma_7 a_7) \int_0^a \alpha_7(\tau_7) Y^{-7}(\gamma_7, \tau_7) d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7 (\rho_0)_7 \int_0^\infty E_7(a_7) da_7.$$

Since $f_7'(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*) \int_0^\infty \beta_7(a_7) p_7^*(a_J) p_7^*(a_7) da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \qquad (78)$$

Where $P_7^* = \int_0^\infty p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0) e^{\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*) \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7} da_7\right) p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7) \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7} da_{,7}$$

And assume that

$$[k_3]_7 \int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7) d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7 a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7^{'}]_7(z_7))\int_0^{\infty} a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}\, da_7$$

$-h^{'}(z_7)\int_0^{\infty}\beta_7(a_7)\, e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}\ da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1 - \int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^{-}(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r, R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_7(a_7) e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^{\infty} m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - \left[p^*\right]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^{\infty} \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M_{,7}$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $\left[f_{7*}^{'}\right]_7$ denoting $\left[f^{'}{}_7(P^*)\right]_7 \left[f_{7*}^{'}\right]_7$ denoting $[f_7]_7$ $(Q_7^*,$ $h_{*}_7$ denoting $h_*^{'}$ denoting $h_7^{'}$ $(Q_7^*)$, and $\left[g_*^{'}\right]_7$

Denoting $\left[g^{'}\right]_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7)m_7(t,a_7)da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\,W_{,7} \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7[Q_7^*]M_7$$

Integrating $[m_7]_7(t,a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t,a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t,a_7)da_7 - \int_0^\infty \mu(a_7)m(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f'_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$$= m_7(t,0) - \int_0^\infty \mu_7(a_7)\,m_7(t,a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 - [f_{7*}']_7 P_7^* W_{.7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}']_7 P_7^*\right)M_7 + \left(\frac{[h_*']_7}{[h_*]^7}\,P_7^*(0) - f_{7*}' P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}\left[g^{'}{}_{*}\right]_7 Q_7^{*}\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t,a_7)da_7 = ([h_{*}]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if $a_7^7[a_7^7]2<4\,[a_7]2<4\,a_7e$,

$\mathrm{Re}\,\lambda_7\left[\lambda_7\right]_7 = \mathrm{Re}\lambda_7\left[\lambda_7\right]_7 = -\frac{7}{7}a_7[a2\,]_7 < 0.$

Let $\Delta=\max_{7<u<\infty}\left|h_{*}\,[h_{*}]_7\beta_7(a_7) - \mu_7(a_7)\right|$. Then$|n_7(t)|$ ⍰⊿∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^{*}(a), \boldsymbol{Q}^{*})$ is asymptotically stable if either $a_7^7 <4\,a_7 e$ and $[a_7] > 1\Delta$,or $[a_7^7]$ ⍰[$a_7$]e and $(a_7 - \sqrt{(a_7^7)_7 - 4(a_7)_7 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+7}\right)$ and $\beta_7(a_7) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7) + d\tau_7 da} = \frac{1}{(m+\mu_0+1)^7} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ 0.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)= [0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)=(1+\omega_7x_7)/(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])_{\,7}^{\,3}\ [\mathrm{K}]_{\,7}[P_0]_{\,7}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_7$i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_{\,7}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^{*}\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{\,7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^{\,7}] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^{\,7}] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma)^{\,7}} + \frac{1\omega_7}{([m_7]_{\,7} + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 + b_7} + \sigma_7)^{\,3}} < 1$$

And

$$\frac{1}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,7}} + \frac{1\omega_7}{(m_7 + [\mu_0]_{\,7} + 1 + \frac{K_7 b_7}{\delta_7 - b_7})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_7 + [\mu_0]_7 + 1 + \frac{K_7 b_7}{\delta_7 - b_7} + (m_7 + [\mu_0]_7 + \frac{K_7 b_7}{\delta_7 - b_7} + \sigma_7)\, Q_7^*)^3$$

$$= m_7 + [\mu_0]_7 + 3 + \frac{K_7 b_7}{\delta_7 - b_7}(1 + Q^*)(1 + \omega_7 Q_7^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,7}\right]e = \left(\frac{1K_7 b_7}{\delta_7 - b_7}\right)^7 + \frac{4K_7 b_7 \sigma_7 Q_7^*}{(\delta_7 - b_7)(7 + Q_7^*)} + \frac{\sigma_7^7 Q_7^{*7}}{(1 + Q_7^*)^7}$$

$$\frac{4\, b_7 \sigma_7 (\delta_7 - b) Q^*}{\delta_7 (1 + Q_7^*)^7} - \frac{4\,(\delta_7 - b_7)^7 (1 - \omega_7) Q_7^*}{\delta_7 (1 + \omega_7 Q_7^*)(1 + Q_7^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_7\, ae^{mu}_7\ \ [\mu_0]_7 - \frac{a_7}{a_7 + 7}$$

Hence if $[a_7^7 > 4(a_7)_7 e$ and

$$[a_7] = \frac{1K_7 b_7}{\delta_7 - b_7} + \frac{\sigma_7 Q_7^*}{1 + Q_7^*} > 1\Delta$$

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| The nontrivial equilibrium is stable. |
| Some of the examples in which stability analysis is of seminal importance are: These are the system to which the<br><br>model becomes applicable |
| **Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014** |

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| (951)Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system. |
| (952) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition. |
| (953)Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon. |
| (954) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society |

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| **Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).** |

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| (955)Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs). |
| (956) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3]. |
| (957)This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States. |
| (958)Humans are not very good at understanding interacting many-body systems. |
| (959)We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them. |
| (960)But what happens when we couple (e&eb) the particles strongly? |
| (961)To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments. |

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| (962)Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water. |
| (963)The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h. |
| (964) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1] |
| (965) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (966)The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (967) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (968)The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (969) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (970) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (971)The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (972)Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (973) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (974) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (975)There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (976)In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (977)When the system is treated quantum mechanically (eb) these orbits are quantized. |

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| (978)The energy levels of these quantized orbitals take on (e) discrete values |
| (979):{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (980) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (981) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (982)Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |
| (983) For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (984)The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (985) They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (986)A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (987)The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (988)Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (989) In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (990)The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (991)Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (992)When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (993) A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (994)In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (995)The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (996)An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (997)The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (998)This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |

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| (999)Another classic example of duality involves (e&eb) vortices. |
| (1000) A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (1001) Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (1002) The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (1003) This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |
| (1004) Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (1005) A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (1006) The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (1007) TIs are distinguished only by (e) their edge states. |
| (1008) The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (1009) A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (1010) But the Dirac cone is not (e) the only possible signature. |
| (1011) Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential “TI-ness.” This essential property is (=) called an anomaly in high-energy theory. |
| (1012) Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (1013) When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (1014) However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (1015) By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a “chiral TI,” which instead has (e) particle-hole symmetry |
| (1016) . But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (1017) The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (1018) The bulk duality implies (eb) a duality between edge theories. |

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| (1019) The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (1020) One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (1021) Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (1022) In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |
| (1023) Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |
| (1024) Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (1025) Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (1026) The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (1027) This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (1028) They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (1029) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $TH=\hbar\kappa/2\pi$. |
| (1030) This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (1031) This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (1032) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (1033) Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (1034) The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

IF we define

$$[a_7]_7 = [f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}]\,P_7^*$$

$$[a_7]_7 = \left[f_{7*}^{7}\right] P_7^* - \frac{h_*^7}{h_*} P^*(0)$$

$$e = [g_*^7]_7\, Q_7^*$$

$$\eta_7(t) = \left[h_*\right]_7 \beta_7 \left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A = \begin{pmatrix} -[a_7]_7 - [a_7]_7 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)= \begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t) = \begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_7(0) + \int_0 \mathrm{e}^{At}$$

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION

## ATTRITION MODELS

## MODULE EIGHT

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

### 97. MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(5615) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(5616) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(5617) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

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| (5618) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related." |
| (5619) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules. |
| (5620) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems. |
| (5621) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe. |
| (5622) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short. |
| (5623) In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (5624) And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (5625) Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (5626) One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (5627) And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (5628) "The underlying physics is (=) chaos, essentially," he said. |
| (5629) The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (5630) But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (5631) The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (5632) To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (5633) Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (5634) Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (5635) By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (5636) The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |

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| (5637) The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (5638) "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |
| (5639) "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (5640) It turns out that they are actually the driving forces behind (eb) thermalization. |
| (5641) "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |
| (5642) The study's findings have fundamental implications for (e) quantum computing. |
| (5643) At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (5644) "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (5645) A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (5646) The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (5647) Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (5648) The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'-the basic component of a quantum computer. |
| (5649) Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (5650) By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (5651) The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (5652) Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

(5653) Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions.

(5654) As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables.

(5655) There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number.

(5656) These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits.

**Theory**

(5657) Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(5658) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(5659) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(5660) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(5661) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(5662) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(5663) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(5664) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(5665) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(5666) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(5667) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(5668) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(5669) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(5670) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(5671) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(5672) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(5673) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(5674) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(5675) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(5676) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(5677) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(5678) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(5679) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(5680) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(5681) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(5682) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(5683) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(5684) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(5685) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(5686) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(5687) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(5688) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(5689) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(5690) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(5691) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

**Black holes**

(5692) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(5693) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(5694) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(5695) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(5696) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(5697) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(5698) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(5699) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(5700) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(5701) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(5702) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that

ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(5703) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(5704) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(5705) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(5706) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(5707) However, when summed the entire series appears to be (=) divergent.

(5708) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(5709) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(5710) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(5711) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(5712) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(5713) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(5714) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(5715) The granular properties of space can be (=) explicitly computed.

(5716) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(5717) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(5718) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(5719) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(5720) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(5721) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(5722) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(5723) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(5724) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(5725) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(5726) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(5727) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(5728) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(5729) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(5730) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(5731) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \; P_8 \; 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \; \infty,$$

$$\rho_8(t,0) = \int_0^\infty n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \le \rho_8(t, a_8) \le \rho_8^*(t, a_8),$$

$$0 \le Q_8(t) \le Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8 Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t, 0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t, a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8 Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t, a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a) e^{-\int_0^a \mu_8(\tau)dt} e^{-ra}\, da_8 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(t)dt}\, da_8 > 1, \qquad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8Q_8 - c_8P_8Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 98. THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8e^{r_8t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^\infty u_8\,(t, a_8)da_8 = \int_0^\infty \rho_8(t, a_8)da_8 - \int_0^\infty (\rho_0)_8(a_8)da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \quad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \quad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t,0) = \rho_8(t,0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^\infty \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8(0) \quad (18)$$

Expanding

$$h_8(Q_8) = 1 + (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0'Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + (h_{a_8}')(\rho_0)_8 Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8 u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^\infty \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8 a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\, o\,)_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t,a_8)da_8$$

$$u_8(t,0) = \int_0^\infty \beta_8\,(a_8)u_8(t,a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t,a_8) = \tilde{\mathrm{u}}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_8)$ gives,

$$\tilde{\mathrm{u}}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{\mathrm{u}}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a(\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8\,(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8=0,$$

(24)

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty} Y_8(\gamma_8,a_8)\int_0^{a}(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\,da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80}^{l}{}_8\left(\int_0^{\infty} Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8, a_8) = E_8(a_8)e^{-\gamma_8 a_8}$. by a simple calculation,

$$\int_0^\infty \beta_8(a_8)Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma}\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}\ da\right);$$

$$\int_0^\infty Y_8(\gamma_8, a_8)I(\gamma_8, a_8)da = \frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^8]\int_0^\infty E_8(a_8)e^{-\gamma_8 a_8}da_8\ (\int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8)$$

$$+\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1 + \frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^\infty E_8(\,a_8)da_8 - \int_0^\infty E_8(\,a_8)e^{-\gamma_8 a_8}da_8\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1 + \frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^\infty E_8(\,a_8)da_8) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)da_8 = 1 - \int_0^\infty \beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8 \neq 0,$$

Since $\int_0^\infty \beta_8(a_8)Y_8(\gamma_8 a_8)\ \int_0^a \alpha_8\ (\tau_8)Y^{-8}(\gamma_8, \tau_8)d\tau_8 da_8 > 0$. Therefore, the characteristic solution is

$$\gamma_8 = -(f_{80}^{'})_8(\rho_0)_8 \int_0^{\infty} E_8(a_8)da_8.$$

Since $f_8'(P_0)_8 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^{\infty} \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^{\infty} p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}da_{,8}$$

And assume that

$$[k_3]_8\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)([f_8']_8(z_8))\int_0^\infty a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}da_{\square}$$

$-h'(z_8)\int_0^\infty \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8) d\tau_8 - [f_8]_8(P_8^*) a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1.$ (H4)

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - [p^*]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8) m_8(t,a_8)\, da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0) W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M_{,8}$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting

$[f_8]_8$ $(Q_8^*,$ $h_{*}^{'}$ denoting $h_{8}^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $\left[g^{'}\right]_8 (P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8) m_8 - \left[f^{'}{}_{8*}\right]_8 [p_8^*](a_8) M_8 - \left[f_{8*}^{'}\right]_8 p_8^*(a_8) W_8$

$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8) m_8(t,a_8) da_8 + \frac{[h_*^8]_8}{[h^*]_8} [p^*]_8(0)\, W_{,8}$ (30)

$$\frac{dW_8}{dt} = [g_*^8]_8 [Q_8^*] M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to ∞, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8) da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8) da_8 - \int_0^\infty \mu(a_8) m(t,a_8) da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + \left[f^{'}{}_{8*}\right]_8 \int_0^\infty p_8^*(a_8) da_8\right) M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8) da_8 W_8$$

$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8) da_8$

$$- \left([f_{8*}]_8 + [f_{8*}]_8 + \left[f_{8*}^{\cdot}\right]_8 P_8^*\right) M_8 - \left[f_{8*}^{\cdot}\right]_8 P_8^* W_{\cdot 8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8 \beta_8(a_8) - \mu_8(a_8)) m_8(t,a_8) da_8$$

$$- \left([f_{8*}] + [f_{8*}]_8 + \left[f_{8*}^{\cdot}\right]_8 P_8^*\right) M_8 + \left(\frac{[h_*^{'}]_8}{[h_*]^8}\, P_8^*(0) - f_{8*}^{'} P_8^*\right) W_8,$$

(31)

$$\frac{dW_8(t)}{dt} \left[g^{\cdot}{}_*\right]_8 Q_8^*\, M_{\cdot 8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{\cdot *}]_8 \beta_8(a_8) - \mu_8(a_8)) m_8(t,a_8) da_8 = ([h_*]_8 \beta_8(\theta_8) - \mu_8(\theta_8)) M_{\cdot 8}$$

Where $0 < \theta_8 = \theta_8(t) < \infty$.

And if a $^8_8$[a $^8_8$]2<4 [a $_8$]2<4 a $_8$e,

Re $\lambda_8 [\lambda_8]_8 = \text{Re}\lambda_8 [\lambda_8]_8 = -\frac{8}{8} a_8 [\text{a2}\ ]_8 < 0$.

Let $\Delta = \max_{8<u<\infty} \left|h_* [h_*]_8 \beta_8(a_8) - \mu_8(a_8)\right|$. Then $|n_8(t)|$ ⊠⊿∀t ⊠. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_8^8 < 4\,a_8\,e$ and $[a_8] > 1\Delta$,or $[a_8^8] \geq 4[a_8]e$ and $(a_8 - \sqrt{(a_8^8)_8 - 4(a_8)_8\,e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+8}\right)$ and $\beta_8(a_8) = ae^{-ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8) + d\tau_8}da = \frac{1}{(m+\mu_0+1)^8} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0] \geq 0.5$. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\,x_8/(1 + x_8)$and $h_8(x_8)=(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [K][P_0])}{[m_8]_8 + (\mu_0)_8 + [K][P_0])_8^3\ [K]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)\ [P_0]_8(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ P^*\ [f_{8*}] = \frac{\sigma Q_8^*}{8+Q_8^*} \quad [h_*] = \frac{8+\omega_8 Q_8^*}{8+Q_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}^{\ 8}_{8*}] = \frac{\sigma_8}{(8+\mathrm{Q}^{*}_{8})^{8}}[\ \mathrm{g}^{\ 8}_{*}] = \frac{\sigma_8}{(8+\mathrm{P}^{*}_{8})^{8}} \quad [\mathrm{h}^{\ 8}_{*}] = \frac{\omega_8 - 8}{(8+\mathrm{Q}^{*}_{8})^{8}}$

In order to have (H3) and (H4) hold, $m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma)^{\ 8}} + \frac{1\omega_8}{([m_8]_{\ 8} + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 + b_8} + \sigma_8)^{\ 3}} < 1$$

And

$$\frac{1}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 8}} + \frac{1\omega_8}{(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^{\ 3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the following equation:

$(m_8 + [\mu_0]_{\ 8} + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_{\ 8} + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\ \mathrm{Q}^{*}_{8})^{\ 3}$

$= m_8 + [\mu_0]_{\ 8} + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + \mathrm{Q}^{*})(1 + \omega_8 \mathrm{Q}^{*}_{8}$

Now

$$\left[\mathrm{a}^{\ 1}_{1}\right] - 4\left[\mathrm{a}_{\ 8}\right]\mathrm{e} = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^{8} + \frac{4K_8 b_8 \sigma_8\ \mathrm{Q}^{*}_{8}}{(\delta_8 - b_8)(8 + \mathrm{Q}^{*}_{8})} + \frac{\sigma^{\ 8}_{8}\mathrm{Q}^{*8}_{\ 8}}{(1 + \mathrm{Q}^{*}_{8})^{\ 8}}$$

$$\frac{4\,b_8\sigma_8\,(\delta_{\,8}-\mathrm{b})\mathrm{Q}^{\,*}}{\delta_{\,8}\,(1+\mathrm{Q}_8^{\,*})^{\,8}}-\frac{4\,(\delta_{\,8}-b_8)^{\,8}\,(1-\omega_8)\mathrm{Q}_{\,8}^{\,*}}{\delta_{\,8}\,(1+\omega_8\mathrm{Q}_8^{\,*})(1+\mathrm{Q}_8^{\,*})}\mathrm{P}^{\,*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{\,8}\,ae_{\,8}^{\,mu}\;\;[\mu_0]_{\,8}-\frac{a_8}{a_8+8}$$

Hence if $[\mathrm{a}_8^{\,8}>4(a_8)_{\,8}\mathrm{e}$ and

$$[a_8]=\frac{1K_8b_8}{\delta_8-b_8}+\frac{\sigma_8\,\mathrm{Q}_8^{\,*}}{1+\mathrm{Q}_8^{\,*}}>1\Delta$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8}=[f_{8*}]_{\,8}+[f_{8*}]_{\,8}+[f_{8*}]\,\mathrm{P}_8^{\,*}$$

$$[a_8]_{\,8}=\left[\mathrm{f}_{8\,*}^{\,8}\right]\mathrm{P}_8^{\,*}-\frac{\mathrm{h}_{\,*}^{\,8}}{\mathrm{h}_{\,*}}\mathrm{P}^{\,*}(0)$$

$$\mathrm{e}=[\,\mathrm{g}_*^{\,8}]_{\,8}\,\mathrm{Q}_8^{\,*}$$

$$\eta_8(\mathrm{t})=\left[\mathrm{h}_{\,*}\right]_{\,8}\beta_{\,8}\left(\theta_{\,8}(\mathrm{t})\right)-\mu_{\,8}\left(\theta(\mathrm{t})\right),$$

The system(32)can be written as the following system:

$$[\mathrm{M(t)}]\left[\mathrm{a}_{8}\right]M_8 - [a_8]_{\,8}W_8 - [\mathrm{n(t)}]M_8$$

$$w_8(\mathrm{t}) = \mathrm{e}M_8$$

Write

$$\mathrm{A}=\begin{pmatrix} -[a_8]_{\,8} & -[a_8]_{\,8} \\ e & 0 \end{pmatrix}$$

$$\mathrm{B}=(\mathrm{t})=\begin{pmatrix} [\eta(t)]_{\,8} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$\mathrm{X(t)}=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$$

The solution of (32)is

$$X_8(\mathrm{t}) = \mathrm{e}^{A,t}[\mathrm{X}]_{\,8}(0) + \int_0 \mathrm{e}^{At}$$

This equation notwithstanding its importance in all the models stand out for its importance for the following models

to which it is applied:

**Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016**

(641)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.

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| (642)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (643) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (644) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (645)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (646) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (647)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (648)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (649)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems. |
| (650)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions. |
| (651) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field. |
| (652)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface. |
| (653)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator. |
| (654)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states. |
| (655) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions. |
| (656)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(657)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(658)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(659)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(660)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(661)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(662)Bringing them to thermal equilibrium requires (e) an external bath.

(663) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(664)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(665)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(666)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(667)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(668)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(669) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(670) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(671) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(672)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(673)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

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| (674) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL. |
| (675)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked. |
| (676)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice. |
| (677) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site. |
| (678)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself). |
| (679)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition. |
| (680)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (681) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (682)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (683) This is a realistic scenario for (e) all known systems. |
| (684)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (685)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (686)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (687) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (688)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (689)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (690)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (691)So in those systems, the quantum phase transition does indeed exist. |
| (692)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (693)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus |

find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity**

**avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2} \in [-0.99, 0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2} \in [2,48] M_\odot$, and (e&eb) total mass $M_{total} \leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced**

**LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 99. MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \ P_9 \ 0, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \ \infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ 0 $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth

rate with $\beta(a) \geq 0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h'(Q) \leq 0, \forall Q \geq 0$, and $\lim_{Q\to\infty} h(Q) = k_3 \geq 0$.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9 P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9) \int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \leq \rho_9(t, a_9) \leq \rho_9^*(t, a_9),$$

$$0 \leq Q_9(t) \leq Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9)da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9)da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \tag{10}$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \tag{$H1$}$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9Q_9 + c_9P_9Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9P_9 - \lambda_9P_9Q_9$$

$$\frac{dQ_9}{dt} = b_9Q_9 - c_9P_9Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9\,(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9\,(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$

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PREDATOR-PREY POPULATION MODELS

The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9 u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t, a_9) = \tilde{u}_9(a_9)^{e^{\gamma(t-a_9)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_9}\, da_9$. Then,

A solution of the first differential equation for $\tilde{u}(a_9)$ gives,

$$\tilde{u}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{u}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})\tau_9}\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^\infty \beta_9(a_9)e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a (\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds}\,d\tau_9\, da_9$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\,da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1 - \int_0^\infty e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9 + \int_0^\tau \mu_9(s_9)ds + y\tau}\,\alpha_9(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_9 a_9 - (f_{90})_9 a_9 - \int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty}Y_9(\gamma_9,a_9)\int_0^{a}(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^{a}(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90_9}^{l}\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^{a}\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9, a_9) = \int_0^a (\rho_0)_9(\tau_9) e^{\gamma_9\tau_9 + (f_{90})_9\tau + \int_0^\tau \mu_9(s_9)ds_9} d\tau$$

$$= \frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9 a_9} - 1)$$

Let

$$E_9(a_9) = e^{-(f_{90})_9 a_9 - \int_0^a \mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ \left(\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(a_9)da_9 - \int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) da_9 = 1 - \int_0^\infty \beta_9(a_9) E_9(a_9) e^{-\gamma_9 a_9} da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9) Y_9(\gamma_9 a_9) \int_0^a \alpha_9(\tau_9) Y^{-9}(\gamma_9, \tau_9) d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9 (\rho_0)_9 \int_0^\infty E_9(a_9) da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$p_9^*(0) = h_9(Q_9^*) \int_0^\infty \beta_9(a_9) p_9^*(a_J) p_9^*(a_9) da_9$ (27)

$g_9(P_9^*) - b_9 = 0,$ (28)

Where $P_9^* = \int_0^\infty p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b),$

From (26),

$p_9^*(a) = p_9^*(0) e^{\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9}$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*) \int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(Q_9^*) a_9} da_9\right) p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9) \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9) d\tau_9 - [f_9]_9(P_9^*) a_9 - [f_9]_9(z_9) a_9} da_{,9}$$

And assume that

$$[k_3]_9 \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9 a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left([f_9^{'}]_9(z_9)\right)\int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9)\int_0^{\infty}\beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}\, da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty}\beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty}\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y_9^a} da_9}$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r$, $R\left(f_9(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \quad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9)da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - \left[p^*\right]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9)m_9(t,a_9)\,da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0)W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $\left[\dot{f}_{9*}\right]_9$ denoting $\left[f^{'}{}_9(P^*)\right]_9 \left[\dot{f}_{9*}\right]_9$ denoting $[f_9]_9$ $(Q_9^*,$ $h_{9}^{*}$ denoting $\dot{h}_*$ denoting $\dot{h}_9$ $(Q_9^*)$, and $\left[\dot{g}_*\right]_9$

Denoting $\left[\dot{g}\right]_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9[p_9^*](a_9)M_9 - [f'_{9*}]_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9)m_9(t,a_9)da_9 + \frac{[h_*^9]_9}{[h^*]_9}[p^*]_9(0)\,W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9[Q_9^*]M_9$$

Integrating $[m_9]_9(t,a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t,a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t,a_9)da_9 - \int_0^\infty \mu(a_9)m(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f'_{9*}]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$$= m_9(t,0) - \int_0^\infty \mu_9(a_9)\,m_9(t,a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + [f'_{9*}]_9 P_9^*\right)M_9 - [f'_{9*}]_9 P_9^* W_{,9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + [f'_{9*}]_9 P_9^*\right)M_9 + \left(\frac{[h'_*]_9}{[h_*]^9}\,P_9^*(0) - f'_{9*}P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_{*}\right]_9 Q_9^{*} M_{.9}$$

By the mean value theorem,

$$\int_0^{\infty} ([h_{*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t,a_9)da_9 = ([h_{*}]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9[a_9^9]2<4\ [a_9]2<4\ a_9e$,

$\text{Re}\,\lambda_9\left[\lambda_9\right]_9 = \text{Re}\lambda_9\left[\lambda_9\right]_9 = -\frac{9}{9}a_9[a2\ ]_9 < 0.$

Let $\Delta=\max_{9<u<\infty}\left|h_{*}[h_{*}]_9\beta_9(a_9) - \mu_9(a_9)\right|$. Then$|n_9(t)|$ ?Δ∀t ?. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^{*}(a), \boldsymbol{Q}^{*}$) is asymptotically stable if either $a_9^9 <4\ a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ?4$[a_9]e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae\text{-}^{ma}$ Then $\Delta=[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9) + d\tau_9}da = \frac{1}{(m+\mu_0+1)^9} + \frac{1}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ 0.5. Assume$[f_9](x_9)= K_9x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9)= [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9)=(1+\omega_9x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])\,{}^{3}_{9}\ [\mathrm{K}]_{\,9}[P_0]\,{}^{3}_{9}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)\ [P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ \left[\mathrm{f}\,{}^{9}_{9*}\right] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\,[\,\mathrm{g}\,{}^{9}_{*}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad \left[\mathrm{h}\,{}^{9}_{*}\right] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_9 + [\mu_0]_9 + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_9 + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\, Q_9^*)^3$$

$$= m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1 + Q^*)(1 + \omega_9 Q_9^*$$

Now

$$\left[a\begin{matrix}1\\1\end{matrix}\right] - 4\left[a_{\,9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9\, Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9\, Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9\, (\delta_9 - b)Q^*}{\delta_9\, (1 + Q_9^*)^9} - \frac{4\, (\delta_9 - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_9 (1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{mu}_9\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a_9^9 > 4(a_9)_9 e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9\, Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^9_{9*}\right]P^*_9 - \frac{h^9_*}{h_*}P^*(0)$$

$$e=[g^9_*]_9\,Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9\beta_9\left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(t) = e^{A,t}[X]_9(0) + \int_0 e^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual**

**oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of**

**classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed**

**huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy**

**holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION FOUR

## Universal Operator Theoretic Framework For Quantum Fault Tolerance

### INTRODUCTION—VARIABLES USED

**A Universal Operator Theoretic Framework for Quantum Fault Tolerance Gerald Gilbert (1), Michael Hamrick (1), Yaakov S. Weinstein (1), Vaneet Aggarwal (2), A. Robert Calderbank (2) ((1) MITRE Quantum Information Science Group, (2) Princeton University) (Submitted on 2 Sep 2007 (v1), last revised 12 Feb 2008 (this version, v3))**

(1) In this paper authors introduce a universal operator theoretic framework for (e) quantum fault tolerance.
(2) This incorporates a top-down approach that implements (eb) a system-level criterion based on (e) specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation.
(3) This leads to (eb) more accurate determinations of (e) error thresholds than could previously be obtained.
(4) This is demonstrated both formally and with an explicit numerical example. The basis for approach is the Quantum Computer Condition (QCC), an inequality governing (e&eb) the evolution of a quantum computer.
(5) Authors show that all known coding schemes are (=) actually special cases of the QCC.
**(6)** They demonstrate this by introducing a new, operator theoretic form of (e) entanglement assisted quantum error correction, which incorporates as (=) special cases all known error correcting protocols, and is itself a special case of the QCC. Subjects: Quantum Physics (quant-ph) Cite as:arXiv: 0709.0128 [quant-ph] (or arXiv: 0709.0128v3 [quant-ph] for this version)

### NOTATION

**Module One**

In this paper authors introduce a universal operator theoretic framework for (e) quantum fault tolerance

| | |
|---|---|
| $G_{13}$ : Category one of quantum fault tolerance<br>$G_{14}$ : Category two of SAS<br>$G_{15}$ : Category three of SAS | |
| $T_{13}$ : Category one of universal operator theoretic framework<br>$T_{14}$ : Category two of SAS<br>$T_{15}$ : Category three of SAS | |
| **Module Two**<br>This incorporates a top-down approach that implements (eb) a system-level criterion based on (e) specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation | |
| $G_{16}$ : Category one of top-down approach<br>$G_{17}$ : Category two of SAS<br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one of system-level criterion based on (e) specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation<br>$T_{17}$ : Category two of SAS<br>$T_{18}$ : Category three of SAS | |
| **Module three**<br>This incorporates a top-down approach that implements a system-level criterion based on (e) specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation | |
| $G_{20}$ : Category one of specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation<br>$G_{21}$ : Category two of SAS<br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of top-down approach that implements a system-level criterion<br>$T_{21}$ : Category two of SAS<br>$T_{22}$ : Category three of SAS | |
| **Module four**<br>This incorporates a top-down approach that implements a system-level criterion based on specification of the full system dynamics, applied at (e&eb) every level of error correction concatenation | |
| $G_{24}$ : Category one of **top-down approach that implements a system-level criterion based on** | |

| | |
|---|---|
| **specification of the full system dynamics;** every level of error correction concatenation<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one of every level of error correction concatenation; **top-down approach that implements a system-level criterion based on specification of the full system dynamics**<br><br>$T_{25}$ : Category two of SAS<br><br>$T_{26}$ : Category three of SAS | |
| **Module five**<br><br>This leads to (eb) more accurate determinations of (e) error thresholds than could previously be obtained | |
| $G_{28}$ : Category one of error thresholds<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of accurate determinations<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>This is demonstrated both formally and with an explicit numerical example<br><br>The basis for approach is the Quantum Computer Condition (QCC), an inequality governing (e&eb) the evolution of a quantum computer | |
| $G_{32}$ : Category one of **Quantum Computer Condition (QCC), an inequality;** evolution of a quantum computer<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of evolution of a quantum computer **;Quantum Computer Condition (QCC), an inequality**<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven** | |

| | |
|---|---|
| Authors show that all known coding schemes are (=) actually special cases of the QCC | |
| $G_{36}$ : Category one of all known coding schemes<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of actually special cases of the QCC<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>They demonstrate this by introducing a new, operator theoretic form of (e) entanglement assisted quantum error correction, which incorporates as (=) special cases all known error correcting protocols, and is itself a special case of the QCC.<br><br>Subjects: Quantum Physics (quant-ph) Cite as:arXiv: 0709.0128 [quant-ph] (or arXiv: 0709.0128v3 [quant-ph] for this version) | |
| $G_{40}$ : Category one of entanglement assisted quantum error correction, which incorporates as (=) special cases all known error correcting protocols, and is itself a special case of the QCC.<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of operator theoretic form<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>They demonstrate this by introducing a new, operator theoretic form of entanglement assisted quantum error correction, which incorporates as special cases all known error correcting protocols, and is itself a special case of the QCC.<br><br>Subjects: Quantum Physics (quant-ph) Cite as:arXiv: 0709.0128 [quant-ph] (or arXiv: 0709.0128v3 [quant-ph] for this version) | |
| $G_{44}$ : Category one of special cases all known error correcting protocols, and is itself a special case of the QCC.<br><br>$G_{45}$ : Category two of SAS | |

| | |
|---|---|
| $G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of operator theoretic form of entanglement assisted quantum error correction<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |
| **The Coefficients:** | |
| $(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}\ (a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}\ (b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:<br>$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$<br>$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,<br>$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$<br>$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$<br>$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$<br>$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$<br><br>are Accentuation coefficients<br>$(a_{13}^{'})^{(1)}, (a_{14}^{'})^{(1)}, (a_{15}^{'})^{(1)}, (b_{13}^{'})^{(1)}, (b_{14}^{'})^{(1)}, (b_{15}^{'})^{(1)}, (a_{16}^{'})^{(2)}, (a_{17}^{'})^{(2)}, (a_{18}^{'})^{(2)}$, $(b_{16}^{'})^{(2)}, (b_{17}^{'})^{(2)}, (b_{18}^{'})^{(2)}$<br>$, (a_{20}^{'})^{(3)}, (a_{21}^{'})^{(3)}, (a_{22}^{'})^{(3)}, (b_{20}^{'})^{(3)}, (b_{21}^{'})^{(3)}, (b_{22}^{'})^{(3)}$<br>$(a_{24}^{'})^{(4)}, (a_{25}^{'})^{(4)}, (a_{26}^{'})^{(4)}, (b_{24}^{'})^{(4)}, (b_{25}^{'})^{(4)}, (b_{26}^{'})^{(4)}, (b_{28}^{'})^{(5)}, (b_{29}^{'})^{(5)}, (b_{30}^{'})^{(5)}\ (a_{28}^{'})^{(5)}, (a_{29}^{'})^{(5)}, (a_{30}^{'})^{(5)}$ ,<br>$(a_{32}^{'})^{(6)}, (a_{33}^{'})^{(6)}, (a_{34}^{'})^{(6)}, (b_{32}^{'})^{(6)}, (b_{33}^{'})^{(6)}, (b_{34}^{'})^{(6)}$<br>$(a_{36}^{'})^{(7)}, (a_{37}^{'})^{(7)}, (a_{38}^{'})^{(7)}, (b_{36}^{'})^{(7)}, (b_{37}^{'})^{(7)}, (b_{38}^{'})^{(7)}$,<br>$(a_{40}^{'})^{(8)}, (a_{41}^{'})^{(8)}, (a_{42}^{'})^{(8)}, (b_{40}^{'})^{(8)}, (b_{41}^{'})^{(8)}, (b_{42}^{'})^{(8)}$,<br>$(a_{44}^{'})^{(9)}, (a_{45}^{'})^{(9)}, (a_{46}^{'})^{(9)}, (b_{44}^{'})^{(9)}, (b_{45}^{'})^{(9)}, (b_{46}^{'})^{(9)}$,<br>are Dissipation coefficients | |
| **Module Numbered One**<br><br>**The differential system of this model is now (Module Numbered one)** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14}, t)\right]G_{13}$ | 1 |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14}, t)\right]G_{14}$ | 2 |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14}, t)\right]G_{15}$ | 3 |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G, t)\right]T_{13}$ | 4 |
| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G, t)\right]T_{14}$ | 5 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G, t)\right]T_{15}$ | 6 |
| $+(a_{13}^{''})^{(1)}(T_{14}, t) =$ First augmentation factor | |
| $-(b_{13}^{''})^{(1)}(G, t) =$ First detritions factor | |
| **Module Numbered Two**<br><br>**The differential system of this model is now ( Module numbered two)** | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right]G_{16}$ | 7 |

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| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17}, t)\right]G_{17}$ | 8 |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17}, t)\right]G_{18}$ | 9 |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{16}$ | 10 |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{17}$ | 11 |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}\left((G_{19}), t\right)\right]T_{18}$ | 12 |
| $+(a_{16}^{''})^{(2)}(T_{17}, t) =$ First augmentation factor | |
| $-(b_{16}^{''})^{(2)}\left((G_{19}), t\right) =$ First detritions factor | |
| **Module Numbered Three**<br><br>**The differential system of this model is now (Module numbered three)** | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right]G_{20}$ | 13 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21}, t)\right]G_{21}$ | 14 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21}, t)\right]G_{22}$ | 15 |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23}, t)\right]T_{20}$ | 16 |
| $\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23}, t)\right]T_{21}$ | 17 |
| $\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23}, t)\right]T_{22}$ | 18 |
| $+(a_{20}^{''})^{(3)}(T_{21}, t) =$ First augmentation factor | |
| $-(b_{20}^{''})^{(3)}(G_{23}, t) =$ First detritions factor | |
| **Module Numbered Four**<br><br>**The differential system of this model is now (Module numbered Four)** | |
| $\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25}, t)\right]G_{24}$ | 19 |
| $\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25}, t)\right]G_{25}$ | 20 |
| $\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25}, t)\right]G_{26}$ | 21 |
| $\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{24}$ | 22 |
| $\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{25}$ | 23 |
| $\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\left((G_{27}), t\right)\right]T_{26}$ | 24 |
| $+(a_{24}^{''})^{(4)}(T_{25}, t) =$ **First augmentation factor** | |
| $-(b_{24}^{''})^{(4)}\left((G_{27}), t\right) =$ **First detritions factor** | |
| **Module Numbered Five**:<br><br>**The differential system of this model is now (Module number five)** | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right]G_{28}$ | 25 |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29}, t)\right]G_{29}$ | 26 |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}, t)\right]G_{30}$ | 27 |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}\left((G_{31}), t\right)\right]T_{28}$ | 28 |
| $\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}\left((G_{31}), t\right)\right]T_{29}$ | 29 |

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| $\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31}),t)\right]T_{30}$ | 30 |
| $+(a''_{28})^{(5)}(T_{29},t) =$ **First augmentation factor** | |
| $-(b''_{28})^{(5)}((G_{31}),t) =$ **First detritions factor** | |
| **Module Numbered Six**<br><br>**The differential system of this model is now (Module numbered Six)** | |
| $\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right]G_{32}$ | 31 |
| $\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33},t)\right]G_{33}$ | 32 |
| $\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33},t)\right]G_{34}$ | 33 |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35}),t)\right]T_{32}$ | 34 |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}((G_{35}),t)\right]T_{33}$ | 35 |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35}),t)\right]T_{34}$ | 36 |
| $+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor** | |
| **Module Numbered Seven:**<br><br>**The differential system of this model is now (Seventh Module)** | |
| $\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ | 37 |
| $\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ | *38* |
| $\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ | *39* |
| $\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39}),t)\right]T_{36}$ | *40* |
| $\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}((G_{39}),t)\right]T_{37}$ | *41* |
| $\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39}),t)\right]T_{38}$ | *42* |
| $+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor** | |
| **Module Numbered Eight**<br><br>The differential system of this model is now | |
| $\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ | *43* |
| $\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ | *44* |
| $\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ | *45* |
| $\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43}),t)\right]T_{40}$ | *46* |
| $\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}((G_{43}),t)\right]T_{41}$ | *47* |
| $\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43}),t)\right]T_{42}$ | *48* |
| **Module Numbered Nine**<br><br>The differential system of this model is now | |
| $\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ | 49 |
| $\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ | 50 |

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| $\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45},t)\right]G_{46}$ | 51 |
| $\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ | 52 |
| $\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ | 53 |
| $\frac{dT_{46}}{dt} = (b_{46})^{(9)}T_{45} - \left[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}\left((G_{47}),t\right)\right]T_{46}$ | 54 |
| $+(a_{44}^{''})^{(9)}(T_{45},t) =$ **First augmentation factor** | |
| $-(b_{44}^{''})^{(9)}\left((G_{47}),t\right) =$ **First detrition factor** | |
| $\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \begin{bmatrix} (a_{13}^{'})^{(1)}\boxed{+(a_{13}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$ | *55* |
| $\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \begin{bmatrix} (a_{14}^{'})^{(1)}\boxed{+(a_{14}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$ | *56* |
| $\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \begin{bmatrix} (a_{15}^{'})^{(1)}\boxed{+(a_{15}^{''})^{(1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$ | *57* |
| Where $\boxed{(a_{13}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{14}^{''})^{(1)}(T_{14},t)}$, $\boxed{(a_{15}^{''})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3<br>$\boxed{+(a_{16}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2,2,)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{20}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{21}^{''})^{(3,3,)}(T_{21},t)}$, $\boxed{+(a_{22}^{''})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{24}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}^{''})^{(4,4,4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}^{''})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{28}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{29}^{''})^{(5,5,5,5,)}(T_{29},t)}$, $\boxed{+(a_{30}^{''})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{32}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6,6,6,6,)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3<br>$\boxed{+(a_{38}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{37}^{''})^{(7,7)}(T_{37},t)}$ $\boxed{+(a_{36}^{''})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3<br>$\boxed{+(a_{40}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{41}^{''})^{(8,8)}(T_{41},t)}$ $\boxed{+(a_{42}^{''})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3<br>$\boxed{+(a_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \begin{bmatrix} (b_{13}^{'})^{(1)}\boxed{-(b_{13}^{''})^{(1)}(G,t)} & \boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 58 |

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| $\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \begin{bmatrix} (b_{14}^{'})^{(1)}\boxed{-(b_{14}^{''})^{(1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | 59 |
| $\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \begin{bmatrix} (b_{15}^{'})^{(1)}\boxed{-(b_{15}^{''})^{(1)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | *60* |
| Where $\boxed{-(b_{13}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1)}(G,t)}$, $\boxed{-(b_{15}^{''})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{37}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{41}^{''})^{(8,8)}(G_{43},t)}$ $\boxed{-(b_{42}^{''})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |
| | |
| $\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \begin{bmatrix} (a_{16}^{'})^{(2)}\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{13}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{20}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{16}$ | *61* |
| $\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \begin{bmatrix} (a_{17}^{'})^{(2)}\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{14}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{21}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{17}$ | *62* |
| $\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \begin{bmatrix} (a_{18}^{'})^{(2)}\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)} & \boxed{+(a_{15}^{''})^{(1,1,)}(T_{14},t)} & \boxed{+(a_{22}^{''})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9)}(T_{45},t)} \end{bmatrix} G_{18}$ | *63* |
| Where $\boxed{+(a_{16}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{17}^{''})^{(2)}(T_{17},t)}$, $\boxed{+(a_{18}^{''})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3 | |

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| $+(a_{13}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{14}^{''})^{(1,1,)}(T_{14},t)$ , $+(a_{15}^{''})^{(1,1,)}(T_{14},t)$ are second augmentation coefficient for category 1, 2 and 3<br>$+(a_{20}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{21}^{''})^{(3,3,3)}(T_{21},t)$ , $+(a_{22}^{''})^{(3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5)}(T_{29},t)$ , $+(a_{30}^{''})^{(5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6)}(T_{33},t)$ , $+(a_{34}^{''})^{(6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{37}^{''})^{(7,7,7)}(T_{37},t)$ , $+(a_{38}^{''})^{(7,7,7)}(T_{37},t)$ are seventh augmentation coefficient for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8)}(T_{41},t)$ , $+(a_{42}^{''})^{(8,8,8)}(T_{41},t)$ are eight augmentation coefficient for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{45}^{''})^{(9,9)}(T_{45},t)$ , $+(a_{46}^{''})^{(9,9)}(T_{45},t)$ are ninth augmentation coefficient for category 1, 2 and 3 | |
| $\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[\begin{matrix}(b_{16}^{'})^{(2)}\left[-(b_{16}^{''})^{(2)}(G_{19},t)\right] & \left[-(b_{13}^{''})^{(1,1,)}(G,t)\right] & \left[-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)\right] \\ \left[-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{36}^{''})^{(7,7,7)}(G_{39},t)\right] & \left[-(b_{40}^{''})^{(8,8,8)}(G_{43},t)\right] & \left[-(b_{44}^{''})^{(9,9)}(G_{47},t)\right]\end{matrix}\right] T_{16}$ | *64* |
| $\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[\begin{matrix}(b_{17}^{'})^{(2)}\left[-(b_{17}^{''})^{(2)}(G_{19},t)\right] & \left[-(b_{14}^{''})^{(1,1,)}(G,t)\right] & \left[-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)\right] \\ \left[-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{37}^{''})^{(7,7,7)}(G_{39},t)\right] & \left[-(b_{41}^{''})^{(8,8,8)}(G_{43},t)\right] & \left[-(b_{45}^{''})^{(9,9)}(G_{47},t)\right]\end{matrix}\right] T_{17}$ | *65* |
| $\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[\begin{matrix}(b_{18}^{'})^{(2)}\left[-(b_{18}^{''})^{(2)}(G_{19},t)\right] & \left[-(b_{15}^{''})^{(1,1,)}(G,t)\right] & \left[-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)\right] \\ \left[-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)\right] & \left[-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)\right] & \left[-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)\right] \\ \left[-(b_{38}^{''})^{(7,7,7)}(G_{39},t)\right] & \left[-(b_{42}^{''})^{(8,8,8)}(G_{43},t)\right] & \left[-(b_{46}^{''})^{(9,9)}(G_{47},t)\right]\end{matrix}\right] T_{18}$ | *66* |
| where $-(b_{16}^{''})^{(2)}(G_{19},t)$ , $-(b_{17}^{''})^{(2)}(G_{19},t)$ , $-(b_{18}^{''})^{(2)}(G_{19},t)$ are first detrition coefficients for category 1, 2 and 3<br>$-(b_{13}^{''})^{(1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,)}(G,t)$ are second detrition coefficients for category 1,2 and 3<br>$-(b_{20}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,)}(G_{23},t)$ are third detrition coefficients for category 1,2 and 3<br>$-(b_{24}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4)}(G_{27},t)$ are fourth detrition coefficients for category 1,2 and 3<br>$-(b_{28}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5)}(G_{31},t)$ are fifth detrition coefficients for category 1,2 and 3<br>$-(b_{32}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6)}(G_{35},t)$ are sixth detrition coefficients for category 1,2 and 3 | |

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| $-(b_{36}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1,2 and 3<br>$-(b_{40}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1,2 and 3<br>$-(b_{44}^{''})^{(9,9)}(G_{47},t)$, $-(b_{46}^{''})^{(9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1,2 and 3 | |
| $\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \begin{bmatrix} (a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21},t) & +(a_{16}^{''})^{(2,2,2)}(T_{17},t) & +(a_{13}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{36}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{20}$ | 67 |
| $\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \begin{bmatrix} (a_{21}^{'})^{(3)}+(a_{21}^{''})^{(3)}(T_{21},t) & +(a_{17}^{''})^{(2,2,2)}(T_{17},t) & +(a_{14}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{37}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{21}$ | 68 |
| $\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \begin{bmatrix} (a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21},t) & +(a_{18}^{''})^{(2,2,2)}(T_{17},t) & +(a_{15}^{''})^{(1,1,1,)}(T_{14},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t) \\ +(a_{38}^{''})^{(7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9)}(T_{45},t) \end{bmatrix} G_{22}$ | 69 |
| $+(a_{20}^{''})^{(3)}(T_{21},t)$, $+(a_{21}^{''})^{(3)}(T_{21},t)$, $+(a_{22}^{''})^{(3)}(T_{21},t)$ are first augmentation coefficients for category 1, 2 and 3<br>$+(a_{16}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2)}(T_{17},t)$ are second augmentation coefficients for category 1, 2 and 3<br>$+(a_{13}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,)}(T_{14},t)$ are third augmentation coefficients for category 1, 2 and 3<br>$+(a_{24}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficients for category 1, 2 and 3<br>$+(a_{28}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficients for category 1, 2 and 3<br>$+(a_{32}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficients for category 1, 2 and 3<br>$+(a_{36}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7)}(T_{37},t)$ are seventh augmentation coefficients for category 1, 2 and 3<br>$+(a_{40}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8)}(T_{41},t)$ are eight augmentation coefficients for category 1, 2 and 3<br>$+(a_{44}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9)}(T_{45},t)$ are ninth augmentation coefficients for category 1, 2 and 3 | |
| $\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \begin{bmatrix} (b_{20}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t) & -(b_{16}^{''})^{(2,2,2)}(G_{19},t) & -(b_{13}^{''})^{(1,1,1,)}(G,t) \\ -(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{36}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{20}$ | 70 |

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \begin{bmatrix} (b_{21}^{'})^{(3)} & -(b_{21}^{''})^{(3)}(G_{23},t) & -(b_{17}^{''})^{(2,2,2)}(G_{19},t) & -(b_{14}^{''})^{(1,1,1,)}(G,t) \\ -(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{37}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{21} \quad (71)$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \begin{bmatrix} (b_{22}^{'})^{(3)} & -(b_{22}^{''})^{(3)}(G_{23},t) & -(b_{18}^{''})^{(2,2,2)}(G_{19},t) & -(b_{15}^{''})^{(1,1,1,)}(G,t) \\ -(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t) & -(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t) \\ -(b_{38}^{''})^{(7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9)}(G_{47},t) \end{bmatrix} T_{22} \quad (72)$$

$-(b_{20}^{''})^{(3)}(G_{23},t)$, $-(b_{21}^{''})^{(3)}(G_{23},t)$, $-(b_{22}^{''})^{(3)}(G_{23},t)$ *are first detrition coefficients for category 1, 2 and 3*

$-(b_{16}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2)}(G_{19},t)$ *are second detrition coefficients for category 1, 2 and 3*

$-(b_{13}^{''})^{(1,1,1,)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,)}(G,t)$ *are third detrition coefficients for category 1,2 and 3*

$-(b_{24}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,4,4,4)}(G_{27},t)$ *are fourth detrition coefficients for category 1, 2 and 3*

$-(b_{28}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5,5,5,5)}(G_{31},t)$ *are fifth detrition coefficients for category 1, 2 and 3*

$-(b_{32}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,6,6,6,6)}(G_{35},t)$ *are sixth detrition coefficients for category 1, 2 and 3*

$-(b_{36}^{''})^{(7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7)}(G_{39},t)$ $-(b_{38}^{''})^{(7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2 and 3

$-(b_{40}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8)}(G_{43},t)$ are eight detrition coefficients for category 1, 2 and 3

$-(b_{46}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \begin{bmatrix} (a_{24}^{'})^{(4)} & +(a_{24}^{''})^{(4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,)}(T_{29},t) & +(a_{32}^{''})^{(6,6,)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{16}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3)}(T_{21},t) \\ +(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{24} \quad (73)$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a_{25}^{'})^{(4)} & +(a_{25}^{''})^{(4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,)}(T_{29},t) & +(a_{33}^{''})^{(6,6)}(T_{33},t) \\ +(a_{14}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{17}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3)}(T_{21},t) \\ +(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{25} \quad (74)$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a_{26}^{'})^{(4)} & +(a_{26}^{''})^{(4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,)}(T_{29},t) & +(a_{34}^{''})^{(6,6,)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1)}(T_{14},t) & +(a_{18}^{''})^{(2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3)}(T_{21},t) \\ +(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t) & +(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9)}(T_{45},t) \end{bmatrix} G_{26} \quad (75)$$

$(a_{24}^{''})^{(4)}(T_{25},t)$, $(a_{25}^{''})^{(4)}(T_{25},t)$, $(a_{26}^{''})^{(4)}(T_{25},t)$ *are first augmentation coefficients category* 1, 2 3

$+(a_{28}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,)}(T_{29},t)$ *are second augmentation*

$coefficient\ \ for\ category\ 1,2\ and\ 3$
$+(a_{32}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,)}(T_{33},t)$ $are\ third\ \ augmentation$
$coefficient\ for\ category\ 1,2\ and\ 3$
$+(a_{13}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1)}(T_{14},t)$
$are\ fourth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$
$+(a_{16}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2)}(T_{17},t)$
$are\ fifth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$
$+(a_{20}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3)}(T_{21},t)$
$are\ sixth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$
$+(a_{36}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7)}(T_{37},t)$
$are\ seventh\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$
$+(a_{40}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8)}(T_{41},t)$
$are\ eighth\ augmentation\ coefficients\ for\ category\ 1,2\ and\ 3$
$+(a_{46}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9)}(T_{45},t)$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt}=(b_{24})^{(4)}T_{25}-\begin{bmatrix}(b_{24}^{'})^{(4)} & -(b_{24}^{''})^{(4)}(G_{27},t) & -(b_{28}^{''})^{(5,5,)}(G_{31},t) & -(b_{32}^{''})^{(6,6,)}(G_{35},t)\\ & -(b_{13}^{''})^{(1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)\\ & -(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{24} \quad (76)$$

$$\frac{dT_{25}}{dt}=(b_{25})^{(4)}T_{24}-\begin{bmatrix}(b_{25}^{'})^{(4)} & -(b_{25}^{''})^{(4)}(G_{27},t) & -(b_{29}^{''})^{(5,5,)}(G_{31},t) & -(b_{33}^{''})^{(6,6,)}(G_{35},t)\\ & -(b_{14}^{''})^{(1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)\\ & -(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{25} \quad (77)$$

$$\frac{dT_{26}}{dt}=(b_{26})^{(4)}T_{25}-\begin{bmatrix}(b_{26}^{'})^{(4)} & -(b_{26}^{''})^{(4)}(G_{27},t) & -(b_{30}^{''})^{(5,5,)}(G_{31},t) & -(b_{34}^{''})^{(6,6,)}(G_{35},t)\\ & -(b_{15}^{''})^{(1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)\\ & -(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9)}(G_{47},t)\end{bmatrix}T_{26} \quad (78)$$

$Where\ -(b_{24}^{''})^{(4)}(G_{27},t)$, $-(b_{25}^{''})^{(4)}(G_{27},t)$, $-(b_{26}^{''})^{(4)}(G_{27},t)$ $are\ first\ detrition\ coefficients$
$for\ category\ 1,2\ and\ 3$
$-(b_{28}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,)}(G_{31},t)$ $are\ second\ detrition\ coefficients$
$for\ category\ 1,2\ and\ 3$
$-(b_{32}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{33}^{''})^{(6,6,)}(G_{35},t)$, $-(b_{34}^{''})^{(6,6,)}(G_{35},t)$ $are\ third\ detrition\ coefficients$
$for\ category\ 1,2\ and\ 3$
$-(b_{13}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1)}(G,t)$
$are\ fourth\ \ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$
$-(b_{16}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2)}(G_{19},t)$
$are\ fifth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$
$-(b_{20}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3)}(G_{23},t)$
$are\ sixth\ detrition\ coefficients\ for\ category\ 1,2\ and\ 3$
$-(b_{36}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7)}(G_{39},t)$

| | |
|---|---|
| *are seventh detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{40}'')^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}'')^{(8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}'')^{(8,8,8,8,8)}(G_{43},t)}$<br>*are eighth detrition coefficients for category* 1, 2 *and* 3<br>$\boxed{-(b_{46}'')^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}'')^{(9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}'')^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3 | |
| $\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a_{28}')^{(5)}\boxed{+(a_{28}'')^{(5)}(T_{29},t)} & \boxed{+(a_{24}'')^{(4,4,)}(T_{25},t)} & \boxed{+(a_{32}'')^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}'')^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}'')^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}'')^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}'')^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}'')^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28}$ | *79* |
| $\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a_{29}')^{(5)}\boxed{+(a_{29}'')^{(5)}(T_{29},t)} & \boxed{+(a_{25}'')^{(4,4,)}(T_{25},t)} & \boxed{+(a_{33}'')^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}'')^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}'')^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}'')^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29}$ | *80* |
| $\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a_{30}')^{(5)}\boxed{+(a_{30}'')^{(5)}(T_{29},t)} & \boxed{+(a_{26}'')^{(4,4,)}(T_{25},t)} & \boxed{+(a_{34}'')^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}'')^{(7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}'')^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30}$ | *81* |
| *Where* $\boxed{+(a_{28}'')^{(5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3<br>*And* $\boxed{+(a_{24}'')^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{32}'')^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3<br>$\boxed{+(a_{13}'')^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}'')^{(1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3<br>$\boxed{+(a_{16}'')^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3<br>$\boxed{+(a_{20}'')^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{36}'')^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{37}'')^{(7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a_{38}'')^{(7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{40}'')^{(8,8\ ,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}'')^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3<br>$\boxed{+(a_{46}'')^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a_{44}'')^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3 | |
| $\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \begin{bmatrix} (b_{28}')^{(5)}\boxed{-(b_{28}'')^{(5)}(G_{31},t)} & \boxed{-(b_{24}'')^{(4,4,)}(G_{27},t)} & \boxed{-(b_{32}'')^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{36}'')^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}'')^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{28}$ | *82* |

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \begin{bmatrix} (b_{29}^{'})^{(5)}\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{29} \quad (83)$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \begin{bmatrix} (b_{30}^{'})^{(5)}\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)} & \boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)} & \boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{30} \quad (84)$$

*where* $\boxed{-(b_{28}^{''})^{(5)}(G_{31},t)}$ , $\boxed{-(b_{29}^{''})^{(5)}(G_{31},t)}$ , $\boxed{-(b_{30}^{''})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b_{24}^{''})^{(4,4,)}(G_{27},t)}$ , $\boxed{-(b_{25}^{''})^{(4,4,)}(G_{27},t)}$ , $\boxed{-(b_{26}^{''})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b_{32}^{''})^{(6,6,6)}(G_{35},t)}$ , $\boxed{-(b_{33}^{''})^{(6,6,6)}(G_{35},t)}$ , $\boxed{-(b_{34}^{''})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b_{13}^{''})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1)}(G,t)}$ , $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b_{46}^{''})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \begin{bmatrix} (a_{32}^{'})^{(6)}\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{28}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{32} \quad (85)$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \begin{bmatrix} (a_{33}^{'})^{(6)}\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{29}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{33} \quad (86)$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \begin{bmatrix} (a_{34}^{'})^{(6)}\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)} & \boxed{+(a_{30}^{''})^{(5,5,5)}(T_{29},t)} & \boxed{+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)} & \boxed{+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{34} \quad (87)$$

$\boxed{+(a_{32}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{33}^{''})^{(6)}(T_{33},t)}$, $\boxed{+(a_{34}^{''})^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3

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| $+(a_{28}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5)}(T_{29},t)$ *are second augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{24}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,)}(T_{25},t)$ *are third augmentation coefficients for category* 1, 2 *and* 3<br>$+(a_{13}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{14}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1)}(T_{14},t)$ - are fourth augmentation coefficients<br>$+(a_{16}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2)}(T_{17},t)$ - fifth augmentation coefficients<br>$+(a_{20}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3)}(T_{21},t)$ sixth augmentation coefficients<br>$+(a_{36}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{37}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$, $+(a_{38}^{''})^{(7,7,7,7,7,7,7)}(T_{37},t)$ seventh augmentation coefficients<br>$+(a_{40}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{42}^{''})^{(8,8,8,8,8,8,8)}(T_{41},t)$ Eighth augmentation coefficients<br>$+(a_{44}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$, $+(a_{46}^{''})^{(9,9,9,9,9,9)}(T_{45},t)$ ninth augmentation coefficients | |
| $\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \begin{bmatrix} (b_{32}^{'})^{(6)} & -(b_{32}^{''})^{(6)}(G_{35},t) & -(b_{28}^{''})^{(5,5,5)}(G_{31},t) & -(b_{24}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{20}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{40}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{44}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{32}$ | *88* |
| $\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \begin{bmatrix} (b_{33}^{'})^{(6)} & -(b_{33}^{''})^{(6)}(G_{35},t) & -(b_{29}^{''})^{(5,5,5)}(G_{31},t) & -(b_{25}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{21}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{41}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{45}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{33}$ | *89* |
| $\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \begin{bmatrix} (b_{34}^{'})^{(6)} & -(b_{34}^{''})^{(6)}(G_{35},t) & -(b_{30}^{''})^{(5,5,5)}(G_{31},t) & -(b_{26}^{''})^{(4,4,4,)}(G_{27},t) \\ & -(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t) & -(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t) & -(b_{22}^{''})^{(3,3,3,3,3,3)}(G_{23},t) \\ & -(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t) & -(b_{42}^{''})^{(8,8,8,8,8,8,8)}(G_{43},t) & -(b_{46}^{''})^{(9,9,9,9,9,9)}(G_{47},t) \end{bmatrix} T_{34}$ | *90* |
| | |
| $-(b_{32}^{''})^{(6)}(G_{35},t)$, $-(b_{33}^{''})^{(6)}(G_{35},t)$, $-(b_{34}^{''})^{(6)}(G_{35},t)$ *are first detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{28}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{29}^{''})^{(5,5,5)}(G_{31},t)$, $-(b_{30}^{''})^{(5,5,5)}(G_{31},t)$ *are second detrition coefficients for category* 1, 2 *and* 3<br>$-(b_{24}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{25}^{''})^{(4,4,4,)}(G_{27},t)$, $-(b_{26}^{''})^{(4,4,4,)}(G_{27},t)$ *are third detrition coefficients for category* 1,2 *and* 3<br>$-(b_{13}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{14}^{''})^{(1,1,1,1,1,1)}(G,t)$, $-(b_{15}^{''})^{(1,1,1,1,1,1)}(G,t)$ are fourth detrition coefficients for category 1, 2, and 3<br>$-(b_{16}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{17}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$, $-(b_{18}^{''})^{(2,2,2,2,2,2)}(G_{19},t)$ are fifth detrition coefficients for category 1, 2, and 3 | |

$-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$, $-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)$ are sixth detrition coefficients for category 1, 2, and 3

$-(b_{36}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{37}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$, $-(b_{38}^{''})^{(7,7,7,7,7,7,7)}(G_{39},t)$ are seventh detrition coefficients for category 1, 2, and 3

$-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$, $-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)$ are eighth detrition coefficients for category 1, 2, and 3

$-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$, $-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \begin{bmatrix} (a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37},t) & +(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t) & +(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{13} \quad (91)$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \begin{bmatrix} (a_{37}^{'})^{(7)}+(a_{37}^{''})^{(7)}(T_{37},t) & +(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{14} \quad (92)$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \begin{bmatrix} (a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37},t) & +(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t) & +(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t) \\ +(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t) & +(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t) & +(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t) \\ +(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t) & +(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t) & +(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t) \end{bmatrix} G_{15} \quad (93)$$

Where $(a_{36}^{''})^{(7)}(T_{37},t)$, $(a_{37}^{''})^{(7)}(T_{37},t)$, $(a_{38}^{''})^{(7)}(T_{37},t)$ are first augmentation coefficients for category 1, 2 and 3

$+(a_{16}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{17}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$, $+(a_{18}^{''})^{(2,2,2,2,2,2,2)}(T_{17},t)$ are second augmentation coefficient for category 1, 2 and 3

$+(a_{20}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{21}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$, $+(a_{22}^{''})^{(3,3,3,3,3,3,3)}(T_{21},t)$ are third augmentation coefficient for category 1, 2 and 3

$+(a_{24}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{25}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$, $+(a_{26}^{''})^{(4,4,4,4,4,4,4)}(T_{25},t)$ are fourth augmentation coefficient for category 1, 2 and 3

$+(a_{28}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{29}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$, $+(a_{30}^{''})^{(5,5,5,5,5,5,5)}(T_{29},t)$ are fifth augmentation coefficient for category 1, 2 and 3

$+(a_{32}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{33}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$, $+(a_{34}^{''})^{(6,6,6,6,6,6,6)}(T_{33},t)$ are sixth augmentation coefficient for category 1, 2 and 3

$+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{13}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$, $+(a_{15}^{''})^{(1,1,1,1,1,1,1)}(T_{14},t)$ are seventh augmentation coefficient for category 1, 2 and 3

$+(a_{42}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{41}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)$, $+(a_{40}^{''})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)$ are eighth augmentation coefficient for 1,2,3

| | |
|---|---|
| $+(a_{46}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{45}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$, $+(a_{44}^{''})^{(9,9,9,9,9,9,9)}(T_{45},t)$ are ninth augmentation coefficient for 1,2,3 | |
| $\frac{dT_{36}}{dt} =$ $(b_{36})^{(7)}T_{37} - \begin{bmatrix} (b_{36}^{'})^{(7)}\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$ | 94 |
| $\frac{dT_{37}}{dt} =$ $(b_{37})^{(7)}T_{36} - \begin{bmatrix} (b_{37}^{'})^{(7)}\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$ | |
| $\frac{dT_{38}}{dt} =$ $(b_{38})^{(7)}T_{37} - \begin{bmatrix} (b_{38}^{'})^{(7)}\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$ | |
| Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3<br>$\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3 | |



$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a'_{40})^{(8)}\boxed{+(a''_{40})^{(8)}(T_{41},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13}$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a'_{41})^{(8)}\boxed{+(a''_{41})^{(8)}(T_{41},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a'_{42})^{(8)}\boxed{+(a''_{42})^{(8)}(T_{41},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a''_{40})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{40}}{dt} =$

$$(b_{40})^{(8)}T_{41} - \begin{bmatrix} (b_{40}^{'})^{(8)}\boxed{-(b_{40}^{''})^{(8)}(G_{43},t)} & \boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{41}}{dt} =$$

$$(b_{41})^{(8)}T_{40} - \begin{bmatrix} (b_{41}^{'})^{(8)}\boxed{-(b_{41}^{''})^{(8)}(G_{43},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{42}}{dt} =$$

$$(b_{42})^{(8)}T_{41} - \begin{bmatrix} (b_{42}^{'})^{(8)}\boxed{-(b_{42}^{''})^{(8)}(G_{43},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{36}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{38}^{''})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{44}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \begin{bmatrix} (a_{44}^{'})^{(9)}\boxed{+(a_{44}^{''})^{(9)}(T_{45},t)} & \boxed{+(a_{16}^{''})^{(2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{20}^{''})^{(3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}^{''})^{(4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{28}^{''})^{(5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{32}^{''})^{(6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}^{''})^{(1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{36}^{''})^{(7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{40}^{''})^{(8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{13}$$

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$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \begin{bmatrix} (a_{45}')^{(9)}\boxed{+(a_{45}'')^{(9)}(T_{45},t)} & \boxed{+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{14}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \begin{bmatrix} (a_{46}')^{(9)}\boxed{+(a_{46}'')^{(9)}(T_{37},t)} & \boxed{+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a_{44}'')^{(9)}(T_{45},t)}$, $\boxed{+(a_{45}'')^{(9)}(T_{45},t)}$, $\boxed{+(a_{46}'')^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{14}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a_{40}'')^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{42}'')^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a_{41}'')^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \begin{bmatrix} (b_{44}')^{(9)}\boxed{-(b_{44}'')^{(9)}(G_{47},t)} & \boxed{-(b_{16}'')^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{20}'')^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{24}'')^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{28}'')^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{32}'')^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{13}'')^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{36}'')^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{13}$$

$$\frac{dT_{45}}{dt} =$$

$$(b_{45})^{(9)}T_{44} - \begin{bmatrix} (b_{45}^{'})^{(9)}\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{46}}{dt} =$$

$$(b_{46})^{(9)}T_{45} - \begin{bmatrix} (b_{46}^{'})^{(9)}\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)} & \boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b_{44}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{45}^{''})^{(9)}(G_{47},t)}$, $\boxed{-(b_{46}^{''})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{16}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{17}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b_{18}^{''})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{20}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{21}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b_{22}^{''})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{24}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{25}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b_{26}^{''})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{29}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b_{30}^{''})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{33}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b_{34}^{''})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{14}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b_{13}^{''})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{37}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{36}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b_{38}^{''})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{42}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{41}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b_{40}^{''})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$(a_i)^{(1)}, (a_i^{'})^{(1)}, (a_i^{''})^{(1)}, (b_i)^{(1)}, (b_i^{'})^{(1)}, (b_i^{''})^{(1)} > 0, \quad i,j = 13,14,15$ *97*

The functions $(a_i^{''})^{(1)}, (b_i^{''})^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}$, $(r_i)^{(1)}$:

$(a_i^{''})^{(1)}(T_{14},t) \le (p_i)^{(1)} \le (\hat{A}_{13})^{(1)}$

$(b_i^{''})^{(1)}(G,t) \le (r_i)^{(1)} \le (b_i^{'})^{(1)} \le (\hat{B}_{13})^{(1)}$

$lim_{T_2\to\infty}(a_i^{''})^{(1)}(T_{14},t) = (p_i)^{(1)}$ 98

$\lim_{G\to\infty}(b_i^{''})^{(1)}(G,t) = (r_i)^{(1)}$

| | |
|---|---|
| **Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :<br>Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, \ (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)\rvert \leq (\hat{k}_{13})^{(1)} \lvert T_{14} - T_{14}'\rvert e^{-(\hat{M}_{13})^{(1)}t}$<br>$\lvert(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)\rvert < (\hat{k}_{13})^{(1)} \lVert G - G'\rVert e^{-(\hat{M}_{13})^{(1)}t}$ | 99 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$. It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ :<br>$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants<br>$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ | 100 |
| **Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ :<br>There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together With $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, \ (r_i)^{(1)}, i = 13,14,15,$ satisfy the inequalities<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(a_i)^{(1)} + (a_i')^{(1)} + (\hat{A}_{13})^{(1)} + (\hat{P}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$<br>$\frac{1}{(\hat{M}_{13})^{(1)}}[(b_i)^{(1)} + (b_i')^{(1)} + (\hat{B}_{13})^{(1)} + (\hat{Q}_{13})^{(1)} (\hat{k}_{13})^{(1)}] < 1$ | 101 |
| Where we suppose | |
| $(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$ | |
| The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded. | |
| **Definition of** $(p_i)^{(2)}, \ (r_i)^{(2)}$: | |
| $(a_i'')^{(2)}(T_{17}, t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ | 102 |
| $(b_i'')^{(2)}(G_{19}, t) \leq \ (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ | 103 |
| $\lim_{T_2 \to \infty} (a_i'')^{(2)}(T_{17}, t) = (p_i)^{(2)}$ | 104 |
| $\lim_{G \to \infty} (b_i'')^{(2)}((G_{19}), t) = \ (r_i)^{(2)}$ | 105 |

| | |
|---|---|
| **Definition of** $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ :<br><br>Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$ | 106 |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(2)}(T_{17}^{'}, t) - (a_i^{''})^{(2)}(T_{17}, t)\rvert \leq (\hat{k}_{16})^{(2)} \lvert T_{17} - T_{17}^{'}\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 107 |
| $\lvert(b_i^{''})^{(2)}((G_{19})^{'}, t) - (b_i^{''})^{(2)}\big((G_{19}), t\big)\rvert < (\hat{k}_{16})^{(2)} \lvert\lvert(G_{19}) - (G_{19})^{'}\rvert\rvert e^{-(\hat{M}_{16})^{(2)}t}$ | 108 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(2)}(T_{17}^{'}, t)$ and $(a_i^{''})^{(2)}(T_{17}, t)$ . $(T_{17}^{'}, t)$ and $(T_{17}, t)$ are points belonging to the interval $\big[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\big]$ . It is to be noted that $(a_i^{''})^{(2)}(T_{17}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i^{''})^{(2)}(T_{17}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ : | |
| $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants<br><br>$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ | 109 |
| **Definition of** $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :<br><br>There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)} and\ (\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i^{'})^{(2)}, (b_i)^{(2)}, (b_i^{'})^{(2)}, (p_i)^{(2)}, \ (r_i)^{(2)}, i = 16,17,18,$<br><br>satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i^{'})^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 110 |
| $\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i^{'})^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)} (\hat{k}_{16})^{(2)}] < 1$ | 111 |
| Where we suppose | |
| $(a_i)^{(3)}, (a_i^{'})^{(3)}, (a_i^{''})^{(3)}, (b_i)^{(3)}, (b_i^{'})^{(3)}, (b_i^{''})^{(3)} > 0, \quad i, j = 20,21,22$<br><br>The functions $(a_i^{''})^{(3)}, (b_i^{''})^{(3)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(3)}$, $(\mathrm{r_i})^{(3)}$:<br><br>$(a_i^{''})^{(3)}(T_{21}, t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$<br><br>$(b_i^{''})^{(3)}(G_{23}, t) \leq (r_i)^{(3)} \leq (b_i^{'})^{(3)} \leq (\hat{B}_{20})^{(3)}$ | 112 |
| $lim_{T_2 \to \infty}(a_i^{''})^{(3)}(T_{21}, t) = (p_i)^{(3)}$ | 113 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(3)}(G_{23},t) = (r_i)^{(3)}$<br><br>**Definition of** $(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}$ :<br><br>Where $\boxed{(\hat{A}_{20})^{(3)}, (\hat{B}_{20})^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)\rvert \le (\hat{k}_{20})^{(3)}\lvert T_{21} - T_{21}'\rvert e^{-(\hat{M}_{20})^{(3)}t}$<br><br>$\lvert(b_i'')^{(3)}(G_{23}',t) - (b_i'')^{(3)}(G_{23},t)\rvert < (\hat{k}_{20})^{(3)}\lvert\lvert G_{23} - G_{23}'\rvert\rvert e^{-(\hat{M}_{20})^{(3)}t}$ | 114 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\hat{k}_{20})^{(3)}, (\hat{M}_{20})^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{20})^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$, the first augmentation coefficient attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$ :<br><br>$(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}$, are positive constants<br>$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}, \frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ | 115 |
| There exists two constants There exists two constants $(\hat{P}_{20})^{(3)}$ and $(\hat{Q}_{20})^{(3)}$ which together with $(\hat{M}_{20})^{(3)}, (\hat{k}_{20})^{(3)}, (\hat{A}_{20})^{(3)} and\ (\hat{B}_{20})^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)}, (r_i)^{(3)}, i = 20,21,22,$<br>satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(a_i)^{(3)} + (a_i')^{(3)} + (\hat{A}_{20})^{(3)} + (\hat{P}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$<br><br>$\frac{1}{(\hat{M}_{20})^{(3)}}[(b_i)^{(3)} + (b_i')^{(3)} + (\hat{B}_{20})^{(3)} + (\hat{Q}_{20})^{(3)}(\hat{k}_{20})^{(3)}] < 1$ | 116 |
| Where we suppose | |
| $(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$<br><br>The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(4)}$, $(r_i)^{(4)}$:<br><br>$(a_i'')^{(4)}(T_{25},t) \le (p_i)^{(4)} \le (\hat{A}_{24})^{(4)}$<br><br>$(b_i'')^{(4)}\big((G_{27}),t\big) \le (r_i)^{(4)} \le (b_i')^{(4)} \le (\hat{B}_{24})^{(4)}$ | 117 |
| $lim_{T_2\to\infty}(a_i'')^{(4)}(T_{25},t) = (p_i)^{(4)}$ | 118 |

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|---|---|
| $\lim_{G\to\infty}(b_i'')^{(4)}\left((G_{27}),t\right) = \ (r_i)^{(4)}$<br><br>**Definition of** $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :<br><br>Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24{,}25{,}26}$ | |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i'')^{(4)}(T_{25}',t) - (a_i'')^{(4)}(T_{25},t)\rvert \le (\hat{k}_{24})^{(4)}\lvert T_{25} - T_{25}'\rvert e^{-(\hat{M}_{24})^{(4)}t}$<br><br>$\lvert(b_i'')^{(4)}((G_{27})',t) - (b_i'')^{(4)}\left((G_{27}),t\right)\rvert < (\hat{k}_{24})^{(4)}\lvert\lvert(G_{27}) - (G_{27})'\rvert\rvert e^{-(\hat{M}_{24})^{(4)}t}$ | 119 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}',t)$ and$(a_i'')^{(4)}(T_{25},t)$ . $(T_{25}',t)$ and $(T_{25},t)$ are points belonging to the interval $\left[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\right]$. It is to be noted that $(a_i'')^{(4)}(T_{25},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$ then the function $(a_i'')^{(4)}(T_{25},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ :<br><br>$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants<br><br>$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ | 120 |
| **Definition of** $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ :<br><br>There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and\ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24{,}25{,}26$, satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + \ (\hat{P}_{24})^{(4)}\ (\hat{k}_{24})^{(4)}] < 1$<br><br>$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + \ (\hat{Q}_{24})^{(4)}\ \ (\hat{k}_{24})^{(4)}] < 1$ | 121 |
| Where we suppose | |
| $(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i,j = 28{,}29{,}30$<br><br>The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(5)}, \ (r_i)^{(5)}$:<br><br>$(a_i'')^{(5)}(T_{29},t) \le (p_i)^{(5)} \le (\hat{A}_{28})^{(5)}$<br><br>$(b_i'')^{(5)}\left((G_{31}),t\right) \le \ (r_i)^{(5)} \le (b_i')^{(5)} \le (\hat{B}_{28})^{(5)}$ | 122 |

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|---|---|
| $lim_{T_2\to\infty}(a_i^{''})^{(5)}\,(T_{29},t)=(p_i)^{(5)}$<br><br>$\lim_{G\to\infty}(b_i^{''})^{(5)}\,(G_{31},t)=\ (r_i)^{(5)}$<br><br>**Definition of** $(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)}$ :<br><br>Where $\boxed{(\hat{A}_{28})^{(5)},(\hat{B}_{28})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)}}$ are positive constants and $\boxed{i=28,29,30}$ | 123 |
| They satisfy Lipschitz condition:<br><br>$\lvert(a_i^{''})^{(5)}(T_{29}^{'},t)-(a_i^{''})^{(5)}(T_{29},t)\rvert\le(\hat{k}_{28})^{(5)}\lvert T_{29}-T_{29}^{'}\rvert e^{-(\hat{M}_{28})^{(5)}t}$<br><br>$\lvert(b_i^{''})^{(5)}((G_{31})^{'},t)-(b_i^{''})^{(5)}\big((G_{31}),t\big)\rvert<(\hat{k}_{28})^{(5)}\lvert\lvert(G_{31})-(G_{31})^{'}\rvert\rvert e^{-(\hat{M}_{28})^{(5)}t}$ | 124 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(5)}(T_{29}^{'},t)$ and$(a_i^{''})^{(5)}(T_{29},t)$ . $(T_{29}^{'},t)$ and $(T_{29},t)$ are points belonging to the interval $\big[(\hat{k}_{28})^{(5)},(\hat{M}_{28})^{(5)}\big]$. It is to be noted that $(a_i^{''})^{(5)}(T_{29},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)}=1$ then the function $(a_i^{''})^{(5)}(T_{29},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$ :<br><br>$(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)}$, are positive constants<br><br>$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\ ,\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}<1$ | 125 |
| **Definition of** $(\hat{P}_{28})^{(5)},(\hat{Q}_{28})^{(5)}$ :<br><br>There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)},(\hat{k}_{28})^{(5)},(\hat{A}_{28})^{(5)}and\ (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)},(a_i^{'})^{(5)},(b_i)^{(5)},(b_i^{'})^{(5)},(p_i)^{(5)},\ (r_i)^{(5)},i=28,29,30,$ satisfy the inequalities<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[\,(a_i)^{(5)}+(a_i^{'})^{(5)}+\ (\hat{A}_{28})^{(5)}+\ (\hat{P}_{28})^{(5)}\,(\hat{k}_{28})^{(5)}]<1$<br><br>$\frac{1}{(\hat{M}_{28})^{(5)}}[\ (b_i)^{(5)}+(b_i^{'})^{(5)}+\ (\hat{B}_{28})^{(5)}+\ (\hat{Q}_{28})^{(5)}\ (\hat{k}_{28})^{(5)}]<1$ | 126 |
| Where we suppose | |
| $(a_i)^{(6)},(a_i^{'})^{(6)},(a_i^{''})^{(6)},(b_i)^{(6)},(b_i^{'})^{(6)},(b_i^{''})^{(6)}>0,\quad i,j=32,33,34$<br><br>The functions $(a_i^{''})^{(6)},(b_i^{''})^{(6)}$ are positive continuous increasing and bounded.<br><br>**Definition of** $(p_i)^{(6)},\ (r_i)^{(6)}$:<br><br>$(a_i^{''})^{(6)}(T_{33},t)\le(p_i)^{(6)}\le(\hat{A}_{32})^{(6)}$ | 127 |

$(b_i^{''})^{(6)}((G_{35}),t) \leq \ (r_i)^{(6)} \leq (b_i^{'})^{(6)} \leq (\hat{B}_{32})^{(6)}$

$lim_{T_2\to\infty}(a_i^{''})^{(6)}(T_{33},t) = (p_i)^{(6)}$ 128

$\lim_{G\to\infty}(b_i^{''})^{(6)}\left((G_{35}),t\right) = \ (r_i)^{(6)}$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i^{''})^{(6)}(T_{33}^{'},t) - (a_i^{''})^{(6)}(T_{33},t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}^{'}|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i^{''})^{(6)}((G_{35})^{'},t) - (b_i^{''})^{(6)}\left((G_{35}),t\right)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})^{'}||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(6)}(T_{33}^{'},t)$ and$(a_i^{''})^{(6)}(T_{33},t)$ . $(T_{33}^{'},t)$ and $(T_{33},t)$ are points belonging to the interval $\left[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}\right]$. It is to be noted that $(a_i^{''})^{(6)}(T_{33},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i^{''})^{(6)}(T_{33},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and \ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i^{'})^{(6)}, (b_i)^{(6)}, (b_i^{'})^{(6)}, (p_i)^{(6)}, \ (r_i)^{(6)}, i = 32,33,34,$ satisfy the inequalities

$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (a_i)^{(6)} + (a_i^{'})^{(6)} + \ (\hat{A}_{32})^{(6)} + \ (\hat{P}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$

$\frac{1}{(\hat{M}_{32})^{(6)}}[\ (b_i)^{(6)} + (b_i^{'})^{(6)} + \ (\hat{B}_{32})^{(6)} + \ (\hat{Q}_{32})^{(6)}\ \ (\hat{k}_{32})^{(6)}] < 1$

Where we suppose

(OOO) $(a_i)^{(7)}, (a_i^{'})^{(7)}, (a_i^{''})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (b_i^{''})^{(7)} > 0, \quad i,j = 36,37,38$ 131

(PPP) The functions $(a_i^{''})^{(7)}, (b_i^{''})^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)}, \ (r_i)^{(7)}$:

$(a_i^{''})^{(7)}(T_{37},t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i^{''})^{(7)}(G_{39},t) \leq (r_i)^{(7)} \leq (b_i^{'})^{(7)} \leq (\hat{B}_{36})^{(7)}$

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*(QQQ)* $\lim_{T_2\to\infty} (a_i^{''})^{(7)}(T_{37},t) = (p_i)^{(7)}$

*(RRR)*

$\lim_{G\to\infty}(b_i^{''})^{(7)}\big((G_{39}),t\big) = (r_i)^{(7)}$

**Definition of** $(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}$ :

Where $\boxed{(\hat{A}_{36})^{(7)}, (\hat{B}_{36})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

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They satisfy Lipschitz condition:

$|(a_i^{''})^{(7)}(T_{37}^{'},t) - (a_i^{''})^{(7)}(T_{37},t)| \leq (\hat{k}_{36})^{(7)}|T_{37} - T_{37}^{'}|e^{-(\hat{M}_{36})^{(7)}t}$

$|(b_i^{''})^{(7)}((G_{39})^{'},t) - (b_i^{''})^{(7)}\big((G_{39}),t\big)| < (\hat{k}_{36})^{(7)}||(G_{39}) - (G_{39})^{'}||e^{-(\hat{M}_{36})^{(7)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(7)}(T_{37}^{'},t)$ and$(a_i^{''})^{(7)}(T_{37},t)$ . $(T_{37}^{'},t)$ and $(T_{37},t)$ are points belonging to the interval $\left[(\hat{k}_{36})^{(7)}, (\hat{M}_{36})^{(7)}\right]$. It is to be noted that $(a_i^{''})^{(7)}(T_{37},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{36})^{(7)} = 1$ then the function $(a_i^{''})^{(7)}(T_{37},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

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**Definition of** $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}$ :

(SSS) $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)},$ are positive constants

$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}, \frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$

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**Definition of** $(\hat{P}_{36})^{(7)}, (\hat{Q}_{36})^{(7)}$ :

(TTT) There exists two constants $(\hat{P}_{36})^{(7)}$ and $(\hat{Q}_{36})^{(7)}$ which together with $(\hat{M}_{36})^{(7)}, (\hat{k}_{36})^{(7)}, (\hat{A}_{36})^{(7)} and\ (\hat{B}_{36})^{(7)}$ and the constants $(a_i)^{(7)}, (a_i^{'})^{(7)}, (b_i)^{(7)}, (b_i^{'})^{(7)}, (p_i)^{(7)}, (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$\frac{1}{(\hat{M}_{36})^{(7)}}[(a_i)^{(7)} + (a_i^{'})^{(7)} + (\hat{A}_{36})^{(7)} + (\hat{P}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

$\frac{1}{(\hat{M}_{36})^{(7)}}[(b_i)^{(7)} + (b_i^{'})^{(7)} + (\hat{B}_{36})^{(7)} + (\hat{Q}_{36})^{(7)}(\hat{k}_{36})^{(7)}] < 1$

Where we suppose

| | |
|---|---|
| $(a_i)^{(8)}, (a_i^{'})^{(8)}, (a_i^{''})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (b_i^{''})^{(8)} > 0, \quad i,j = 40,41,42$ | 136 |
| The functions $(a_i^{''})^{(8)}, (b_i^{''})^{(8)}$ are positive continuous increasing and bounded | |
| **Definition of** $(p_i)^{(8)}$, $(r_i)^{(8)}$: | 137 |
| $(a_i^{''})^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq (\hat{A}_{40})^{(8)}$ | 138 |
| $(b_i^{''})^{(8)}((G_{43}), t) \leq (r_i)^{(8)} \leq (b_i^{'})^{(8)} \leq (\hat{B}_{40})^{(8)}$ | 139 |
| $\lim_{T_2 \to \infty} (a_i^{''})^{(8)}(T_{41}, t) = (p_i)^{(8)}$ | 140 |
| $\lim_{G \to \infty} (b_i^{''})^{(8)}((G_{43}), t) = (r_i)^{(8)}$ | 141 |
| **Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :<br>Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)}, (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$ | |
| They satisfy Lipschitz condition: | |
| $\lvert(a_i^{''})^{(8)}(T_{41}^{'}, t) - (a_i^{''})^{(8)}(T_{41}, t)\rvert \leq (\hat{k}_{40})^{(8)} \lvert T_{41} - T_{41}^{'}\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 142 |
| $\lvert(b_i^{''})^{(8)}((G_{43})^{'}, t) - (b_i^{''})^{(8)}((G_{43}), t)\rvert < (\hat{k}_{40})^{(8)} \lvert\lvert(G_{43}) - (G_{43})^{'}\rvert\rvert e^{-(\hat{M}_{40})^{(8)} t}$ | 143 |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i^{''})^{(8)}(T_{41}^{'}, t)$ and $(a_i^{''})^{(8)}(T_{41}, t)$ . $(T_{41}^{'}, t)$ and $(T_{41}, t)$ are points belonging to the interval $[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}]$. It is to be noted that $(a_i^{''})^{(8)}(T_{41}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i^{''})^{(8)}(T_{41}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ : | |
| $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants | |
| $\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ | 144 |
| **Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :<br>There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i^{'})^{(8)}, (b_i)^{(8)}, (b_i^{'})^{(8)}, (p_i)^{(8)}$, $(r_i)^{(8)}, i = 40,41,42$,<br>Satisfy the inequalities | |
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(a_i)^{(8)} + (a_i^{'})^{(8)} + (\hat{A}_{40})^{(8)} + (\hat{P}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 145 |

| | |
|---|---|
| $\frac{1}{(\hat{M}_{40})^{(8)}}[(b_i)^{(8)} + (b_i')^{(8)} + (\hat{B}_{40})^{(8)} + (\hat{Q}_{40})^{(8)} (\hat{k}_{40})^{(8)}] < 1$ | 146 |
| Where we suppose | |
| $(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$<br>The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.<br>**Definition of** $(p_i)^{(9)}$, $(r_i)^{(9)}$:<br>$(a_i'')^{(9)}(T_{45},t) \le (p_i)^{(9)} \le (\hat{A}_{44})^{(9)}$<br>$(b_i'')^{(9)}(G_{47},t) \le (r_i)^{(9)} \le (b_i')^{(9)} \le (\hat{B}_{44})^{(9)}$ | 146 A |
| $\lim_{T_2\to\infty}(a_i'')^{(9)}(T_{45},t) = (p_i)^{(9)}$<br>$\lim_{G\to\infty}(b_i'')^{(9)}(G_{47},t) = (r_i)^{(9)}$<br>**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :<br>Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$ | |
| They satisfy Lipschitz condition:<br>$\lvert(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)\rvert \le (\hat{k}_{44})^{(9)}\lvert T_{45} - T_{45}'\rvert e^{-(\hat{M}_{44})^{(9)}t}$<br>$\lvert(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}((G_{47}),t)\rvert < (\hat{k}_{44})^{(9)}\lvert\lvert(G_{47}) - (G_{47})'\rvert\rvert e^{-(\hat{M}_{44})^{(9)}t}$ | |
| With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and $(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$, the **first augmentation coefficient** attributable to the system, would be absolutely continuous. | |
| **Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :<br>$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants<br>$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}, \frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ | |
| **Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :<br>There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)}, (r_i)^{(9)}, i = 44,45,46,$<br>satisfy the inequalities | |

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| $\frac{1}{(\hat{M}_{44})^{(9)}}[(a_i)^{(9)} + (a_i^{'})^{(9)} + (\hat{A}_{44})^{(9)} + (\hat{P}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$<br><br>$\frac{1}{(\hat{M}_{44})^{(9)}}[(b_i)^{(9)} + (b_i^{'})^{(9)} + (\hat{B}_{44})^{(9)} + (\hat{Q}_{44})^{(9)}(\hat{k}_{44})^{(9)}] < 1$ | |
| **Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>Definition of $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 147 |
| **Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$<br><br>$G_i(t) \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$ | 148 |
| **Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>$G_i(t) \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$<br><br>$T_i(t) \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$ | 149 |
| **Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 150 |
| **Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ :<br><br>$G_i(t) \leq (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br><br>$T_i(t) \leq (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 151 |
| **Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br><br>**Definition of** $G_i(0), T_i(0)$ : | 152 |

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| $G_i(t) \le \left(\hat{P}_{32}\right)^{(6)} e^{\left(\hat{M}_{32}\right)^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{32}\right)^{(6)} e^{\left(\hat{M}_{32}\right)^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | |
| **Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{36}\right)^{(7)} e^{\left(\hat{M}_{36}\right)^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{36}\right)^{(7)} e^{\left(\hat{M}_{36}\right)^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 |
| **Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{40}\right)^{(8)} e^{\left(\hat{M}_{40}\right)^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{40}\right)^{(8)} e^{\left(\hat{M}_{40}\right)^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 A |
| **Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions<br>**Definition of** $G_i(0), T_i(0)$ :<br>$G_i(t) \le \left(\hat{P}_{44}\right)^{(9)} e^{\left(\hat{M}_{44}\right)^{(9)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$<br>$T_i(t) \le \left(\hat{Q}_{44}\right)^{(9)} e^{\left(\hat{M}_{44}\right)^{(9)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$ | 153 B |
| **Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 154 |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le \left(\hat{P}_{13}\right)^{(1)}$ , $T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)}$, | 155 |
| $0 \le G_i(t) - G_i^0 \le \left(\hat{P}_{13}\right)^{(1)} e^{\left(\hat{M}_{13}\right)^{(1)} t}$ | 156 |
| $0 \le T_i(t) - T_i^0 \le \left(\hat{Q}_{13}\right)^{(1)} e^{\left(\hat{M}_{13}\right)^{(1)} t}$ | 157 |
| By<br>$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a_{13}')^{(1)} + a_{13}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$ | 158 |
| $\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a_{14}')^{(1)} + (a_{14}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a_{15}')^{(1)} + (a_{15}'')^{(1)} (T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$ | |

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| $\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b_{13}')^{(1)} - (b_{13}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b_{14}')^{(1)} - (b_{14}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$ | |
| $\bar{T}_{15}(t) = T_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b_{15}')^{(1)} - (b_{15}'')^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$<br>Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 159 |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{16})^{(2)}$, $T_i^0 \le (\hat{Q}_{16})^{(2)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ | |
| By<br>$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a_{16}')^{(2)} + a_{16}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$ | 160 |
| $\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}(s_{(16)}) - \left((a_{17}')^{(2)} + (a_{17}'')^{(2)}(T_{17}(s_{(16)}), s_{(17)})\right) G_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}(s_{(16)}) - \left((a_{18}')^{(2)} + (a_{18}'')^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{18}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}(s_{(16)}) - \left((b_{16}')^{(2)} - (b_{16}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{16}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}(s_{(16)}) - \left((b_{17}')^{(2)} - (b_{17}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{17}(s_{(16)})\right] ds_{(16)}$ | |
| $\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}(s_{(16)}) - \left((b_{18}')^{(2)} - (b_{18}'')^{(2)}(G_{19}(s_{(16)}), s_{(16)})\right) T_{18}(s_{(16)})\right] ds_{(16)}$<br>Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \le (\hat{P}_{20})^{(3)}$, $T_i^0 \le (\hat{Q}_{20})^{(3)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ | |

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| By<br>$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}(s_{(20)}) - \left((a'_{20})^{(3)} + a''_{20})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{20}(s_{(20)})\right] ds_{(20)}$ | 161 |
| $\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}(s_{(20)}) - \left((a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}(s_{(20)}) - \left((a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21}(s_{(20)}), s_{(20)})\right) G_{22}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}(s_{(20)}) - \left((b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{20}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}(s_{(20)}) - \left((b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{21}(s_{(20)})\right] ds_{(20)}$ | |
| $\bar{\mathrm{T}}_{22}(\mathrm{t}) = \mathrm{T}_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}(s_{(20)}) - \left((b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23}(s_{(20)}), s_{(20)})\right) T_{22}(s_{(20)})\right] ds_{(20)}$<br>Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \leq (\hat{P}_{24})^{(4)}, T_i^0 \leq (\hat{Q}_{24})^{(4)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\bar{M}_{24})^{(4)} t}$ | |
| By<br>$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$ | 162 |
| $\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$ | |
| $\bar{\mathrm{T}}_{26}(\mathrm{t}) = \mathrm{T}_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$<br>Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |

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| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{28})^{(5)}$ , $T_i^0 \leq (\hat{Q}_{28})^{(5)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)}t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{28})^{(5)} e^{(\bar{M}_{28})^{(5)}t}$ | |
| By<br>$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$ | 163 |
| $\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$ | |
| $\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$<br>Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{32})^{(6)}$ , $T_i^0 \leq (\hat{Q}_{32})^{(6)}$, | |
| $0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)}t}$ | |
| $0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\bar{M}_{32})^{(6)}t}$ | |
| By<br>$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[(a_{32})^{(6)} G_{33}(s_{(32)}) - \left((a'_{32})^{(6)} + a''_{32})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{32}(s_{(32)})\right] ds_{(32)}$ | 164 |
| $\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[(a_{33})^{(6)} G_{32}(s_{(32)}) - \left((a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{33}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[(a_{34})^{(6)} G_{33}(s_{(32)}) - \left((a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}(s_{(32)}), s_{(32)})\right) G_{34}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[(b_{32})^{(6)} T_{33}(s_{(32)}) - \left((b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{32}(s_{(32)})\right] ds_{(32)}$ | |
| $\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[(b_{33})^{(6)} T_{32}(s_{(32)}) - \left((b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{33}(s_{(32)})\right] ds_{(32)}$ | |

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| $\overline{T}_{34}(t) = T_{34}^0 + \int_0^t \left[(b_{34})^{(6)} T_{33}(s_{(32)}) - \left((b_{34}')^{(6)} - (b_{34}'')^{(6)}(G_{35}(s_{(32)}), s_{(32)})\right) T_{34}(s_{(32)})\right] ds_{(32)}$<br><br>Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{36})^{(7)}$ , $T_i^0 \le (\hat{Q}_{36})^{(7)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ | |
| By<br><br>$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[(a_{36})^{(7)} G_{37}(s_{(36)}) - \left((a_{36}')^{(7)} + a_{36}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{36}(s_{(36)})\right] ds_{(36)}$ | 165 |
| $\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[(a_{37})^{(7)} G_{36}(s_{(36)}) - \left((a_{37}')^{(7)} + (a_{37}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[(a_{38})^{(7)} G_{37}(s_{(36)}) - \left((a_{38}')^{(7)} + (a_{38}'')^{(7)}(T_{37}(s_{(36)}), s_{(36)})\right) G_{38}(s_{(36)})\right] ds_{(36)}$ | |
| $\overline{T}_{36}(t) = T_{36}^0 + \int_0^t \left[(b_{36})^{(7)} T_{37}(s_{(36)}) - \left((b_{36}')^{(7)} - (b_{36}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{36}(s_{(36)})\right] ds_{(36)}$ | |
| $\overline{T}_{37}(t) = T_{37}^0 + \int_0^t \left[(b_{37})^{(7)} T_{36}(s_{(36)}) - \left((b_{37}')^{(7)} - (b_{37}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{37}(s_{(36)})\right] ds_{(36)}$ | |
| $\overline{T}_{38}(t) = T_{38}^0 + \int_0^t \left[(b_{38})^{(7)} T_{37}(s_{(36)}) - \left((b_{38}')^{(7)} - (b_{38}'')^{(7)}(G_{39}(s_{(36)}), s_{(36)})\right) T_{38}(s_{(36)})\right] ds_{(36)}$<br><br>Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br><br>Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{40})^{(8)}$ , $T_i^0 \le (\hat{Q}_{40})^{(8)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ | |

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| By<br>$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a'_{40})^{(8)} + a''_{40})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$ | 166 |
| $\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$ | |
| $\bar{T}_{42}(t) = T_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$<br>Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| **Proof:**<br>Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy | 166 A |
| $G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \le (\hat{P}_{44})^{(9)}$ , $T_i^0 \le (\hat{Q}_{44})^{(9)}$, | |
| $0 \le G_i(t) - G_i^0 \le (\hat{P}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| $0 \le T_i(t) - T_i^0 \le (\hat{Q}_{44})^{(9)} e^{(\bar{M}_{44})^{(9)} t}$ | |
| By<br>$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{46}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{44}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{45}(s_{(44)})\right] ds_{(44)}$ | |
| $\bar{T}_{46}(t) = T_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47}(s_{(44)}), s_{(44)})\right) T_{46}(s_{(44)})\right] ds_{(44)}$<br>Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$ | |
| The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | 167 |

$$G_{13}(t) \le G_{13}^0 + \int_0^t \left[(a_{13})^{(1)}\left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$$

$$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$$

From which it follows that

$$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \le \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 1

Analogous inequalities hold also for $G_{14}, G_{15}, T_{13}, T_{14}, T_{15}$

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \le G_{16}^0 + \int_0^t \left[(a_{16})^{(2)}\left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$$

From which it follows that

$$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \le \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}}\left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$$

Analogous inequalities hold also for $G_{17}, G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{20}(t) \le G_{20}^0 + \int_0^t \left[(a_{20})^{(3)}\left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that

$$(G_{20}(t) - G_{20}^0) e^{-(\hat{M}_{20})^{(3)} t} \le \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right) e^{\left(-\frac{(\hat{P}_{20})^{(3)} + G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}, G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that

$$G_{24}(t) \le G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)} t\right) G_{25}^0 + \frac{(a_{24})^{(4)} (\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)} t} - 1\right)$$

From which it follows that

$$(G_{24}(t)-G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \le \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[((\hat{P}_{24})^{(4)}+G_{25}^0)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)}+(\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \le G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0+(\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1+(a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t}-1\right)$$

From which it follows that

$$(G_{28}(t)-G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \le \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[((\hat{P}_{28})^{(5)}+G_{29}^0)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)}+(\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{32}(t) \le G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0+(\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1+(a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t}-1\right)$$

From which it follows that

$$(G_{32}(t)-G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \le \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[((\hat{P}_{32})^{(6)}+G_{33}^0)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)}+(\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}, G_{26}, T_{24}, T_{25}, T_{26}$

(l) The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{36}(t) \le G_{36}^0 + \int_0^t \left[(a_{36})^{(7)}\left(G_{37}^0+(\hat{P}_{36})^{(7)}e^{(\hat{M}_{36})^{(7)}s_{(36)}}\right)\right] ds_{(36)} =$$

$$\left(1+(a_{36})^{(7)}t\right)G_{37}^0 + \frac{(a_{36})^{(7)}(\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left(e^{(\hat{M}_{36})^{(7)}t}-1\right)$$

From which it follows that

$$(G_{36}(t)-G_{36}^0)e^{-(\hat{M}_{36})^{(7)}t} \le \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}}\left[((\hat{P}_{36})^{(7)}+G_{37}^0)e^{\left(-\frac{(\hat{P}_{36})^{(7)}+G_{37}^0}{G_{37}^0}\right)}+(\hat{P}_{36})^{(7)}\right]$$

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| $(G_i^0)$ is as defined in the statement of theorem 7 | |
| The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that | |
| $G_{40}(t) \le G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} \left(G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}}\right)\right] ds_{(40)} =$ $\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left(e^{(\hat{M}_{40})^{(8)} t} - 1\right)$ | 180 |
| From which it follows that $(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \le \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[\left((\hat{P}_{40})^{(8)} + G_{41}^0\right) e^{\left(-\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0}\right)} + (\hat{P}_{40})^{(8)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 8 Analogous inequalities hold also for $G_{41}, G_{42}, T_{40}, T_{41}, T_{42}$ | 181 |
| The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that $G_{44}(t) \le G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} \left(G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}}\right)\right] ds_{(44)} =$ $\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left(e^{(\hat{M}_{44})^{(9)} t} - 1\right)$ | |
| From which it follows that $(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \le \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[\left((\hat{P}_{44})^{(9)} + G_{45}^0\right) e^{\left(-\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0}\right)} + (\hat{P}_{44})^{(9)}\right]$ $(G_i^0)$ is as defined in the statement of theorem 9 Analogous inequalities hold also for $G_{45}, G_{46}, T_{44}, T_{45}, T_{46}$ | |
| It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}, \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have | 182 |
| $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}} \left[(\hat{P}_{13})^{(1)} + \left((\hat{P}_{13})^{(1)} + G_j^0\right) e^{-\left(\frac{(\hat{P}_{13})^{(1)} + G_j^0}{G_j^0}\right)}\right] \le (\hat{P}_{13})^{(1)}$ | 183 |
| $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{Q}_{13})^{(1)} + T_j^0\right) e^{-\left(\frac{(\hat{Q}_{13})^{(1)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{13})^{(1)}\right] \le (\hat{Q}_{13})^{(1)}$ | 184 |
| In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric | 185 |

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| $d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$<br><br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{13})^{(1)}t}\}$ | |
| Indeed if we denote<br><br>**Definition of** $\tilde{G},\tilde{T}:\left(\tilde{G},\tilde{T}\right)=\mathcal{A}^{(1)}(G,T)$<br><br>It results<br><br>$\left\|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{13})^{(1)}\left\|G_{14}^{(1)}-G_{14}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$<br><br>$\int_0^t\{(a_{13}^{'})^{(1)}\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br><br>$(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left\|G_{13}^{(1)}-G_{13}^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$<br><br>$G_{13}^{(2)}\left\|(a_{13}^{''})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a_{13}^{''})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)\right\|\ e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$<br><br>Where $s_{(13)}$ represents integrand that is integrated over the interval [0, t]<br><br>From the hypotheses it follows | |
| $\left\|G^{(1)}-G^{(2)}\right\|e^{-(\widehat{M}_{13})^{(1)}t}\le$<br>$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+(a_{13}^{'})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\ G^{(2)},T^{(2)}\right)\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 186 |
| **Remark 1:** The fact that we supposed $(a_{13}^{''})^{(1)}$ and $(b_{13}^{''})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(1)}$ and $(b_i^{''})^{(1)}, i=13,14,15$ depend only on $T_{14}$ and respectively on $G$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 2:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br><br>From 19 to 24 it results<br><br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(1)}-(a_i^{''})^{(1)}(T_{14}(s_{(13)}),s_{(13)})\}ds_{(13)}\right]}\ge 0$<br><br>$T_i(t)\ge T_i^0e^{\left(-(b_i^{'})^{(1)}t\right)}>0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1,\left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ : | 187 |

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| **Remark 3**: if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if<br>$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}')^{(1)} G_{14}$ and by integrating<br>$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1/(a_{14}')^{(1)}$<br>In the same way , one can obtain<br>$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2/(a_{15}')^{(1)}$<br>If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively. | |
| **Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. | 188 |
| **Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$.<br>**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :<br>Indeed let $t_1$ be so that for $t > t_1$<br>$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}(t) > (m)^{(1)}$ | 189 |
| Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results<br>$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded.<br>The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}(G(t),t) = (b_{15}')^{(1)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | |
| It is now sufficient to take $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}} < 1$ and to choose<br>$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have | 190 |
| $\frac{(a_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + \left((\hat{P}_{16})^{(2)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$ | 191 |
| $\frac{(b_i)^{(2)}}{(\widehat{M}_{16})^{(2)}}\left[\left((\hat{Q}_{16})^{(2)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$ | 192 |
| In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself | 193 |

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| The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric<br>$d\left(((G_{19})^{(1)},(T_{19})^{(1)}),((G_{19})^{(2)},(T_{19})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{16})^{(2)}t}\}$ | 194 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{19}},\widetilde{T_{19}}$ : $\left(\widetilde{G_{19}},\widetilde{T_{19}}\right)=\mathcal{A}^{(2)}(G_{19},T_{19})$ | 195 |
| It results<br>$\left\|\tilde{G}_{16}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{16})^{(2)}\left\|G_{17}^{(1)}-G_{17}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}ds_{(16)}+$<br>$\int_0^t\{(a_{16}^{'})^{(2)}\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)\left\|G_{16}^{(1)}-G_{16}^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}+$<br>$G_{16}^{(2)}\left\|(a_{16}^{''})^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)-(a_{16}^{''})^{(2)}\left(T_{17}^{(2)},s_{(16)}\right)\right\|\ e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$ | 196 |
| Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses it follows | 197 |
| $\left\|(G_{19})^{(1)}-(G_{19})^{(2)}\right\|e^{-(\widehat{M}_{16})^{(2)}t}\le$<br>$\frac{1}{(\widehat{M}_{16})^{(2)}}\left((a_{16})^{(2)}+(a_{16}^{'})^{(2)}+(\widehat{A}_{16})^{(2)}+(\widehat{P}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)d\left(((G_{19})^{(1)},(T_{19})^{(1)};\ (G_{19})^{(2)},(T_{19})^{(2)})\right)$ | |
| And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 198 |
| **Remark 6:** The fact that we supposed $(a_{16}^{''})^{(2)}$ and $(b_{16}^{''})^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ and $(\widehat{Q}_{16})^{(2)}e^{(\widehat{M}_{16})^{(2)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}, i=16,17,18$ depend only on $T_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition. | 199 |
| **Remark 7:** There does not exist any t where $G_i(t)=0$ and $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(2)}-(a_i^{''})^{(2)}(T_{17}(s_{(16)}),s_{(16)})\}ds_{(16)}\right]}\ge 0$<br>$T_i(t)\ge T_i^0e^{(-(b_i^{'})^{(2)}t)}>0$ for $t>0$ | 200 |

**Definition of** $\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_1, \left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_2$ and $\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_3$ : 201

**Remark 8:** if $\mathrm{G}_{16}$ is bounded, the same property have also $\mathrm{G}_{17}$ and $\mathrm{G}_{18}$ . indeed if

$\mathrm{G}_{16} < (\widehat{\mathrm{M}}_{16})^{(2)}$ it follows $\frac{\mathrm{dG}_{17}}{\mathrm{dt}} \leq \left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_1 - (a_{17}')^{(2)}\mathrm{G}_{17}$ and by integrating

$\mathrm{G}_{17} \leq \left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_2 = \mathrm{G}_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_1/(a_{17}')^{(2)}$

In the same way , one can obtain

$\mathrm{G}_{18} \leq \left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_3 = \mathrm{G}_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_2/(a_{18}')^{(2)}$

If $\mathrm{G}_{17}$ or $\mathrm{G}_{18}$ is bounded, the same property follows for $\mathrm{G}_{16}$ , $\mathrm{G}_{18}$ and $\mathrm{G}_{16}$ , $\mathrm{G}_{17}$ respectively.

**Remark 9**: If $\mathrm{G}_{16}$ is bounded, from below, the same property holds for $\mathrm{G}_{17}$ and $\mathrm{G}_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $\mathrm{G}_{17}$ is bounded from below. 202

**Remark 10:** If $\mathrm{T}_{16}$ is bounded from below and $\lim_{\mathrm{t}\to\infty}((b_i'')^{(2)}((G_{19})(\mathrm{t}),\mathrm{t})) = (b_{17}')^{(2)}$ then $\mathrm{T}_{17} \to \infty$. 203

**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :

Indeed let $\mathrm{t}_2$ be so that for $\mathrm{t} > \mathrm{t}_2$

$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(\mathrm{t}),\mathrm{t}) < \varepsilon_2, \mathrm{T}_{16}(\mathrm{t}) > (m)^{(2)}$

Then $\frac{\mathrm{dT}_{17}}{\mathrm{dt}} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2\mathrm{T}_{17}$ which leads to 204

$\mathrm{T}_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - \mathrm{e}^{-\varepsilon_2\mathrm{t}}) + \mathrm{T}_{17}^0\mathrm{e}^{-\varepsilon_2\mathrm{t}}$ If we take t such that $\mathrm{e}^{-\varepsilon_2\mathrm{t}} = \frac{1}{2}$ it results

$\mathrm{T}_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right)$, $t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $\mathrm{T}_{17}$ is unbounded. 205
The same property holds for $\mathrm{T}_{18}$ if $\lim_{t\to\infty}(b_{18}'')^{(2)}\left((G_{19})(\mathrm{t}),\mathrm{t}\right) = (b_{18}')^{(2)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}} < 1$ and to choose 207

$(\hat{\mathrm{P}}_{20})^{(3)}$ and $(\hat{\mathrm{Q}}_{20})^{(3)}$ large to have

$$\frac{(a_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)} \quad 208$$

$$\frac{(b_i)^{(3)}}{(\bar{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)} \quad 209$$

In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 210

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| Equations into itself | |
| The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric<br>$d\left(((G_{23})^{(1)},(T_{23})^{(1)}),((G_{23})^{(2)},(T_{23})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{20})^{(3)}t}\}$ | 211 |
| Indeed if we denote<br>**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}:(\widetilde{(G_{23})},\widetilde{(T_{23})})=\mathcal{A}^{(3)}((G_{23}),(T_{23}))$ | 212 |
| It results<br>$\left\|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{20})^{(3)}\left\|G_{21}^{(1)}-G_{21}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$<br>$\int_0^t\{(a_{20}^{'})^{(3)}\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$(a_{20}^{''})^{(3)}(T_{21}^{(1)},s_{(20)})\left\|G_{20}^{(1)}-G_{20}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$<br>$G_{20}^{(2)}\left\|(a_{20}^{''})^{(3)}(T_{21}^{(1)},s_{(20)})-(a_{20}^{''})^{(3)}(T_{21}^{(2)},s_{(20)})\right\|\ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$<br>Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$<br>From the hypotheses it follows | 213 |
| $\left\|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right\|e^{-(\widehat{M}_{20})^{(3)}t}\le$<br>$\frac{1}{(\widehat{M}_{20})^{(3)}}((a_{20})^{(3)}+(a_{20}^{'})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)})d\left(((G_{23})^{(1)},(T_{23})^{(1)};(G_{23})^{(2)},(T_{23})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 214 |
| **Remark 11:** The fact that we supposed $(a_{20}^{''})^{(3)}$ and $(b_{20}^{''})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(3)}$ and $(b_i^{''})^{(3)},i=20,21,22$ depend only on $\mathrm{T}_{21}$ and respectively on $(G_{23})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition. | 215 |
| **Remark 12:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(3)}-(a_i^{''})^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\ge 0$ | 216 |

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| $T_i\,(t) \geq T_i^0 e^{\left(-(b_i^{'})^{(3)}t\right)} > 0$ for t > 0 | |
| **Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1, \left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ :<br><br>**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if<br><br>$G_{20} < (\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt} \leq \left((\widehat{M}_{20})^{(3)}\right)_1 - (a_{21}^{'})^{(3)} G_{21}$ and by integrating<br><br>$G_{21} \leq \left((\widehat{M}_{20})^{(3)}\right)_2 = G_{21}^0 + 2(a_{21})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_1/(a_{21}^{'})^{(3)}$<br><br>In the same way , one can obtain<br><br>$G_{22} \leq \left((\widehat{M}_{20})^{(3)}\right)_3 = G_{22}^0 + 2(a_{22})^{(3)}\left((\widehat{M}_{20})^{(3)}\right)_2/(a_{22}^{'})^{(3)}$<br><br>If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively. | 217 |
| **Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. | 218 |
| **Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{21}^{'})^{(3)}$ then $T_{21} \to \infty$.<br><br>**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :<br><br>Indeed let $t_3$ be so that for $t > t_3$<br><br>$(b_{21})^{(3)} - (b_i^{''})^{(3)}\left((G_{23})(t),t\right) < \varepsilon_3, T_{20}\,(t) > (m)^{(3)}$ | 219 |
| Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)}(m)^{(3)} - \varepsilon_3 T_{21}$ which leads to<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{\varepsilon_3}\right)(1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results<br><br>$T_{21} \geq \left(\frac{(a_{21})^{(3)}(m)^{(3)}}{2}\right)$, $t = log\frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded.<br>The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b_{22}^{''})^{(3)}\left((G_{23})(t),t\right) = (b_{22}^{'})^{(3)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 220 |
| It is now sufficient to take $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}} < 1$ and to choose<br><br>$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have | 221 |
| $\frac{(a_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[(\widehat{P}_{24})^{(4)} + \left((\widehat{P}_{24})^{(4)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{24})^{(4)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{24})^{(4)}$ | 222 |
| $\frac{(b_i)^{(4)}}{(\widehat{M}_{24})^{(4)}}\left[\left((\widehat{Q}_{24})^{(4)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{24})^{(4)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{24})^{(4)}\right] \leq (\widehat{Q}_{24})^{(4)}$ | 223 |

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| In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | 224 |
| The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric<br>$d\left(((G_{27})^{(1)},(T_{27})^{(1)}),((G_{27})^{(2)},(T_{27})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{24})^{(4)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{27}}),(\widetilde{T_{27}})$ : $\left((\widetilde{G_{27}}),(\widetilde{T_{27}})\right)=\mathcal{A}^{(4)}((G_{27}),(T_{27}))$<br>It results<br>$\left\|\tilde{G}_{24}^{(1)}-\tilde{G}_i^{(2)}\right\|\le\int_0^t(a_{24})^{(4)}\left\|G_{25}^{(1)}-G_{25}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\,ds_{(24)}+$<br>$\int_0^t\{(a_{24}^{'})^{(4)}\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)\left\|G_{24}^{(1)}-G_{24}^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}+$<br>$G_{24}^{(2)}\left\|(a_{24}^{''})^{(4)}\left(T_{25}^{(1)},s_{(24)}\right)-(a_{24}^{''})^{(4)}\left(T_{25}^{(2)},s_{(24)}\right)\right\|\ e^{-(\widehat{M}_{24})^{(4)}s_{(24)}}e^{(\widehat{M}_{24})^{(4)}s_{(24)}}\}ds_{(24)}$<br>Where $s_{(24)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses on Equations it follows | 225 |
| $\left\|(G_{27})^{(1)}-(G_{27})^{(2)}\right\|e^{-(\widehat{M}_{24})^{(4)}t}\le$<br>$\frac{1}{(\widehat{M}_{24})^{(4)}}\left((a_{24})^{(4)}+(a_{24}^{'})^{(4)}+(\widehat{A}_{24})^{(4)}+(\widehat{P}_{24})^{(4)}(\widehat{k}_{24})^{(4)}\right)d\left(((G_{27})^{(1)},(T_{27})^{(1)};\ (G_{27})^{(2)},(T_{27})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 226 |
| **Remark 16:** The fact that we supposed $(a_{24}^{''})^{(4)}$ and $(b_{24}^{''})^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ *and* $(\widehat{Q}_{24})^{(4)}e^{(\widehat{M}_{24})^{(4)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(4)}$ and $(b_i^{''})^{(4)}, i=24,25,26$ depend only on $T_{25}$ respectively on $(G_{27})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 227 |
| **Remark 17:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$<br>it results<br>$G_i(t)\ge G_i^0e^{\left[-\int_0^t\{(a_i^{'})^{(4)}-(a_i^{''})^{(4)}(T_{25}(s_{(24)}),s_{(24)})\}ds_{(24)}\right]}\ge 0$ | 228 |

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|---|---|
| $T_i(t) \geq T_i^0 e^{(-(b_i')^{(4)}t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{24})^{(4)})_1, ((\widehat{M}_{24})^{(4)})_2$ *and* $((\widehat{M}_{24})^{(4)})_3$ :<br><br>**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if<br><br>$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq ((\widehat{M}_{24})^{(4)})_1 - (a_{25}')^{(4)} G_{25}$ and by integrating<br><br>$G_{25} \leq ((\widehat{M}_{24})^{(4)})_2 = G_{25}^0 + 2(a_{25})^{(4)} ((\widehat{M}_{24})^{(4)})_1 / (a_{25}')^{(4)}$<br><br>In the same way , one can obtain<br><br>$G_{26} \leq ((\widehat{M}_{24})^{(4)})_3 = G_{26}^0 + 2(a_{26})^{(4)} ((\widehat{M}_{24})^{(4)})_2 / (a_{26}')^{(4)}$<br><br>If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively. | 229 |
| **Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. | 230 |
| **Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b_{25}')^{(4)}$ then $T_{25} \to \infty$.<br><br>**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :<br><br>Indeed let $t_4$ be so that for $t > t_4$<br><br>$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$ | 231 |
| Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results<br><br>$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right)$, $t = log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b_{26}'')^{(4)}((G_{27})(t),t) = (b_{26}')^{(4)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42<br><br>Analogous inequalities hold also for $G_{29}, G_{30}, T_{28}, T_{29}, T_{30}$ | 232 |
| It is now sufficient to take $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}} < 1$ and to choose<br><br>$(\widehat{P}_{28})^{(5)}$ and $(\widehat{Q}_{28})^{(5)}$ large to have | 233 |
| $\frac{(a_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[(\widehat{P}_{28})^{(5)} + ((\widehat{P}_{28})^{(5)} + G_j^0)e^{-\left(\frac{(\widehat{P}_{28})^{(5)} + G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{28})^{(5)}$ | 234 |

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| $\frac{(b_i)^{(5)}}{(\widehat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{28})^{(5)}\right]\leq(\hat{Q}_{28})^{(5)}$ | 235 |
| In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric<br>$d\left(((G_{31})^{(1)},(T_{31})^{(1)}),((G_{31})^{(2)},(T_{31})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{28})^{(5)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{31}}),(\widetilde{T_{31}})$ : $\left((\widetilde{G_{31}}),(\widetilde{T_{31}})\right)=\mathcal{A}^{(5)}((G_{31}),(T_{31}))$<br>It results<br>$\left\|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{28})^{(5)}\left\|G_{29}^{(1)}-G_{29}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}ds_{(28)}+$<br>$\int_0^t\{(a_{28}')^{(5)}\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)\left\|G_{28}^{(1)}-G_{28}^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$<br>$G_{28}^{(2)}\left\\|(a_{28}'')^{(5)}\left(T_{29}^{(1)},s_{(28)}\right)-(a_{28}'')^{(5)}\left(T_{29}^{(2)},s_{(28)}\right)\right\\|\ e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$<br>Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,t]$<br>From the hypotheses on it follows | 236 |
| $\left\|(G_{31})^{(1)}-(G_{31})^{(2)}\right\|e^{-(\widehat{M}_{28})^{(5)}t}\leq$<br>$\frac{1}{(\widehat{M}_{28})^{(5)}}\left((a_{28})^{(5)}+(a_{28}')^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\right)d\left(((G_{31})^{(1)},(T_{31})^{(1)};\ (G_{31})^{(2)},(T_{31})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 237 |
| **Remark 21:** The fact that we supposed $(a_{28}'')^{(5)}$ and $(b_{28}'')^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(5)}$ and $(b_i'')^{(5)}, i=28,29,30$ depend only on $T_{29}$ and respectively on $(G_{31})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | 238 |
| **Remark 22:** There does not exist any $t$ where $G_i(t)=0$ *and* $T_i(t)=0$ | 239 |

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| it results<br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(5)} - (a_i'')^{(5)}(T_{29}(s_{(28)}), s_{(28)})\} ds_{(28)}\right]} \geq 0$<br>$T_i(t) \geq T_i^0 e^{(-(b_i')^{(5)} t)} > 0$ for t > 0 | |
| **Definition of** $((\widehat{M}_{28})^{(5)})_1, ((\widehat{M}_{28})^{(5)})_2$ *and* $((\widehat{M}_{28})^{(5)})_3$ :<br>**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if<br>$G_{28} < (\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt} \leq ((\widehat{M}_{28})^{(5)})_1 - (a_{29}')^{(5)} G_{29}$ and by integrating<br>$G_{29} \leq ((\widehat{M}_{28})^{(5)})_2 = G_{29}^0 + 2(a_{29})^{(5)} ((\widehat{M}_{28})^{(5)})_1 / (a_{29}')^{(5)}$<br>In the same way , one can obtain<br>$G_{30} \leq ((\widehat{M}_{28})^{(5)})_3 = G_{30}^0 + 2(a_{30})^{(5)} ((\widehat{M}_{28})^{(5)})_2 / (a_{30}')^{(5)}$<br>If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively. | 240 |
| **Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. | 241 |
| **Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t), t)) = (b_{29}')^{(5)}$ then $T_{29} \to \infty$.<br>**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :<br>Indeed let $t_5$ be so that for $t > t_5$<br>$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t), t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$ | 242 |
| Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results<br>$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right)$, $t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.<br>The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b_{30}'')^{(5)}((G_{31})(t), t) = (b_{30}')^{(5)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations<br>Analogous inequalities hold also for $G_{33}$ , $G_{34}, T_{32}, T_{33}, T_{34}$ | 243 |
| It is now sufficient to take $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}$ , $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}} < 1$ and to choose<br>$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have | 244 |

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| $\frac{(a_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)}+\left((\hat{P}_{32})^{(6)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{32})^{(6)}$ | 245 |
| $\frac{(b_i)^{(6)}}{(\widehat{M}_{32})^{(6)}}\left[\left((\hat{Q}_{32})^{(6)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{32})^{(6)}\right]\leq(\hat{Q}_{32})^{(6)}$ | 246 |
| In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric<br>$d\left(((G_{35})^{(1)},(T_{35})^{(1)}),((G_{35})^{(2)},(T_{35})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{32})^{(6)}t}\}$<br>Indeed if we denote<br>**Definition of** $(\widetilde{G_{35}}),(\widetilde{T_{35}})$ : $\left((\widetilde{G_{35}}),(\widetilde{T_{35}})\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$<br>It results<br>$\left\|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{32})^{(6)}\left\|G_{33}^{(1)}-G_{33}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}ds_{(32)}+$<br>$\int_0^t\{(a'_{32})^{(6)}\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left\|G_{32}^{(1)}-G_{32}^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$<br>$G_{32}^{(2)}\left\|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)\right\|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$<br>Where $s_{(32)}$ represents integrand that is integrated over the interval [0, t]<br>From the hypotheses it follows | 247 |
| $\left\|(G_{35})^{(1)}-(G_{35})^{(2)}\right\|e^{-(\widehat{M}_{32})^{(6)}t}\leq$<br>$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)})\right)$<br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 248 |
| **Remark 26:** The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$.<br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(6)}$ and $(b_i'')^{(6)}, i=32,33,34$ depend only on $T_{33}$ and respectively on | 249 |

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| $(G_{35})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 27:** There does not exist any $t$ where $G_i\,(t) = 0\ and\ T_i\,(t) = 0$<br><br>it results<br><br>$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(6)} - (a_i'')^{(6)}(T_{33}(s_{(32)}), s_{(32)})\} ds_{(32)}\right]} \geq 0$<br><br>$T_i\,(t) \geq T_i^0 e^{\left(-(b_i')^{(6)} t\right)} > 0$ for t > 0 | 250 |
| **Definition of** $\left((\widehat{M}_{32})^{(6)}\right)_1, \left((\widehat{M}_{32})^{(6)}\right)_2\ and\ \left((\widehat{M}_{32})^{(6)}\right)_3$ :<br><br>**Remark 28:** if $G_{32}$ is bounded, the same property have also $G_{33}\ and\ G_{34}$ . indeed if<br><br>$G_{32} < (\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt} \leq \left((\widehat{M}_{32})^{(6)}\right)_1 - (a_{33}')^{(6)} G_{33}$ and by integrating<br><br>$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_1 / (a_{33}')^{(6)}$<br><br>In the same way , one can obtain<br><br>$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)} \left((\widehat{M}_{32})^{(6)}\right)_2 / (a_{34}')^{(6)}$<br><br>If $G_{33}\ or\ G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively. | 251 |
| **Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}\ and\ G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. | 252 |
| **Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty} ((b_i'')^{(6)}\ ((G_{35})(t), t)) = (b_{33}')^{(6)}$ then $T_{33}\ \to \infty$.<br><br>**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :<br><br>Indeed let $t_6$ be so that for $t\ > t_6$<br><br>$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t), t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$ | 253 |
| Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results<br><br>$T_{33}\ \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right),\quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded.<br>The same property holds for $T_{34}$ if $\lim_{t\to\infty} (b_{34}'')^{(6)}\left((G_{35})(t), t(t), t\right) = (b_{34}')^{(6)}$<br><br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 254 |
| Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$<br><br>It is now sufficient to take $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}} < 1$ and to choose<br>$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have | 255 |

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| $\frac{(a_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)}+\left((\hat{P}_{36})^{(7)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{36})^{(7)}$ | 256 |
| $\frac{(b_i)^{(7)}}{(\widehat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{36})^{(7)}\right]\leq(\hat{Q}_{36})^{(7)}$ | 257 |
| In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric<br><br>$d\left(((G_{39})^{(1)},(T_{39})^{(1)}),((G_{39})^{(2)},(T_{39})^{(2)})\right)=$<br>$\sup_i\{\max_{t\in\mathbb{R}_+}\left\|G_i^{(1)}(t)-G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t},\max_{t\in\mathbb{R}_+}\left\|T_i^{(1)}(t)-T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{36})^{(7)}t}\}$<br><br>Indeed if we denote<br><br>**Definition of** $\widetilde{(G_{39})},\widetilde{(T_{39})}:\left(\widetilde{(G_{39})},\widetilde{(T_{39})}\right)=\mathcal{A}^{(7)}((G_{39}),(T_{39}))$<br>It results<br><br>$\left\|\tilde{G}_{36}^{(1)}-\tilde{G}_i^{(2)}\right\|\leq\int_0^t(a_{36})^{(7)}\left\|G_{37}^{(1)}-G_{37}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}ds_{(36)}+$<br>$\int_0^t\{(a_{36}^{'})^{(7)}\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)\left\|G_{36}^{(1)}-G_{36}^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}+$<br>$G_{36}^{(2)}\left\|(a_{36}^{''})^{(7)}\left(T_{37}^{(1)},s_{(36)}\right)-(a_{36}^{''})^{(7)}\left(T_{37}^{(2)},s_{(36)}\right)\right\|\ e^{-(\widehat{M}_{36})^{(7)}s_{(36)}}e^{(\widehat{M}_{36})^{(7)}s_{(36)}}\}ds_{(36)}$<br><br>Where $s_{(36)}$ represents integrand that is integrated over the interval [0, t]<br><br>From the hypotheses on it follows | 258 |
| $\left\|(G_{39})^{(1)}-(G_{39})^{(2)}\right\|e^{-(\widehat{M}_{36})^{(7)}t}\leq$<br>$\frac{1}{(\widehat{M}_{36})^{(7)}}\left((a_{36})^{(7)}+(a_{36}^{'})^{(7)}+(\widehat{A}_{36})^{(7)}+(\widehat{P}_{36})^{(7)}(\widehat{k}_{36})^{(7)}\right)d\left(((G_{39})^{(1)},(T_{39})^{(1)};\ (G_{39})^{(2)},(T_{39})^{(2)})\right)$<br><br>And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows | 259 |
| **Remark 31:** The fact that we supposed $(a_{36}^{''})^{(7)}$ and $(b_{36}^{''})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ *and* $(\widehat{Q}_{36})^{(7)}e^{(\widehat{M}_{36})^{(7)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it | 260 |

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| suffices to consider that $(a_i^{''})^{(7)}$ and $(b_i^{''})^{(7)}, i = 36,37,38$ depend only on $T_{37}$ and respectively on $(G_{39})(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 32:** There does not exist any $t$ where $G_i\ (t) = 0\ and\ T_i\ (t) = 0$<br><br>it results<br><br>$G_i\ (t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(7)} - (a_i^{''})^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$<br><br>$T_i\ (t) \geq T_i^0 e^{\left(-(b_i^{'})^{(7)} t\right)} > 0$ for t > 0 | 261 |
| **Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2\ and\ \left((\widehat{M}_{36})^{(7)}\right)_3$ :<br><br>**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}\ and\ G_{38}$ . indeed if<br><br>$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a_{37}^{'})^{(7)} G_{37}$ and by integrating<br><br>$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_1 / (a_{37}^{'})^{(7)}$<br><br>In the same way , one can obtain<br><br>$G_{38} \leq \left((\widehat{M}_{36})^{(7)}\right)_3 = G_{38}^0 + 2(a_{38})^{(7)} \left((\widehat{M}_{36})^{(7)}\right)_2 / (a_{38}^{'})^{(7)}$<br><br>If $G_{37}\ or\ G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively. | 262 |
| **Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}\ and\ G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. | 263 |
| **Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty} ((b_i^{''})^{(7)}\ ((G_{39})(t), t)) = (b_{37}^{'})^{(7)}$ then $T_{37} \to \infty$.<br><br>**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :<br><br>Indeed let $t_7$ be so that for $t > t_7$<br><br>$(b_{37})^{(7)} - (b_i^{''})^{(7)}((G_{39})(t), t) < \varepsilon_7, T_{36}\ (t) > (m)^{(7)}$ | 264 |
| Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7}\right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results<br><br>$T_{37} \geq \left(\frac{(a_{37})^{(7)}(m)^{(7)}}{2}\right),\ \ t = log\frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.<br>The same property holds for $T_{38}$ if $\lim_{t\to\infty} (b_{38}^{''})^{(7)}\ \left((G_{39})(t), t\right) = (b_{38}^{'})^{(7)}$<br>We now state a more precise theorem about the behaviors at infinity of the solutions of equations | 265 |

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| It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}$ , $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose $(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have | 266 |
| $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[(\widehat{P}_{40})^{(8)} + \left((\hat{P}_{40})^{(8)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{40})^{(8)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{40})^{(8)}$ | 267 |
| $\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[\left((\hat{Q}_{40})^{(8)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{40})^{(8)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{40})^{(8)}\right] \leq (\hat{Q}_{40})^{(8)}$ | 268 |
| In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself | |
| The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric | |
| $d\left(((G_{43})^{(1)}, (T_{43})^{(1)}), ((G_{43})^{(2)}, (T_{43})^{(2)})\right) =$ $\sup_i\{\max_{t\in\mathbb{R}_+} \left\|G_i^{(1)}(t) - G_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}, \max_{t\in\mathbb{R}_+} \left\|T_i^{(1)}(t) - T_i^{(2)}(t)\right\|e^{-(\widehat{M}_{40})^{(8)}t}\}$ | 269 |
| Indeed if we denote <br> **Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left(\widetilde{(G_{43})}, \widetilde{(T_{43})}\right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$ | 270 |
| It results <br> $\left\|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right\| \leq \int_0^t (a_{40})^{(8)} \left\|G_{41}^{(1)} - G_{41}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\, ds_{(40)} +$ $\int_0^t \{(a'_{40})^{(8)}\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{-(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left\|G_{40}^{(1)} - G_{40}^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}} +$ $G_{40}^{(2)}\left\|(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)\right\| \; e^{-(\widehat{M}_{40})^{(8)}s_{(40)}}e^{(\widehat{M}_{40})^{(8)}s_{(40)}}\}ds_{(40)}$ | 271 |
| Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, t]$ <br> From the hypotheses it follows | 272 |
| $\left\|(G_{43})^{(1)} - (G_{43})^{(2)}\right\|e^{-(\widehat{M}_{40})^{(8)}t} \leq$ $\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)}(\hat{k}_{40})^{(8)}\right)d\left(((G_{43})^{(1)}, (T_{43})^{(1)}; (G_{43})^{(2)}, (T_{43})^{(2)})\right)$ <br> And analogous inequalities for $G_i$ and $T_i$. Taking into account the hypothesis the result follows | 273 |
| **Remark 36:** The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate | 274 |

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| condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ *and* $(\widehat{Q}_{40})^{(8)}e^{(\widehat{M}_{40})^{(8)}t}$ respectively of $\mathbb{R}_+$.<br><br>If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(8)}$ and $(b_i^{''})^{(8)}, i = 40,41,42$ depend only on $T_{41}$ and respectively on $(G_{43})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition. | |
| **Remark 37** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$<br><br>it results<br><br>$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(8)} - (a_i^{''})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$<br><br>$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(8)}t)} > 0$ for t > 0 | 275 |
| **Definition of** $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2\ and\ \left((\widehat{M}_{40})^{(8)}\right)_3$ :<br><br>**Remark 38:** if $G_{40}$ is bounded, the same property have also $G_{41}\ and\ G_{42}$ . indeed if<br><br>$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a_{41}^{'})^{(8)} G_{41}$ and by integrating<br><br>$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a_{41}^{'})^{(8)}$<br><br>In the same way , one can obtain<br><br>$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a_{42}^{'})^{(8)}$<br><br>If $G_{41}\ or\ G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively. | 276 |
| **Remark 39:** If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}\ and\ G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. | 277 |
| **Remark 40:** If $T_{40}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(8)}((G_{43})(t), t)) = (b_{41}^{'})^{(8)}$ then $T_{41} \to \infty$.<br><br>**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :<br><br>Indeed let $t_8$ be so that for $t > t_8$<br><br>$(b_{41})^{(8)} - (b_i^{''})^{(8)}\left((G_{43})(t), t\right) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$ | 278 |
| Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to<br><br>$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results | 279 |

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}^{''})^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}^{'})^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}} < 1$ and to choose $(\widehat{P}_{44})^{(9)}$ and $(\widehat{Q}_{44})^{(9)}$ large to have 279 A

$$\frac{(a_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[(\widehat{P}_{44})^{(9)} + \left((\widehat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{44})^{(9)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\widehat{M}_{44})^{(9)}}\left[\left((\widehat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{44})^{(9)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{44})^{(9)}\right] \leq (\widehat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(((G_{47})^{(1)}, (T_{47})^{(1)}), ((G_{47})^{(2)}, (T_{47})^{(2)})\right) =$$
$$sup_i\{max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}, max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\widehat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$
It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$
$$\int_0^t\{(a_{44}^{'})^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right)\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$$
$$G_{44}^{(2)}\left|(a_{44}^{''})^{(9)}\left(T_{45}^{(1)}, s_{(44)}\right) - (a_{44}^{''})^{(9)}\left(T_{45}^{(2)}, s_{(44)}\right)\right|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$$\left|(G_{47})^{(1)} - G^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}t} \leq$$
$$\frac{1}{(\widehat{M}_{44})^{(9)}}\left((a_{44})^{(9)} + (a_{44}^{'})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\right)d\left(((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)})\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a_{44}^{''})^{(9)}$ and $(b_{44}^{''})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by

$(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i^{''})^{(9)}$ and $(b_i^{''})^{(9)}, i = 44{,}45{,}46$ depend only on $T_{45}$ and respectively on $(G_{47})(and\ not\ on\ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i(t) = 0\ and\ T_i(t) = 0$

From 99 to 44 it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i^{'})^{(9)} - (a_i^{''})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\} ds_{(44)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b_i^{'})^{(9)}t)} > 0 \quad \text{for t} > 0$$

**Definition of** $\left((\widehat{M}_{44})^{(9)}\right)_1, \left((\widehat{M}_{44})^{(9)}\right)_2\ and\ \left((\widehat{M}_{44})^{(9)}\right)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}\ and\ G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \left((\widehat{M}_{44})^{(9)}\right)_1 - (a_{45}^{'})^{(9)}G_{45}$ and by integrating

$$G_{45} \leq \left((\widehat{M}_{44})^{(9)}\right)_2 = G_{45}^0 + 2(a_{45})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_1/(a_{45}^{'})^{(9)}$$

In the same way , one can obtain

$$G_{46} \leq \left((\widehat{M}_{44})^{(9)}\right)_3 = G_{46}^0 + 2(a_{46})^{(9)}\left((\widehat{M}_{44})^{(9)}\right)_2/(a_{46}^{'})^{(9)}$$

If $G_{45}\ or\ G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}\ and\ G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{45}^{'})^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$$(b_{45})^{(9)} - (b_i^{''})^{(9)}\left((G_{47})(t), t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1 - e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded.
The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}^{''})^{(9)}\left((G_{47})(t), t\right) = (b_{46}^{'})^{(9)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying
$-(\sigma_2)^{(1)} \le -(a'_{13})^{(1)} + (a'_{14})^{(1)} - (a''_{13})^{(1)}(T_{14}, t) + (a''_{14})^{(1)}(T_{14}, t) \le -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \le -(b'_{13})^{(1)} + (b'_{14})^{(1)} - (b''_{13})^{(1)}(G, t) - (b''_{14})^{(1)}(G, t) \le -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0$ , $(\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0$ , $(u_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)}, , (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0$ , $(\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0$ , $(\bar{u}_2)^{(1)} < 0$ the roots of the equations
$(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by
$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}$, $if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)}$,

and $\boxed{(\nu_0)^{(1)} = \frac{G^0_{13}}{G^0_{14}}}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)}$, $if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)}$, $if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}, if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)}$,

and $\boxed{(u_0)^{(1)} = \frac{T^0_{13}}{T^0_{14}}}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

are defined

Then the solution of global equations satisfies the inequalities 285

$G^0_{13} e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \le G_{13}(t) \le G^0_{13} e^{(S_1)^{(1)}t}$

where $(p_i)^{(1)}$ is defined by equation

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| $\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)}-(p_{13})^{(1)})t} \le G_{14}(t) \le \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)}t}$ | |
| $\left(\frac{(a_{15})^{(1)}G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)}-(p_{13})^{(1)}-(S_2)^{(1)})}\left[e^{((S_1)^{(1)}-(p_{13})^{(1)})t} - e^{-(S_2)^{(1)}t}\right] + G_{15}^0 e^{-(S_2)^{(1)}t} \le G_{15}(t) \le \right.$ $\left.\frac{(a_{15})^{(1)}G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)}-(a_{15}')^{(1)})}\left[e^{(S_1)^{(1)}t} - e^{-(a_{15}')^{(1)}t}\right] + G_{15}^0 e^{-(a_{15}')^{(1)}t}\right)$ | 286 |
| $\boxed{T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}}$ | 287 |
| $\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)}t} \le T_{13}(t) \le \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)}+(r_{13})^{(1)})t}$ | 288 |
| $\frac{(b_{15})^{(1)}T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)}-(b_{15}')^{(1)})}\left[e^{(R_1)^{(1)}t} - e^{-(b_{15}')^{(1)}t}\right] + T_{15}^0 e^{-(b_{15}')^{(1)}t} \le T_{15}(t) \le$ $\frac{(a_{15})^{(1)}T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)}+(r_{13})^{(1)}+(R_2)^{(1)})}\left[e^{((R_1)^{(1)}+(r_{13})^{(1)})t} - e^{-(R_2)^{(1)}t}\right] + T_{15}^0 e^{-(R_2)^{(1)}t}$ | 289 |
| **Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- Where $(S_1)^{(1)} = (a_{13})^{(1)}(m_2)^{(1)} - (a_{13}')^{(1)}$ $(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$ $(R_1)^{(1)} = (b_{13})^{(1)}(\mu_2)^{(1)} - (b_{13}')^{(1)}$ $(R_2)^{(1)} = (b_{15}')^{(1)} - (r_{15})^{(1)}$ | 290 |
| **Behavior of the solutions of equation** | 291 |
| **Theorem 2:** If we denote and define | |
| **Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying | 292 |
| $-(\sigma_2)^{(2)} \le -(a_{16}')^{(2)} + (a_{17}')^{(2)} - (a_{16}'')^{(2)}(T_{17}, t) + (a_{17}'')^{(2)}(T_{17}, t) \le -(\sigma_1)^{(2)}$ | 293 |
| $-(\tau_2)^{(2)} \le -(b_{16}')^{(2)} + (b_{17}')^{(2)} - (b_{16}'')^{(2)}((G_{19}), t) - (b_{17}'')^{(2)}((G_{19}), t) \le -(\tau_1)^{(2)}$ | 294 |
| **Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : | 295 |
| By $(\nu_1)^{(2)} > 0$, $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$, $(u_2)^{(2)} < 0$ the roots | 296 |
| of the equations $(a_{17})^{(2)}(\nu^{(2)})^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 297 |
| and $(b_{14})^{(2)}(u^{(2)})^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and | 298 |
| **Definition of** $(\bar{\nu}_1)^{(2)},, (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : | 299 |
| By $(\bar{\nu}_1)^{(2)} > 0$, $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$, $(\bar{u}_2)^{(2)} < 0$ the | 300 |

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| roots of the equations $(a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ | 301 |
| and $(b_{17})^{(2)}\left(u^{(2)}\right)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ | 302 |
| **Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- | 303 |
| If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by | 304 |
| $(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, ***if*** $(\nu_0)^{(2)} < (\nu_1)^{(2)}$ | 305 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, ***if*** $(\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$,<br>and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$ | 306 |
| $(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, ***if*** $(\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ | 307 |
| and analogously<br>$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, ***if*** $(u_0)^{(2)} < (u_1)^{(2)}$<br>$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, ***if*** $(u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,<br>and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$ | 308 |
| $(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, ***if*** $(\bar{u}_1)^{(2)} < (u_0)^{(2)}$ | 309 |
| Then the solution of global equations satisfies the inequalities<br>$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{16}(t) \le G_{16}^0 e^{(S_1)^{(2)}t}$ | 310 |
| $(p_i)^{(2)}$ is defined by equation | |
| $\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \le G_{17}(t) \le \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ | 311 |
| $\left(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)})}\left[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\right] + G_{18}^0 e^{-(S_2)^{(2)}t} \le G_{18}(t) \le \right.$<br>$\left.\frac{(a_{18})^{(2)} G_{16}^0}{(m_2)^{(2)}((S_1)^{(2)} - (a_{18}')^{(2)})}\left[e^{(S_1)^{(2)}t} - e^{-(a_{18}')^{(2)}t}\right] + G_{18}^0 e^{-(a_{18}')^{(2)}t}\right)$ | 312 |
| $\boxed{T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}}$ | 313 |
| $\frac{1}{(\mu_1)^{(2)}} T_{16}^0 e^{(R_1)^{(2)}t} \le T_{16}(t) \le \frac{1}{(\mu_2)^{(2)}} T_{16}^0 e^{((R_1)^{(2)} + (r_{16})^{(2)})t}$ | 314 |
| $\frac{(b_{18})^{(2)} T_{16}^0}{(\mu_1)^{(2)}((R_1)^{(2)} - (b_{18}')^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b_{18}')^{(2)}t}\right] + T_{18}^0 e^{-(b_{18}')^{(2)}t} \le T_{18}(t) \le$ | 315 |

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| $\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t}-e^{-(R_2)^{(2)}t}\right]+T_{18}^{0}e^{-(R_2)^{(2)}t}$ | |
| **Definition of** $(S_1)^{(2)},(S_2)^{(2)},(R_1)^{(2)},(R_2)^{(2)}$:- | 316 |
| Where $(S_1)^{(2)}=(a_{16})^{(2)}(m_2)^{(2)}-(a_{16}^{'})^{(2)}$<br>$(S_2)^{(2)}=(a_{18})^{(2)}-(p_{18})^{(2)}$ | 317 |
| $(R_1)^{(2)}=(b_{16})^{(2)}(\mu_2)^{(1)}-(b_{16}^{'})^{(2)}$<br>$(R_2)^{(2)}=(b_{18}^{'})^{(2)}-(r_{18})^{(2)}$ | 318 |
| **Behavior of the solutions**<br>**Theorem 3:** If we denote and define<br>**Definition of** $(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ :<br>$(\sigma_1)^{(3)},(\sigma_2)^{(3)},(\tau_1)^{(3)},(\tau_2)^{(3)}$ four constants satisfying<br>$-(\sigma_2)^{(3)}\le-(a_{20}^{'})^{(3)}+(a_{21}^{'})^{(3)}-(a_{20}^{''})^{(3)}(T_{21},t)+(a_{21}^{''})^{(3)}(T_{21},t)\le-(\sigma_1)^{(3)}$<br>$-(\tau_2)^{(3)}\le-(b_{20}^{'})^{(3)}+(b_{21}^{'})^{(3)}-(b_{20}^{''})^{(3)}(G_{23},t)-(b_{21}^{''})^{(3)}((G_{23}),t)\le-(\tau_1)^{(3)}$ | 319 |
| **Definition of** $(v_1)^{(3)},(v_2)^{(3)},(u_1)^{(3)},(u_2)^{(3)}$ :<br>By $(v_1)^{(3)}>0$ , $(v_2)^{(3)}<0$ and respectively $(u_1)^{(3)}>0$ , $(u_2)^{(3)}<0$ the roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_1)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_1)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ and<br>By $(\bar{v}_1)^{(3)}>0$ , $(\bar{v}_2)^{(3)}<0$ and respectively $(\bar{u}_1)^{(3)}>0$ , $(\bar{u}_2)^{(3)}<0$ the<br>roots of the equations $(a_{21})^{(3)}(v^{(3)})^2+(\sigma_2)^{(3)}v^{(3)}-(a_{20})^{(3)}=0$<br>and $(b_{21})^{(3)}(u^{(3)})^2+(\tau_2)^{(3)}u^{(3)}-(b_{20})^{(3)}=0$ | 320 |
| **Definition of** $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ :-<br>If we define $(m_1)^{(3)},(m_2)^{(3)},(\mu_1)^{(3)},(\mu_2)^{(3)}$ by<br>$(m_2)^{(3)}=(v_0)^{(3)},(m_1)^{(3)}=(v_1)^{(3)}$, ***if*** $(v_0)^{(3)}<(v_1)^{(3)}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(\bar{v}_1)^{(3)}$ , ***if*** $(v_1)^{(3)}<(v_0)^{(3)}<(\bar{v}_1)^{(3)}$,<br>and $\boxed{(v_0)^{(3)}=\frac{G_{20}^{0}}{G_{21}^{0}}}$<br>$(m_2)^{(3)}=(v_1)^{(3)},(m_1)^{(3)}=(v_0)^{(3)}$, ***if*** $(\bar{v}_1)^{(3)}<(v_0)^{(3)}$ | 321 |

and analogously

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)},\ \boldsymbol{if}\ (u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}, \boldsymbol{if}\ (u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)},$ and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)},\ \boldsymbol{if}\ (\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{20}(t) \le G_{20}^0 e^{(S_1)^{(3)} t}$

$(p_i)^{(3)}$ is defined by equation

(322)

$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{((S_1)^{(3)} - (p_{20})^{(3)})t} \le G_{21}(t) \le \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)} t}$ (323)

$\left(\frac{(a_{22})^{(3)} G_{20}^0}{(m_1)^{(3)}((S_1)^{(3)} - (p_{20})^{(3)} - (S_2)^{(3)})}\left[e^{((S_1)^{(3)} - (p_{20})^{(3)})t} - e^{-(S_2)^{(3)} t}\right] + G_{22}^0 e^{-(S_2)^{(3)} t} \le G_{22}(t) \le \right.$

$\left.\frac{(a_{22})^{(3)} G_{20}^0}{(m_2)^{(3)}((S_1)^{(3)} - (a_{22}')^{(3)})}\left[e^{(S_1)^{(3)} t} - e^{-(a_{22}')^{(3)} t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)} t}\right)$ (324)

$\boxed{T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}}$ (325)

$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)} t} \le T_{20}(t) \le \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{((R_1)^{(3)} + (r_{20})^{(3)})t}$ (326)

$\frac{(b_{22})^{(3)} T_{20}^0}{(\mu_1)^{(3)}((R_1)^{(3)} - (b_{22}')^{(3)})}\left[e^{(R_1)^{(3)} t} - e^{-(b_{22}')^{(3)} t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)} t} \le T_{22}(t) \le$

$\frac{(a_{22})^{(3)} T_{20}^0}{(\mu_2)^{(3)}((R_1)^{(3)} + (r_{20})^{(3)} + (R_2)^{(3)})}\left[e^{((R_1)^{(3)} + (r_{20})^{(3)})t} - e^{-(R_2)^{(3)} t}\right] + T_{22}^0 e^{-(R_2)^{(3)} t}$ (327)

**Definition of** $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- (328)

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$

$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$

**Behavior of the solutions of equation**

**Theorem:** If we denote and define

**Definition of** $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$-(\sigma_2)^{(4)} \le -(a_{24}')^{(4)} + (a_{25}')^{(4)} - (a_{24}'')^{(4)}(T_{25}, t) + (a_{25}'')^{(4)}(T_{25}, t) \le -(\sigma_1)^{(4)}$

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| $-(\tau_2)^{(4)} \le -(b_{24}^{'})^{(4)} + (b_{25}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27}),t\big) - (b_{25}^{''})^{(4)}\big((G_{27}),t\big) \le -(\tau_1)^{(4)}$ | |
| **Definition of** $(v_1)^{(4)},(v_2)^{(4)},(u_1)^{(4)},(u_2)^{(4)},v^{(4)},u^{(4)}$ :<br><br>By $(v_1)^{(4)} > 0$ , $(v_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0$ , $(u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_1)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and | 329 |
| **Definition of** $(\bar{v}_1)^{(4)},,(\bar{v}_2)^{(4)},(\bar{u}_1)^{(4)},(\bar{u}_2)^{(4)}$ :<br><br>By $(\bar{v}_1)^{(4)} > 0$ , $(\bar{v}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0$ , $(\bar{u}_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(v^{(4)}\big)^2 + (\sigma_2)^{(4)}v^{(4)} - (a_{24})^{(4)} = 0$<br>and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$<br>**Definition of** $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)},(\nu_0)^{(4)}$ :-<br><br>If we define $(m_1)^{(4)}$ , $(m_2)^{(4)}$ , $(\mu_1)^{(4)},(\mu_2)^{(4)}$ by<br><br>$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)}$, ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$<br><br>$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ , ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,<br>and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$<br><br>$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)}$, ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$ | 330 |
| and analogously<br><br>$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)}$, ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)}$ , ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,<br>and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$<br><br>$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)}$, ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$ | 331 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{24}(t) \le G_{24}^0 e^{(S_1)^{(4)}t}$<br><br>where $(p_i)^{(4)}$ is defined by equation | 332 |
| $\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)} - (p_{24})^{(4)})t} \le G_{25}(t) \le \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$ | 333 |
| $\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)} - (p_{24})^{(4)} - (S_2)^{(4)})}\left[e^{((S_1)^{(4)} - (p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \le G_{26}(t) \le$<br>*(a26)4G240(m2)4(S1)4−(a26′)4e(S1)4t−e−(a26′)4t+ G260e−(a26′)4t* | 334 |

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|---|---|
| $T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | |
| $\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \le T_{24}(t) \le \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$ | 335 |
| $\frac{(b_{26})^{(4)}T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b_{26}')^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b_{26}')^{(4)}t}\right] + T_{26}^0 e^{-(b_{26}')^{(4)}t} \le T_{26}(t) \le$ $\frac{(a_{26})^{(4)}T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$ | 336 |
| **Definition of** $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a_{24}')^{(4)}$ $(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$ $(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b_{24}')^{(4)}$ $(R_2)^{(4)} = (b_{26}')^{(4)} - (r_{26})^{(4)}$ | 337 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define | 338 |
| **Definition of** $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ : $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying $-(\sigma_2)^{(5)} \le -(a_{28}')^{(5)} + (a_{29}')^{(5)} - (a_{28}'')^{(5)}(T_{29}, t) + (a_{29}'')^{(5)}(T_{29}, t) \le -(\sigma_1)^{(5)}$ $-(\tau_2)^{(5)} \le -(b_{28}')^{(5)} + (b_{29}')^{(5)} - (b_{28}'')^{(5)}((G_{31}), t) - (b_{29}'')^{(5)}((G_{31}), t) \le -(\tau_1)^{(5)}$ | 339 |
| **Definition of** $(v_1)^{(5)}, (v_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, v^{(5)}, u^{(5)}$ : By $(v_1)^{(5)} > 0$, $(v_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0$, $(u_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and | 340 |
| **Definition of** $(\bar{v}_1)^{(5)},, (\bar{v}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : By $(\bar{v}_1)^{(5)} > 0$, $(\bar{v}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$, $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)} = 0$ and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ **Definition of** $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}, (v_0)^{(5)}$ :- If we define $(m_1)^{(5)}, (m_2)^{(5)}, (\mu_1)^{(5)}, (\mu_2)^{(5)}$ by $(m_2)^{(5)} = (v_0)^{(5)}, (m_1)^{(5)} = (v_1)^{(5)}$, ***if*** $(v_0)^{(5)} < (v_1)^{(5)}$ $(m_2)^{(5)} = (v_1)^{(5)}, (m_1)^{(5)} = (\bar{v}_1)^{(5)}$, ***if*** $(v_1)^{(5)} < (v_0)^{(5)} < (\bar{v}_1)^{(5)}$, | |

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| and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$<br>$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)},$ ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$ | |
| and analogously<br>$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},$ ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)}$,<br>and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$<br>$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)},$ ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$ | 341 |
| Then the solution of global equations satisfies the inequalities<br>$G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{28}(t) \leq G_{28}^0 e^{(S_1)^{(5)} t}$<br>where $(p_i)^{(5)}$ is defined by equation | 342 |
| $\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{((S_1)^{(5)} - (p_{28})^{(5)})t} \leq G_{29}(t) \leq \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)} t}$ | 343 |
| $\left(\frac{(a_{30})^{(5)} G_{28}^0}{(m_1)^{(5)}((S_1)^{(5)} - (p_{28})^{(5)} - (S_2)^{(5)})}\left[e^{((S_1)^{(5)} - (p_{28})^{(5)})t} - e^{-(S_2)^{(5)} t}\right] + G_{30}^0 e^{-(S_2)^{(5)} t} \leq G_{30}(t) \leq$<br>*(a30)5G280(m2)5(S1)5−(a30′)5e(S1)5t−e−(a30′)5t+ G300e−(a30′)5t* | 344 |
| $\boxed{T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}}$ | 345 |
| $\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)} t} \leq T_{28}(t) \leq \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)} + (r_{28})^{(5)})t}$ | 346 |
| $\frac{(b_{30})^{(5)} T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)} - (b_{30}')^{(5)})}\left[e^{(R_1)^{(5)} t} - e^{-(b_{30}')^{(5)} t}\right] + T_{30}^0 e^{-(b_{30}')^{(5)} t} \leq T_{30}(t) \leq$<br>$\frac{(a_{30})^{(5)} T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)} + (r_{28})^{(5)} + (R_2)^{(5)})}\left[e^{((R_1)^{(5)} + (r_{28})^{(5)})t} - e^{-(R_2)^{(5)} t}\right] + T_{30}^0 e^{-(R_2)^{(5)} t}$ | 347 |
| **Definition of** $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:-<br>Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a_{28}')^{(5)}$<br>$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$<br>$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b_{28}')^{(5)}$<br>$(R_2)^{(5)} = (b_{30}')^{(5)} - (r_{30})^{(5)}$ | 348 |
| **Behavior of the solutions of equation** | 349 |

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying

$$-(\sigma_2)^{(6)} \le -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \le -(\sigma_1)^{(6)}$$

$$-(\tau_2)^{(6)} \le -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}\big((G_{35}), t\big) - (b''_{33})^{(6)}\big((G_{35}), t\big) \le -(\tau_1)^{(6)}$$

**Definition of** $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ : 350

By $(\nu_1)^{(6)} > 0$ , $(\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0$ , $(u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(6)}, , (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)} > 0$ , $(\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0$ , $(\bar{u}_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}\big(\nu^{(6)}\big)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}\big(u^{(6)}\big)^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$

**Definition of** $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by

$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)}$, ***if*** $(\nu_0)^{(6)} < (\nu_1)^{(6)}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}$ , ***if*** $(\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)}$,
and $\boxed{(\nu_0)^{(6)} = \frac{G^0_{32}}{G^0_{33}}}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)}$, ***if*** $(\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$

and analogously 352

$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)}$, ***if*** $(u_0)^{(6)} < (u_1)^{(6)}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}$ , ***if*** $(u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)}$,
and $\boxed{(u_0)^{(6)} = \frac{T^0_{32}}{T^0_{33}}}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}$, ***if*** $(\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$

Then the solution of global equations satisfies the inequalities 353

$$G^0_{32}e^{((S_1)^{(6)} - (p_{32})^{(6)})t} \le G_{32}(t) \le G^0_{32}e^{(S_1)^{(6)}t}$$

where $(p_i)^{(6)}$ is defined by equation

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| $\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \le G_{33}(t) \le \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)}t}$ | 354 |
| $\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G_{34}^0 e^{-(S_2)^{(6)}t} \le G_{34}(t) \le$ *(a34)6G320(m2)6(S1)6−(a34′)6e(S1)6t−e−(a34′)6t+ G340e−(a34′)6t* | 355 |
| $\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ | 356 |
| $\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \le T_{32}(t) \le \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ | 357 |
| $\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b_{34}^{'})^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b_{34}^{'})^{(6)}t}\right] + T_{34}^0 e^{-(b_{34}^{'})^{(6)}t} \le T_{34}(t) \le$ $\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$ | 358 |
| **Definition of** $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a_{32}^{'})^{(6)}$ $(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$ $(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b_{32}^{'})^{(6)}$ $(R_2)^{(6)} = (b_{34}^{'})^{(6)} - (r_{34})^{(6)}$ | 359 |
| **Behavior of the solutions of equation** **Theorem 2:** If we denote and define **Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$: $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying $-(\sigma_2)^{(7)} \le -(a_{36}^{'})^{(7)} + (a_{37}^{'})^{(7)} - (a_{36}^{''})^{(7)}(T_{37}, t) + (a_{37}^{''})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$ $-(\tau_2)^{(7)} \le -(b_{36}^{'})^{(7)} + (b_{37}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39}), t) - (b_{37}^{''})^{(7)}((G_{39}), t) \le -(\tau_1)^{(7)}$ | |
| **Definition of** $(v_1)^{(7)}, (v_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, v^{(7)}, u^{(7)}$: By $(v_1)^{(7)} > 0, (v_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}(v^{(7)})^2 + (\sigma_1)^{(7)} v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}(u^{(7)})^2 + (\tau_1)^{(7)} u^{(7)} - (b_{36})^{(7)} = 0$ and | 361 |
| **Definition of** $(\bar{v}_1)^{(7)}, , (\bar{v}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$: By $(\bar{v}_1)^{(7)} > 0, (\bar{v}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the | 362 |

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| roots of the equations $(a_{37})^{(7)}\left(v^{(7)}\right)^2 + (\sigma_2)^{(7)}v^{(7)} - (a_{36})^{(7)} = 0$ and $(b_{37})^{(7)}\left(u^{(7)}\right)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$<br>**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-<br>If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by<br>$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}$, ***if*** $(\nu_0)^{(7)} < (\nu_1)^{(7)}$<br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}$, ***if*** $(\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,<br>and $\boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$<br>$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}$, ***if*** $(\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$ | |
| and analogously<br>$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}$, ***if*** $(u_0)^{(7)} < (u_1)^{(7)}$<br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}$, ***if*** $(u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,<br>and $\boxed{(u_0)^{(7)} = \frac{T_{36}^0}{T_{37}^0}}$<br>$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}$, ***if*** $(\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$ | 363 |
| Then the solution of global equations satisfies the inequalities<br>$G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{36}(t) \le G_{36}^0 e^{(S_1)^{(7)}t}$<br>where $(p_i)^{(7)}$ is defined by equation | 364 |
| $\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)} - (p_{36})^{(7)})t} \le G_{37}(t) \le \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ | 365 |
| $\left(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)} - (p_{36})^{(7)} - (S_2)^{(7)})}\left[e^{((S_1)^{(7)} - (p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \le G_{38}(t) \le \right.$<br>$\left.\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)} - (a'_{38})^{(7)})}[e^{(S_1)^{(7)}t} - e^{-(a'_{38})^{(7)}t}] + G_{38}^0 e^{-(a'_{38})^{(7)}t}\right)$ | 366 |
| $\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}}$ | 367 |
| $\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \le T_{36}(t) \le \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)} + (r_{36})^{(7)})t}$ | 368 |
| $\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)} - (b'_{38})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b'_{38})^{(7)}t}\right] + T_{38}^0 e^{-(b'_{38})^{(7)}t} \le T_{38}(t) \le$ | 369 |

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| $\frac{(a_{38})^{(7)}T_{36}^{0}}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t}-e^{-(R_2)^{(7)}t}\right]+T_{38}^{0}e^{-(R_2)^{(7)}t}$ | |
| **Definition of** $(S_1)^{(7)},(S_2)^{(7)},(R_1)^{(7)},(R_2)^{(7)}$:-<br>Where $(S_1)^{(7)}=(a_{36})^{(7)}(m_2)^{(7)}-(a_{36}')^{(7)}$<br>$(S_2)^{(7)}=(a_{38})^{(7)}-(p_{38})^{(7)}$<br>$(R_1)^{(7)}=(b_{36})^{(7)}(\mu_2)^{(7)}-(b_{36}')^{(7)}$<br>$(R_2)^{(7)}=(b_{38}')^{(7)}-(r_{38})^{(7)}$ | 370 |
| **Behavior of the solutions of equation**<br>**Theorem 2:** If we denote and define | 371 |
| **Definition of** $(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ :<br>$(\sigma_1)^{(8)},(\sigma_2)^{(8)},(\tau_1)^{(8)},(\tau_2)^{(8)}$ four constants satisfying<br>$-(\sigma_2)^{(8)}\le-(a_{40}')^{(8)}+(a_{41}')^{(8)}-(a_{40}'')^{(8)}(T_{41},t)+(a_{41}'')^{(8)}(T_{41},t)\le-(\sigma_1)^{(8)}$<br>$-(\tau_2)^{(8)}\le-(b_{40}')^{(8)}+(b_{41}')^{(8)}-(b_{40}'')^{(8)}((G_{43}),t)-(b_{41}'')^{(8)}((G_{43}),t)\le-(\tau_1)^{(8)}$ | 372 |
| **Definition of** $(\nu_1)^{(8)},(\nu_2)^{(8)},(u_1)^{(8)},(u_2)^{(8)},\nu^{(8)},u^{(8)}$ :<br>By $(\nu_1)^{(8)}>0,(\nu_2)^{(8)}<0$ and respectively $(u_1)^{(8)}>0,(u_2)^{(8)}<0$ the roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_1)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_1)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$ and | |
| **Definition of** $(\bar{\nu}_1)^{(8)},,(\bar{\nu}_2)^{(8)},(\bar{u}_1)^{(8)},(\bar{u}_2)^{(8)}$ :<br>By $(\bar{\nu}_1)^{(8)}>0,(\bar{\nu}_2)^{(8)}<0$ and respectively $(\bar{u}_1)^{(8)}>0,(\bar{u}_2)^{(8)}<0$ the<br>roots of the equations $(a_{41})^{(8)}(\nu^{(8)})^2+(\sigma_2)^{(8)}\nu^{(8)}-(a_{40})^{(8)}=0$<br>and $(b_{41})^{(8)}(u^{(8)})^2+(\tau_2)^{(8)}u^{(8)}-(b_{40})^{(8)}=0$<br>**Definition of** $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)},(\nu_0)^{(8)}$ :-<br>If we define $(m_1)^{(8)},(m_2)^{(8)},(\mu_1)^{(8)},(\mu_2)^{(8)}$ by<br>$(m_2)^{(8)}=(\nu_0)^{(8)},(m_1)^{(8)}=(\nu_1)^{(8)}$, ***if*** $(\nu_0)^{(8)}<(\nu_1)^{(8)}$ | |

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| $(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , $\boldsymbol{if}$ $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$, and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$<br><br>$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)}$, $\boldsymbol{if}$ $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$ | |
| and analogously<br><br>$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)}$, $\boldsymbol{if}$ $(u_0)^{(8)} < (u_1)^{(8)}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , $\boldsymbol{if}$ $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$, and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$<br><br>$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)}, \boldsymbol{if}$ $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$ | 374 |
| Then the solution of global equations satisfies the inequalities<br><br>$G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)} t}$<br><br>where $(p_i)^{(8)}$ is defined by equation | 375 |
| $\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)} - (p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)} t}$ | 376 |
| $\left(\frac{(a_{42})^{(8)} G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)} - (p_{40})^{(8)} - (S_2)^{(8)})}\right)\left[e^{((S_1)^{(8)} - (p_{40})^{(8)})t} - e^{-(S_2)^{(8)} t}\right] + G_{42}^0 e^{-(S_2)^{(8)} t} \leq G_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)} - (a_{42}')^{(8)})}[e^{(S_1)^{(8)} t} - e^{-(a_{42}')^{(8)} t}] + G_{42}^0 e^{-(a_{42}')^{(8)} t})$ | 377 |
| $\boxed{T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}}$ | 378 |
| $\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)} t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)} + (r_{40})^{(8)})t}$ | 379 |
| $\frac{(b_{42})^{(8)} T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)} - (b_{42}')^{(8)})}\left[e^{(R_1)^{(8)} t} - e^{-(b_{42}')^{(8)} t}\right] + T_{42}^0 e^{-(b_{42}')^{(8)} t} \leq T_{42}(t) \leq$<br><br>$\frac{(a_{42})^{(8)} T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)} + (r_{40})^{(8)} + (R_2)^{(8)})}\left[e^{((R_1)^{(8)} + (r_{40})^{(8)})t} - e^{-(R_2)^{(8)} t}\right] + T_{42}^0 e^{-(R_2)^{(8)} t}$ | 380 |
| **Definition of** $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:-<br><br>Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a_{40}')^{(8)}$<br><br>$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$ | 381 |

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

**Behavior of the solutions of equation 37 to 92** 382

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \le -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \le -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \le -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \le -(\tau_1)^{(9)}$

**Definition of** $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0$ , $(\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0$ , $(u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(9)}, , (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0$ , $(\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0$ , $(\bar{u}_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\big(\nu^{(9)}\big)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$ and $(b_{45})^{(9)}\big(u^{(9)}\big)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

**Definition of** $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}$, ***if*** $(\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ , ***if*** $(\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)}$,

and $\boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}$, ***if*** $(\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)}$, ***if*** $(u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}$ , ***if*** $(u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)}$,

and $\boxed{(u_0)^{(9)} = \frac{T^0_{44}}{T^0_{45}}}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}$, ***if*** $(\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$ are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{44}(t) \le G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \le G_{45}(t) \le \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})} \left[ e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t} \right] + G_{46}^0 e^{-(S_2)^{(9)}t} \le G_{46}(t) \le \right.$$

$$\left. \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a'_{46})^{(9)})} [e^{(S_1)^{(9)}t} - e^{-(a'_{46})^{(9)}t}] + G_{46}^0 e^{-(a'_{46})^{(9)}t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \le T_{44}(t) \le \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b'_{46})^{(9)})} \left[ e^{(R_1)^{(9)}t} - e^{-(b'_{46})^{(9)}t} \right] + T_{46}^0 e^{-(b'_{46})^{(9)}t} \le T_{46}(t) \le$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})} \left[ e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t} \right] + T_{46}^0 e^{-(R_2)^{(9)}t}$$

**Definition of** $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a'_{44})^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b'_{44})^{(9)}$

$(R_2)^{(9)} = (b'_{46})^{(9)} - (r_{46})^{(9)}$

**Proof :** From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left( (a'_{13})^{(1)} - (a'_{14})^{(1)} + (a''_{13})^{(1)}(T_{14}, t) \right) - (a''_{14})^{(1)}(T_{14}, t)v^{(1)} - (a_{14})^{(1)}v^{(1)}$$

**Definition of** $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_2)^{(1)}v^{(1)} - (a_{13})^{(1)} \right) \le \frac{dv^{(1)}}{dt} \le -\left( (a_{14})^{(1)}\left(v^{(1)}\right)^2 + (\sigma_1)^{(1)}v^{(1)} - (a_{13})^{(1)} \right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(1)}, (v_0)^{(1)}$ :-

For $0 < \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}} < (\nu_1)^{(1)} < (\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t) \geq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_0)^{(1)}\right)t\right]}} \quad , \quad \boxed{(C)^{(1)} = \frac{(\nu_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(1)} = \frac{(\bar{\nu}_1)^{(1)} - (\nu_0)^{(1)}}{(\nu_0)^{(1)} - (\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)} \leq \nu^{(1)}(t) \leq (\bar{\nu}_1)^{(1)}$

If $0 < (\nu_1)^{(1)} < (\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0} < (\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)} \leq \frac{(\nu_1)^{(1)} + (C)^{(1)}(\nu_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}}{1 + (C)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)} - (\nu_2)^{(1)}\right)t\right]}} \leq \nu^{(1)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\bar{\nu}_1)^{(1)}$$

If $0 < (\nu_1)^{(1)} \leq (\bar{\nu}_1)^{(1)} \leq \boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)} \leq \nu^{(1)}(t) \leq \frac{(\bar{\nu}_1)^{(1)} + (\bar{C})^{(1)}(\bar{\nu}_2)^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}}{1 + (\bar{C})^{(1)} e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)} - (\bar{\nu}_2)^{(1)}\right)t\right]}} \leq (\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)} \leq \nu^{(1)}(t) \leq (m_1)^{(1)}, \quad \boxed{\nu^{(1)}(t) = \frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)} \leq u^{(1)}(t) \leq (\mu_1)^{(1)}, \quad \boxed{u^{(1)}(t) = \frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{13}^{''})^{(1)} = (a_{14}^{''})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b_{13}^{''})^{(1)} = (b_{14}^{''})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a_{16}^{'})^{(2)} - (a_{17}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17}, t)\right) - (a_{17}^{''})^{(2)}(T_{17}, t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \leq \frac{d\nu^{(2)}}{dt} \leq -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \geq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \leq \nu^{(2)}(t) \leq (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \leq \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \leq \nu^{(2)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\bar{\nu}_1)^{(2)}$$

| | |
|---|---|
| If $0 < (v_1)^{(2)} \leq (\bar{v}_1)^{(2)} \leq (v_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain<br><br>$(v_1)^{(2)} \leq v^{(2)}(t) \leq \frac{(\bar{v}_1)^{(2)} + (\bar{C})^{(2)}(\bar{v}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)} - (\bar{v}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{v}_1)^{(2)} - (\bar{v}_2)^{(2)}\right)t\right]}} \leq (v_0)^{(2)}$<br><br>And so with the notation of the first part of condition (c) , we have | 394 |
| **Definition of** $v^{(2)}(t)$ :-<br><br>$(m_2)^{(2)} \leq v^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{v^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$ | 395 |
| In a completely analogous way, we obtain<br><br>**Definition of** $u^{(2)}(t)$ :-<br><br>$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$ | 396 |
| Now, using this result and replacing it in global equations we get easily the result stated in the theorem. | |
| **Particular case :**<br><br>If $(a_{16}^{''})^{(2)} = (a_{17}^{''})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(v_1)^{(2)} = (\bar{v}_1)^{(2)}$ if in addition $(v_0)^{(2)} = (v_1)^{(2)}$ then $v^{(2)}(t) = (v_0)^{(2)}$ and as a consequence $G_{16}(t) = (v_0)^{(2)} G_{17}(t)$<br><br>Analogously if $(b_{16}^{''})^{(2)} = (b_{17}^{''})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then<br><br>$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(v_1)^{(2)}$ and $(\bar{v}_1)^{(2)}$ | 397 |
| **Proof** : From global equations we obtain<br><br>$\frac{dv^{(3)}}{dt} = (a_{20})^{(3)} - \left((a_{20}^{'})^{(3)} - (a_{21}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21}, t)\right) - (a_{21}^{''})^{(3)}(T_{21}, t) v^{(3)} - (a_{21})^{(3)} v^{(3)}$ | 398 |
| **Definition of** $v^{(3)}$ :- $\boxed{v^{(3)} = \frac{G_{20}}{G_{21}}}$<br><br>It follows<br><br>$-\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_2)^{(3)} v^{(3)} - (a_{20})^{(3)}\right) \leq \frac{dv^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(v^{(3)}\right)^2 + (\sigma_1)^{(3)} v^{(3)} - (a_{20})^{(3)}\right)$ | 399 |
| From which one obtains<br><br>For $0 < (v_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (v_1)^{(3)} < (\bar{v}_1)^{(3)}$ | 400 |

$$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}} \quad , \quad \boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**Definition of** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a_{20}^{''})^{(3)} = (a_{21}^{''})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)} G_{21}(t)$

Analogously if $(b_{20}^{''})^{(3)} = (b_{21}^{''})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$ if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)} T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**Proof** : From global equations we obtain

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a_{24}')^{(4)} - (a_{25}')^{(4)} + (a_{24}'')^{(4)}(T_{25}, t)\right) - (a_{25}'')^{(4)}(T_{25}, t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

**Definition of** $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

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In the same manner , we get

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

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If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case,

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

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If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

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And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(4)}(t)$ :-

$(m_2)^{(4)} \leq v^{(4)}(t) \leq (m_1)^{(4)}$, $\boxed{v^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq u^{(4)}(t) \leq (\mu_1)^{(4)}$, $\boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{24}^{''})^{(4)} = (a_{25}^{''})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $v^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case** .

Analogously if $(b_{24}^{''})^{(4)} = (b_{25}^{''})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then $(u_1)^{(4)} = (\bar{u}_4)^{(4)}$ if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

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**Proof** : From global equations we obtain

$$\frac{dv^{(5)}}{dt} = (a_{28})^{(5)} - \left((a_{28}^{'})^{(5)} - (a_{29}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}, t)\right) - (a_{29}^{''})^{(5)}(T_{29}, t)v^{(5)} - (a_{29})^{(5)}v^{(5)}$$

**Definition of** $v^{(5)}$ :- $\boxed{v^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_2)^{(5)}v^{(5)} - (a_{28})^{(5)}\right) \leq \frac{dv^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(v^{(5)}\right)^2 + (\sigma_1)^{(5)}v^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$v^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq v^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get 409

$$\nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)}-(\nu_0)^{(5)}}{(\nu_0)^{(5)}-(\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \le \nu^{(5)}(t) \le (\bar{\nu}_5)^{(5)}$

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If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case,

$$(\nu_1)^{(5)} \le \frac{(\nu_1)^{(5)}+(C)^{(5)}(\nu_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}} \le \nu^{(5)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\bar{\nu}_1)^{(5)}$$

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If $0 < (\nu_1)^{(5)} \le (\bar{\nu}_1)^{(5)} \le \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain

$$(\nu_1)^{(5)} \le \nu^{(5)}(t) \le \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \le (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$$(m_2)^{(5)} \le \nu^{(5)}(t) \le (m_1)^{(5)}, \quad \boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$$(\mu_2)^{(5)} \le u^{(5)}(t) \le (\mu_1)^{(5)}, \quad \boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{28}^{''})^{(5)} = (a_{29}^{''})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case** .

Analogously if $(b_{28}^{''})^{(5)} = (b_{29}^{''})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then $(u_1)^{(5)} = (\bar{u}_1)^{(5)}$ if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a_{32}^{'})^{(6)} - (a_{33}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33},t)\right) - (a_{33}^{''})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_2)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)\le\frac{dv^{(6)}}{dt}\le-\left((a_{33})^{(6)}\left(v^{(6)}\right)^2+(\sigma_1)^{(6)}v^{(6)}-(a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{v}_1)^{(6)},(v_0)^{(6)}$ :-

For $0<\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}<(v_1)^{(6)}<(\bar{v}_1)^{(6)}$

$$v^{(6)}(t)\ge\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_0)^{(6)}\right)t\right]}}\quad,\quad\boxed{(C)^{(6)}=\frac{(v_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(v_2)^{(6)}}}$$

it follows $(v_0)^{(6)}\le v^{(6)}(t)\le(v_1)^{(6)}$

In the same manner , we get

$$v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(6)}=\frac{(\bar{v}_1)^{(6)}-(v_0)^{(6)}}{(v_0)^{(6)}-(\bar{v}_2)^{(6)}}}$$

From which we deduce $(v_0)^{(6)}\le v^{(6)}(t)\le(\bar{v}_1)^{(6)}$

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If $0<(v_1)^{(6)}<(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}<(\bar{v}_1)^{(6)}$ we find like in the previous case,

$$(v_1)^{(6)}\le\frac{(v_1)^{(6)}+(C)^{(6)}(v_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}{1+(C)^{(6)}e^{\left[-(a_{33})^{(6)}\left((v_1)^{(6)}-(v_2)^{(6)}\right)t\right]}}\le v^{(6)}(t)\le$$

$$\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(\bar{v}_1)^{(6)}$$

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If $0<(v_1)^{(6)}\le(\bar{v}_1)^{(6)}\le\boxed{(v_0)^{(6)}=\frac{G_{32}^0}{G_{33}^0}}$ , we obtain

$$(v_1)^{(6)}\le v^{(6)}(t)\le\frac{(\bar{v}_1)^{(6)}+(\bar{C})^{(6)}(\bar{v}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{v}_1)^{(6)}-(\bar{v}_2)^{(6)}\right)t\right]}}\le(v_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $v^{(6)}(t)$ :-

$$(m_2)^{(6)}\le v^{(6)}(t)\le(m_1)^{(6)},\quad\boxed{v^{(6)}(t)=\frac{G_{32}(t)}{G_{33}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$$(\mu_2)^{(6)}\le u^{(6)}(t)\le(\mu_1)^{(6)},\quad\boxed{u^{(6)}(t)=\frac{T_{32}(t)}{T_{33}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

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**Particular case :**

If $(a_{32}^{''})^{(6)} = (a_{33}^{''})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)} G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case** .
Analogously if $(b_{32}^{''})^{(6)} = (b_{33}^{''})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then
$(u_1)^{(6)} = (\bar{u}_1)^{(6)}$ if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)} T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

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**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a_{36}^{'})^{(7)} - (a_{37}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}, t)\right) - (a_{37}^{''})^{(7)}(T_{37}, t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

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In the same manner , we get

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)} + (\bar{C})^{(7)}(\bar{\nu}_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}}{1 + (\bar{C})^{(7)} e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)} - (\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)} - (\nu_0)^{(7)}}{(\nu_0)^{(7)} - (\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

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If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case,

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)} + (C)^{(7)}(\nu_2)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}}{1 + (C)^{(7)} e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)} - (\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$, we obtain

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$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :-

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$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{36}^{''})^{(7)} = (a_{37}^{''})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case** .

Analogously if $(b_{36}^{''})^{(7)} = (b_{37}^{''})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition $(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)}T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain

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$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a_{40}^{'})^{(8)} - (a_{41}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41},t)\right) - (a_{41}^{''})^{(8)}(T_{41},t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get

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$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

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If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case,

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

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If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(8)}(t)$ :-

$$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}, \quad \boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}, \quad \boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a_{40}^{''})^{(8)} = (a_{41}^{''})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case** .

Analogously if $(b_{40}^{''})^{(8)} = (b_{41}^{''})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$ if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

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**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a_{44}^{'})^{(9)} - (a_{45}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}, t)\right) - (a_{45}^{''})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)} + (\bar{C})^{(9)}(\bar{\nu}_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}}{1 + (\bar{C})^{(9)} e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)} - (\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

**Particular case :**

If $(a_{44}^{''})^{(9)} = (a_{45}^{''})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case** .

Analogously if $(b_{44}^{''})^{(9)} = (b_{45}^{''})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then $(u_1)^{(9)} = (\bar{u}_1)^{(9)}$ if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem :** If $(a_i^{''})^{(1)} and\ (b_i^{''})^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}^{'})^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}^{'})^{(1)}(r_{14})^{(1)} - (b_{14}^{'})^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem :** If $(a_i^{''})^{(2)} and\ (b_i^{''})^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}^{'})^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

| | |
|---|---|
| $(b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}^{'})^{(2)}(r_{17})^{(2)} - (b_{17}^{'})^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$<br>$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system | 430 |
| **Theorem :** If $(a_i^{''})^{(3)} and\ (b_i^{''})^{(3)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$<br>$(a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}^{'})^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,<br>$(b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}^{'})^{(3)}(r_{21})^{(3)} - (b_{21}^{'})^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$<br>$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system | 431 |
| We can prove the following<br>**Theorem :** If $(a_i^{''})^{(4)} and\ (b_i^{''})^{(4)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$<br>$(a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}^{'})^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,<br>$(b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}^{'})^{(4)}(r_{25})^{(4)} - (b_{25}^{'})^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$<br>$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system | 432 |
| **Theorem :** If $(a_i^{''})^{(5)} and\ (b_i^{''})^{(5)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$<br>$(a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a_{29}^{'})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,<br>$(b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b_{28}^{'})^{(5)}(r_{29})^{(5)} - (b_{29}^{'})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$<br>$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system | 433 |
| **Theorem** If $(a_i^{''})^{(6)} and\ (b_i^{''})^{(6)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$<br>$(a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a_{33}^{'})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,<br>$(b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b_{32}^{'})^{(6)}(r_{33})^{(6)} - (b_{33}^{'})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$ | 434 |

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| $with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system | |
| **Theorem :** If $(a_i^{''})^{(7)} and\ (b_i^{''})^{(7)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$<br>$(a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a_{37}^{'})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,<br>$(b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b_{36}^{'})^{(7)}(r_{37})^{(7)} - (b_{37}^{'})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$<br>$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system | 435 |
| **Theorem :** If $(a_i^{''})^{(8)} and\ (b_i^{''})^{(8)}$ are independent on $t$ , and the conditions with the notations<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$<br>$(a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a_{41}^{'})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,<br>$(b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b_{40}^{'})^{(8)}(r_{41})^{(8)} - (b_{41}^{'})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$<br>$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system | 436 |
| **Theorem :** If $(a_i^{''})^{(9)} and\ (b_i^{''})^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28)<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$<br>$(a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a_{45}^{'})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,<br>$(b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b_{44}^{'})^{(9)}(r_{45})^{(9)} - (b_{45}^{'})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$<br>$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system | 436 A |
| $(a_{13})^{(1)}G_{14} - \left[(a_{13}^{'})^{(1)} + (a_{13}^{''})^{(1)}(T_{14})\right]G_{13} = 0$ | 437 |
| $(a_{14})^{(1)}G_{13} - \left[(a_{14}^{'})^{(1)} + (a_{14}^{''})^{(1)}(T_{14})\right]G_{14} = 0$ | 438 |
| $(a_{15})^{(1)}G_{14} - \left[(a_{15}^{'})^{(1)} + (a_{15}^{''})^{(1)}(T_{14})\right]G_{15} = 0$ | 439 |
| $(b_{13})^{(1)}T_{14} - [(b_{13}^{'})^{(1)} - (b_{13}^{''})^{(1)}(G)\ ]T_{13} = 0$ | 440 |

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| $(b_{14})^{(1)}T_{13} - [(b_{14}^{'})^{(1)} - (b_{14}^{''})^{(1)}(G)\,]T_{14} = 0$ | 441 |
| $(b_{15})^{(1)}T_{14} - [(b_{15}^{'})^{(1)} - (b_{15}^{''})^{(1)}(G)\,]T_{15} = 0$ | 442 |
| has a unique positive solution , which is an equilibrium solution for the system | |
| $(a_{16})^{(2)}G_{17} - \left[(a_{16}^{'})^{(2)} + (a_{16}^{''})^{(2)}(T_{17})\right]G_{16} = 0$ | 443 |
| $(a_{17})^{(2)}G_{16} - \left[(a_{17}^{'})^{(2)} + (a_{17}^{''})^{(2)}(T_{17})\right]G_{17} = 0$ | 444 |
| $(a_{18})^{(2)}G_{17} - \left[(a_{18}^{'})^{(2)} + (a_{18}^{''})^{(2)}(T_{17})\right]G_{18} = 0$ | 445 |
| $(b_{16})^{(2)}T_{17} - [(b_{16}^{'})^{(2)} - (b_{16}^{''})^{(2)}(G_{19})\,]T_{16} = 0$ | 446 |
| $(b_{17})^{(2)}T_{16} - [(b_{17}^{'})^{(2)} - (b_{17}^{''})^{(2)}(G_{19})\,]T_{17} = 0$ | 447 |
| $(b_{18})^{(2)}T_{17} - [(b_{18}^{'})^{(2)} - (b_{18}^{''})^{(2)}(G_{19})\,]T_{18} = 0$ | 448 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{20})^{(3)}G_{21} - \left[(a_{20}^{'})^{(3)} + (a_{20}^{''})^{(3)}(T_{21})\right]G_{20} = 0$ | 449 |
| $(a_{21})^{(3)}G_{20} - \left[(a_{21}^{'})^{(3)} + (a_{21}^{''})^{(3)}(T_{21})\right]G_{21} = 0$ | 450 |
| $(a_{22})^{(3)}G_{21} - \left[(a_{22}^{'})^{(3)} + (a_{22}^{''})^{(3)}(T_{21})\right]G_{22} = 0$ | 451 |
| $(b_{20})^{(3)}T_{21} - [(b_{20}^{'})^{(3)} - (b_{20}^{''})^{(3)}(G_{23})\,]T_{20} = 0$ | 452 |
| $(b_{21})^{(3)}T_{20} - [(b_{21}^{'})^{(3)} - (b_{21}^{''})^{(3)}(G_{23})\,]T_{21} = 0$ | 453 |
| $(b_{22})^{(3)}T_{21} - [(b_{22}^{'})^{(3)} - (b_{22}^{''})^{(3)}(G_{23})\,]T_{22} = 0$ | 454 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{24})^{(4)}G_{25} - \left[(a_{24}^{'})^{(4)} + (a_{24}^{''})^{(4)}(T_{25})\right]G_{24} = 0$ | 455 |
| $(a_{25})^{(4)}G_{24} - \left[(a_{25}^{'})^{(4)} + (a_{25}^{''})^{(4)}(T_{25})\right]G_{25} = 0$ | 456 |
| $(a_{26})^{(4)}G_{25} - \left[(a_{26}^{'})^{(4)} + (a_{26}^{''})^{(4)}(T_{25})\right]G_{26} = 0$ | 457 |
| $(b_{24})^{(4)}T_{25} - [(b_{24}^{'})^{(4)} - (b_{24}^{''})^{(4)}\big((G_{27})\big)\,]T_{24} = 0$ | 458 |
| $(b_{25})^{(4)}T_{24} - [(b_{25}^{'})^{(4)} - (b_{25}^{''})^{(4)}\big((G_{27})\big)\,]T_{25} = 0$ | 459 |
| $(b_{26})^{(4)}T_{25} - [(b_{26}^{'})^{(4)} - (b_{26}^{''})^{(4)}\big((G_{27})\big)\,]T_{26} = 0$ | 460 |
| has a unique positive solution , which is an equilibrium solution | |

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| $(a_{28})^{(5)}G_{29} - \left[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29})\right]G_{28} = 0$ | 461 |
| $(a_{29})^{(5)}G_{28} - \left[(a_{29}^{'})^{(5)} + (a_{29}^{''})^{(5)}(T_{29})\right]G_{29} = 0$ | 462 |
| $(a_{30})^{(5)}G_{29} - \left[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29})\right]G_{30} = 0$ | 463 |
| $(b_{28})^{(5)}T_{29} - [(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}(G_{31})\,]T_{28} = 0$ | 464 |
| $(b_{29})^{(5)}T_{28} - [(b_{29}^{'})^{(5)} - (b_{29}^{''})^{(5)}(G_{31})\,]T_{29} = 0$ | 465 |
| $(b_{30})^{(5)}T_{29} - [(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}(G_{31})\,]T_{30} = 0$ | 466 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{32})^{(6)}G_{33} - \left[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33})\right]G_{32} = 0$ | 467 |
| $(a_{33})^{(6)}G_{32} - \left[(a_{33}^{'})^{(6)} + (a_{33}^{''})^{(6)}(T_{33})\right]G_{33} = 0$ | 468 |
| $(a_{34})^{(6)}G_{33} - \left[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33})\right]G_{34} = 0$ | 469 |
| $(b_{32})^{(6)}T_{33} - [(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}(G_{35})\,]T_{32} = 0$ | 470 |
| $(b_{33})^{(6)}T_{32} - [(b_{33}^{'})^{(6)} - (b_{33}^{''})^{(6)}(G_{35})\,]T_{33} = 0$ | 471 |
| $(b_{34})^{(6)}T_{33} - [(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}(G_{35})\,]T_{34} = 0$ | 472 |
| has a unique positive solution , which is an equilibrium solution | |
| $(a_{36})^{(7)}G_{37} - \left[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37})\right]G_{36} = 0$ | 473 |
| $(a_{37})^{(7)}G_{36} - \left[(a_{37}^{'})^{(7)} + (a_{37}^{''})^{(7)}(T_{37})\right]G_{37} = 0$ | 474 |
| $(a_{38})^{(7)}G_{37} - \left[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37})\right]G_{38} = 0$ | 475 |
| $(b_{36})^{(7)}T_{37} - [(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}(G_{39})]T_{36} = 0$ | 476 |
| $(b_{37})^{(7)}T_{36} - [(b_{37}^{'})^{(7)} - (b_{37}^{''})^{(7)}(G_{39})\,]T_{37} = 0$ | 477 |
| $(b_{38})^{(7)}T_{37} - [(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}(G_{39})\,]T_{38} = 0$ | 478 |

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| $(a_{40})^{(8)}G_{41} - \left[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41})\right]G_{40} = 0$ | 479 |
| $(a_{41})^{(8)}G_{40} - \left[(a_{41}^{'})^{(8)} + (a_{41}^{''})^{(8)}(T_{41})\right]G_{41} = 0$ | 480 |
| $(a_{42})^{(8)}G_{41} - \left[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41})\right]G_{42} = 0$ | 481 |
| $(b_{40})^{(8)}T_{41} - [(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}(G_{43})\,]T_{40} = 0$ | 482 |
| $(b_{41})^{(8)}T_{40} - [(b_{41}^{'})^{(8)} - (b_{41}^{''})^{(8)}(G_{43})\,]T_{41} = 0$ | 483 |
| $(b_{42})^{(8)}T_{41} - [(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}(G_{43})\,]T_{42} = 0$ | 484 |
| $(a_{44})^{(9)}G_{45} - \left[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45})\right]G_{44} = 0$ | 484 A |
| $(a_{45})^{(9)}G_{44} - \left[(a_{45}^{'})^{(9)} + (a_{45}^{''})^{(9)}(T_{45})\right]G_{45} = 0$ | |
| $(a_{46})^{(9)}G_{45} - \left[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45})\right]G_{46} = 0$ | |
| $(b_{44})^{(9)}T_{45} - [(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}(G_{47})\,]T_{44} = 0$ | |
| $(b_{45})^{(9)}T_{44} - [(b_{45}^{'})^{(9)} - (b_{45}^{''})^{(9)}(G_{47})\,]T_{45} = 0$ | |
| $(b_{46})^{(9)}T_{45} - [(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}(G_{47})\,]T_{46} = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if<br><br>$F(T) = (a_{13}^{'})^{(1)}(a_{14}^{'})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13}^{'})^{(1)}(a_{14}^{''})^{(1)}(T_{14}) + (a_{14}^{'})^{(1)}(a_{13}^{''})^{(1)}(T_{14}) + (a_{13}^{''})^{(1)}(T_{14})(a_{14}^{''})^{(1)}(T_{14}) = 0$ | 485 |
| **Proof:**<br><br>(l) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if<br><br>$\mathrm{F}(T_{19}) = (a_{16}^{'})^{(2)}(a_{17}^{'})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16}^{'})^{(2)}(a_{17}^{''})^{(2)}(T_{17}) + (a_{17}^{'})^{(2)}(a_{16}^{''})^{(2)}(T_{17}) + (a_{16}^{''})^{(2)}(T_{17})(a_{17}^{''})^{(2)}(T_{17}) = 0$ | 486 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if<br><br>$F(T_{23}) = (a_{20}^{'})^{(3)}(a_{21}^{'})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20}^{'})^{(3)}(a_{21}^{''})^{(3)}(T_{21}) + (a_{21}^{'})^{(3)}(a_{20}^{''})^{(3)}(T_{21}) + (a_{20}^{''})^{(3)}(T_{21})(a_{21}^{''})^{(3)}(T_{21}) = 0$ | 487 |
| **Proof:** | 488 |

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| (a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if<br><br>$F(T_{27}) = (a_{24}^{'})^{(4)}(a_{25}^{'})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24}^{'})^{(4)}(a_{25}^{''})^{(4)}(T_{25}) + (a_{25}^{'})^{(4)}(a_{24}^{''})^{(4)}(T_{25}) + (a_{24}^{''})^{(4)}(T_{25})(a_{25}^{''})^{(4)}(T_{25}) = 0$ | |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if<br><br>$F(T_{31}) = (a_{28}^{'})^{(5)}(a_{29}^{'})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28}^{'})^{(5)}(a_{29}^{''})^{(5)}(T_{29}) + (a_{29}^{'})^{(5)}(a_{28}^{''})^{(5)}(T_{29}) + (a_{28}^{''})^{(5)}(T_{29})(a_{29}^{''})^{(5)}(T_{29}) = 0$ | 489 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if<br><br>$F(T_{35}) = (a_{32}^{'})^{(6)}(a_{33}^{'})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32}^{'})^{(6)}(a_{33}^{''})^{(6)}(T_{33}) + (a_{33}^{'})^{(6)}(a_{32}^{''})^{(6)}(T_{33}) + (a_{32}^{''})^{(6)}(T_{33})(a_{33}^{''})^{(6)}(T_{33}) = 0$ | 490 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if<br><br>$F(T_{39}) = (a_{36}^{'})^{(7)}(a_{37}^{'})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36}^{'})^{(7)}(a_{37}^{''})^{(7)}(T_{37}) + (a_{37}^{'})^{(7)}(a_{36}^{''})^{(7)}(T_{37}) + (a_{36}^{''})^{(7)}(T_{37})(a_{37}^{''})^{(7)}(T_{37}) = 0$ | 491 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if<br><br>$F(T_{43}) = (a_{40}^{'})^{(8)}(a_{41}^{'})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40}^{'})^{(8)}(a_{41}^{''})^{(8)}(T_{41}) + (a_{41}^{'})^{(8)}(a_{40}^{''})^{(8)}(T_{41}) + (a_{40}^{''})^{(8)}(T_{41})(a_{41}^{''})^{(8)}(T_{41}) = 0$ | 492 |
| **Proof:**<br><br>(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if<br><br>$F(T_{47}) = (a_{44}^{'})^{(9)}(a_{45}^{'})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44}^{'})^{(9)}(a_{45}^{''})^{(9)}(T_{45}) + (a_{45}^{'})^{(9)}(a_{44}^{''})^{(9)}(T_{45}) + (a_{44}^{''})^{(9)}(T_{45})(a_{45}^{''})^{(9)}(T_{45}) = 0$ | 492 A |
| **Definition and uniqueness of** $T_{14}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}^{'})^{(1)}+(a_{13}^{''})^{(1)}(T_{14}^{*})]}$ , $G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}^{'})^{(1)}+(a_{15}^{''})^{(1)}(T_{14}^{*})]}$ | 493 |
| **Definition and uniqueness of** $T_{17}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(2)}(T_{17})$ being increasing, it follows that | 494 |

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| there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*})=0$. With this value , we obtain from the three first equations | |
| $G_{16}=\frac{(a_{16})^{(2)}G_{17}}{[(a_{16}^{'})^{(2)}+(a_{16}^{''})^{(2)}(T_{17}^{*})]}$ , $G_{18}=\frac{(a_{18})^{(2)}G_{17}}{[(a_{18}^{'})^{(2)}+(a_{18}^{''})^{(2)}(T_{17}^{*})]}$ | 495 |
| **Definition and uniqueness of** $T_{21}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{20}=\frac{(a_{20})^{(3)}G_{21}}{[(a_{20}^{'})^{(3)}+(a_{20}^{''})^{(3)}(T_{21}^{*})]}$ , $G_{22}=\frac{(a_{22})^{(3)}G_{21}}{[(a_{22}^{'})^{(3)}+(a_{22}^{''})^{(3)}(T_{21}^{*})]}$ | 496 |
| **Definition and uniqueness of** $T_{25}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{24}=\frac{(a_{24})^{(4)}G_{25}}{[(a_{24}^{'})^{(4)}+(a_{24}^{''})^{(4)}(T_{25}^{*})]}$ , $G_{26}=\frac{(a_{26})^{(4)}G_{25}}{[(a_{26}^{'})^{(4)}+(a_{26}^{''})^{(4)}(T_{25}^{*})]}$ | 497 |
| **Definition and uniqueness of** $T_{29}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{28}=\frac{(a_{28})^{(5)}G_{29}}{[(a_{28}^{'})^{(5)}+(a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}=\frac{(a_{30})^{(5)}G_{29}}{[(a_{30}^{'})^{(5)}+(a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | 498 |
| **Definition and uniqueness of** $T_{33}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{32}=\frac{(a_{32})^{(6)}G_{33}}{[(a_{32}^{'})^{(6)}+(a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}=\frac{(a_{34})^{(6)}G_{33}}{[(a_{34}^{'})^{(6)}+(a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 499 |
| **Definition and uniqueness of** $T_{37}^{*}$ :- <br><br> After hypothesis $f(0)<0, f(\infty)>0$ and the functions $(a_i^{''})^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*})=0$. With this value , we obtain from the three first equations <br><br> $G_{36}=\frac{(a_{36})^{(7)}G_{37}}{[(a_{36}^{'})^{(7)}+(a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}=\frac{(a_{38})^{(7)}G_{37}}{[(a_{38}^{'})^{(7)}+(a_{38}^{''})^{(7)}(T_{37}^{*})]}$ | 500 |

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| **Definition and uniqueness of** $T_{41}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$ | 501 |
| **Definition and uniqueness of** $T_{45}^{*}$ :-<br><br>After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i^{''})^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations<br><br>$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 501 A |
| By the same argument, the equations admit solutions $G_{13}, G_{14}$ if<br>$\varphi(G) = (b_{13}^{'})^{(1)}(b_{14}^{'})^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$<br>$[(b_{13}^{'})^{(1)}(b_{14}^{''})^{(1)}(G) + (b_{14}^{'})^{(1)}(b_{13}^{''})^{(1)}(G)] + (b_{13}^{''})^{(1)}(G)(b_{14}^{''})^{(1)}(G) = 0$<br><br>Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$ | 502 |
| By the same argument, the equations admit solutions $G_{16}, G_{17}$ if<br><br>$\varphi(G_{19}) = (b_{16}^{'})^{(2)}(b_{17}^{'})^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$<br>$[(b_{16}^{'})^{(2)}(b_{17}^{''})^{(2)}(G_{19}) + (b_{17}^{'})^{(2)}(b_{16}^{''})^{(2)}(G_{19})] + (b_{16}^{''})^{(2)}(G_{19})(b_{17}^{''})^{(2)}(G_{19}) = 0$ | 503 |
| Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ | 504 |
| By the same argument, the equations admit solutions $G_{20}, G_{21}$ if<br><br>$\varphi(G_{23}) = (b_{20}^{'})^{(3)}(b_{21}^{'})^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$<br>$[(b_{20}^{'})^{(3)}(b_{21}^{''})^{(3)}(G_{23}) + (b_{21}^{'})^{(3)}(b_{20}^{''})^{(3)}(G_{23})] + (b_{20}^{''})^{(3)}(G_{23})(b_{21}^{''})^{(3)}(G_{23}) = 0$<br><br>Where in $G_{23}(G_{20}, G_{21}, G_{22}), G_{20}, G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*}) = 0$ | 505 |

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| By the same argument, the equations admit solutions $G_{24}, G_{25}$ if<br><br>$\varphi(G_{27}) = (b_{24}^{'})^{(4)}(b_{25}^{'})^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} -$<br><br>$\left[(b_{24}^{'})^{(4)}(b_{25}^{''})^{(4)}(G_{27}) + (b_{25}^{'})^{(4)}(b_{24}^{''})^{(4)}(G_{27})\right]+(b_{24}^{''})^{(4)}(G_{27})(b_{25}^{''})^{(4)}(G_{27}) = 0$<br><br>Where in $(G_{27})(G_{24}, G_{25}, G_{26}), G_{24}, G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*}) = 0$ | 506 |
| By the same argument, the equations admit solutions $G_{28}, G_{29}$ if<br>$\varphi(G_{31}) = (b_{28}^{'})^{(5)}(b_{29}^{'})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} -$<br><br>$\left[(b_{28}^{'})^{(5)}(b_{29}^{''})^{(5)}(G_{31}) + (b_{29}^{'})^{(5)}(b_{28}^{''})^{(5)}(G_{31})\right]+(b_{28}^{''})^{(5)}(G_{31})(b_{29}^{''})^{(5)}(G_{31}) = 0$<br><br>Where in $(G_{31})(G_{28}, G_{29}, G_{30}), G_{28}, G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*}) = 0$ | 507 |
| By the same argument, the equations admit solutions $G_{32}, G_{33}$ if<br>$\varphi(G_{35}) = (b_{32}^{'})^{(6)}(b_{33}^{'})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} -$<br><br>$\left[(b_{32}^{'})^{(6)}(b_{33}^{''})^{(6)}(G_{35}) + (b_{33}^{'})^{(6)}(b_{32}^{''})^{(6)}(G_{35})\right]+(b_{32}^{''})^{(6)}(G_{35})(b_{33}^{''})^{(6)}(G_{35}) = 0$<br><br>Where in $(G_{35})(G_{32}, G_{33}, G_{34}), G_{32}, G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi({G_{35}}^{*}) = 0$ | 508 |
| By the same argument, the equations admit solutions $G_{36}, G_{37}$ if<br>$\varphi(G_{39}) = (b_{36}^{'})^{(7)}(b_{37}^{'})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} -$<br>$\left[(b_{36}^{'})^{(7)}(b_{37}^{''})^{(7)}(G_{39}) + (b_{37}^{'})^{(7)}(b_{36}^{''})^{(7)}(G_{39})\right]+(b_{36}^{''})^{(7)}(G_{39})(b_{37}^{''})^{(7)}(G_{39}) = 0$<br><br>Where in $(G_{39})(G_{36}, G_{37}, G_{38}), G_{36}, G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi({G_{39}}^{*}) = 0$ | 509 |
| By the same argument, the equations admit solutions $G_{40}, G_{41}$ if<br>$\varphi(G_{43}) = (b_{40}^{'})^{(8)}(b_{41}^{'})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} -$<br>$\left[(b_{40}^{'})^{(8)}(b_{41}^{''})^{(8)}(G_{43}) + (b_{41}^{'})^{(8)}(b_{40}^{''})^{(8)}(G_{43})\right]+(b_{40}^{''})^{(8)}(G_{43})(b_{41}^{''})^{(8)}(G_{43}) = 0$<br><br>Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^{*}$ such that $\varphi({G_{43}}^{*}) = 0$ | 510 |
| By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if<br>$\varphi(G_{47}) = (b_{44}^{'})^{(9)}(b_{45}^{'})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$<br><br>$\left[(b_{44}^{'})^{(9)}(b_{45}^{''})^{(9)}(G_{47}) + (b_{45}^{'})^{(9)}(b_{44}^{''})^{(9)}(G_{47})\right]+(b_{44}^{''})^{(9)}(G_{47})(b_{45}^{''})^{(9)}(G_{47}) = 0$ | |

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| Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^*$ such that $\varphi((G_{47})^*) = 0$ | |
| Finally we obtain the unique solution<br>$G_{14}^*$ given by $\varphi(G^*) = 0$ , $T_{14}^*$ given by $f(T_{14}^*) = 0$ and<br>$G_{13}^* = \frac{(a_{13})^{(1)} G_{14}^*}{[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14}^*)]}$ , $G_{15}^* = \frac{(a_{15})^{(1)} G_{14}^*}{[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}^*)]}$<br>$T_{13}^* = \frac{(b_{13})^{(1)} T_{14}^*}{[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G^*)]}$ , $T_{15}^* = \frac{(b_{15})^{(1)} T_{14}^*}{[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G^*)]}$<br>Obviously, these values represent an equilibrium solution | 511 |
| Finally we obtain the unique solution | |
| $G_{17}^*$ given by $\varphi((G_{19})^*) = 0$ , $T_{17}^*$ given by $f(T_{17}^*) = 0$ and | 512 |
| $G_{16}^* = \frac{(a_{16})^{(2)} G_{17}^*}{[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17}^*)]}$ , $G_{18}^* = \frac{(a_{18})^{(2)} G_{17}^*}{[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17}^*)]}$ | 513 |
| $T_{16}^* = \frac{(b_{16})^{(2)} T_{17}^*}{[(b'_{16})^{(2)} - (b''_{16})^{(2)}((G_{19})^*)]}$ , $T_{18}^* = \frac{(b_{18})^{(2)} T_{17}^*}{[(b'_{18})^{(2)} - (b''_{18})^{(2)}((G_{19})^*)]}$ | 514 |
| Obviously, these values represent an equilibrium solution | |
| Finally we obtain the unique solution<br>$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and<br>$G_{20}^* = \frac{(a_{20})^{(3)} G_{21}^*}{[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21}^*)]}$ , $G_{22}^* = \frac{(a_{22})^{(3)} G_{21}^*}{[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21}^*)]}$<br>$T_{20}^* = \frac{(b_{20})^{(3)} T_{21}^*}{[(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23}{}^*)]}$ , $T_{22}^* = \frac{(b_{22})^{(3)} T_{21}^*}{[(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23}{}^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 515 |
| Finally we obtain the unique solution<br>$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and<br>$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}^*)]}$ | 516 |
| $T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b'_{24})^{(4)} - (b''_{24})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27})^*)]}$<br>Obviously, these values represent an equilibrium solution of global equations | 517 |
| Finally we obtain the unique solution | 518 |

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| $G_{29}^{*}$ given by $\varphi((G_{31})^{*}) = 0$ , $T_{29}^{*}$ given by $f(T_{29}^{*}) = 0$ and<br>$G_{28}^{*} = \frac{(a_{28})^{(5)} G_{29}^{*}}{[(a_{28}^{'})^{(5)} + (a_{28}^{''})^{(5)}(T_{29}^{*})]}$ , $G_{30}^{*} = \frac{(a_{30})^{(5)} G_{29}^{*}}{[(a_{30}^{'})^{(5)} + (a_{30}^{''})^{(5)}(T_{29}^{*})]}$ | |
| $T_{28}^{*} = \frac{(b_{28})^{(5)} T_{29}^{*}}{[(b_{28}^{'})^{(5)} - (b_{28}^{''})^{(5)}((G_{31})^{*})]}$ , $T_{30}^{*} = \frac{(b_{30})^{(5)} T_{29}^{*}}{[(b_{30}^{'})^{(5)} - (b_{30}^{''})^{(5)}((G_{31})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 519 |
| Finally we obtain the unique solution<br>$G_{33}^{*}$ given by $\varphi((G_{35})^{*}) = 0$ , $T_{33}^{*}$ given by $f(T_{33}^{*}) = 0$ and<br>$G_{32}^{*} = \frac{(a_{32})^{(6)} G_{33}^{*}}{[(a_{32}^{'})^{(6)} + (a_{32}^{''})^{(6)}(T_{33}^{*})]}$ , $G_{34}^{*} = \frac{(a_{34})^{(6)} G_{33}^{*}}{[(a_{34}^{'})^{(6)} + (a_{34}^{''})^{(6)}(T_{33}^{*})]}$ | 520 |
| $T_{32}^{*} = \frac{(b_{32})^{(6)} T_{33}^{*}}{[(b_{32}^{'})^{(6)} - (b_{32}^{''})^{(6)}((G_{35})^{*})]}$ , $T_{34}^{*} = \frac{(b_{34})^{(6)} T_{33}^{*}}{[(b_{34}^{'})^{(6)} - (b_{34}^{''})^{(6)}((G_{35})^{*})]}$<br>Obviously, these values represent an equilibrium solution of global equations | 521 |
| Finally we obtain the unique solution<br>$G_{37}^{*}$ given by $\varphi((G_{39})^{*}) = 0$ , $T_{37}^{*}$ given by $f(T_{37}^{*}) = 0$ and<br>$G_{36}^{*} = \frac{(a_{36})^{(7)} G_{37}^{*}}{[(a_{36}^{'})^{(7)} + (a_{36}^{''})^{(7)}(T_{37}^{*})]}$ , $G_{38}^{*} = \frac{(a_{38})^{(7)} G_{37}^{*}}{[(a_{38}^{'})^{(7)} + (a_{38}^{''})^{(7)}(T_{37}^{*})]}$<br>$T_{36}^{*} = \frac{(b_{36})^{(7)} T_{37}^{*}}{[(b_{36}^{'})^{(7)} - (b_{36}^{''})^{(7)}((G_{39})^{*})]}$ , $T_{38}^{*} = \frac{(b_{38})^{(7)} T_{37}^{*}}{[(b_{38}^{'})^{(7)} - (b_{38}^{''})^{(7)}((G_{39})^{*})]}$ | 522 |
| Finally we obtain the unique solution<br>$G_{41}^{*}$ given by $\varphi((G_{43})^{*}) = 0$ , $T_{41}^{*}$ given by $f(T_{41}^{*}) = 0$ and<br>$G_{40}^{*} = \frac{(a_{40})^{(8)} G_{41}^{*}}{[(a_{40}^{'})^{(8)} + (a_{40}^{''})^{(8)}(T_{41}^{*})]}$ , $G_{42}^{*} = \frac{(a_{42})^{(8)} G_{41}^{*}}{[(a_{42}^{'})^{(8)} + (a_{42}^{''})^{(8)}(T_{41}^{*})]}$<br>$T_{40}^{*} = \frac{(b_{40})^{(8)} T_{41}^{*}}{[(b_{40}^{'})^{(8)} - (b_{40}^{''})^{(8)}((G_{43})^{*})]}$ , $T_{42}^{*} = \frac{(b_{42})^{(8)} T_{41}^{*}}{[(b_{42}^{'})^{(8)} - (b_{42}^{''})^{(8)}((G_{43})^{*})]}$ | 523 |
| Finally we obtain the unique solution of 89 to 99<br>$G_{45}^{*}$ given by $\varphi((G_{47})^{*}) = 0$ , $T_{45}^{*}$ given by $f(T_{45}^{*}) = 0$ and<br>$G_{44}^{*} = \frac{(a_{44})^{(9)} G_{45}^{*}}{[(a_{44}^{'})^{(9)} + (a_{44}^{''})^{(9)}(T_{45}^{*})]}$ , $G_{46}^{*} = \frac{(a_{46})^{(9)} G_{45}^{*}}{[(a_{46}^{'})^{(9)} + (a_{46}^{''})^{(9)}(T_{45}^{*})]}$ | 523 A |

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| $T_{44}^{*} = \frac{(b_{44})^{(9)} T_{45}^{*}}{[(b_{44}^{'})^{(9)} - (b_{44}^{''})^{(9)}((G_{47})^{*})]}$ , $T_{46}^{*} = \frac{(b_{46})^{(9)} T_{45}^{*}}{[(b_{46}^{'})^{(9)} - (b_{46}^{''})^{(9)}((G_{47})^{*})]}$ | |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(1)}$ *and* $(b_i^{''})^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>**Proof:** Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$<br><br>$\frac{\partial (a_{14}^{''})^{(1)}}{\partial T_{14}} (T_{14}^{*}) = (q_{14})^{(1)}$ , $\frac{\partial (b_i^{''})^{(1)}}{\partial G_j} (G^{*}) = s_{ij}$ | 524 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^{*} \mathbb{T}_{14}$ | 525 |
| $\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^{*} \mathbb{T}_{14}$ | 526 |
| $\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^{*} \mathbb{T}_{14}$ | 527 |
| $\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^{*} \mathbb{G}_j\right)$ | 528 |
| $\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}^{'})^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^{*} \mathbb{G}_j\right)$ | 529 |
| $\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^{*} \mathbb{G}_j\right)$ | 530 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(2)}$ and $(b_i^{''})^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable | 531 |
| Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | |
| $G_i = G_i^{*} + \mathbb{G}_i \quad , T_i = T_i^{*} + \mathbb{T}_i$ | 532 |
| $\frac{\partial (a_{17}^{''})^{(2)}}{\partial T_{17}} (T_{17}^{*}) = (q_{17})^{(2)}$ , $\frac{\partial (b_i^{''})^{(2)}}{\partial G_j} ((G_{19})^{*}) = s_{ij}$ | 533 |
| taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{16}}{dt} = -\left((a_{16}^{'})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)} G_{16}^{*} \mathbb{T}_{17}$ | 534 |

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| $\frac{d\mathbb{G}_{17}}{dt} = -\left((a'_{17})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G^*_{17}\mathbb{T}_{17}$ | 535 |
| $\frac{d\mathbb{G}_{18}}{dt} = -\left((a'_{18})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G^*_{18}\mathbb{T}_{17}$ | 536 |
| $\frac{d\mathbb{T}_{16}}{dt} = -\left((b'_{16})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T^*_{16}\mathbb{G}_j\right)$ | 537 |
| $\frac{d\mathbb{T}_{17}}{dt} = -\left((b'_{17})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T^*_{17}\mathbb{G}_j\right)$ | 538 |
| $\frac{d\mathbb{T}_{18}}{dt} = -\left((b'_{18})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T^*_{18}\mathbb{G}_j\right)$ | 539 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(3)}$ *and* $(b''_i)^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote<br><br>**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G^*_i + \mathbb{G}_i \quad , T_i = T^*_i + \mathbb{T}_i$<br><br>$\frac{\partial(a''_{21})^{(3)}}{\partial T_{21}}(T^*_{21}) = (q_{21})^{(3)} \quad , \frac{\partial(b''_i)^{(3)}}{\partial G_j}((G_{23})^*) = s_{ij}$ | 540 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{20}}{dt} = -\left((a'_{20})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G^*_{20}\mathbb{T}_{21}$ | 541 |
| $\frac{d\mathbb{G}_{21}}{dt} = -\left((a'_{21})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G^*_{21}\mathbb{T}_{21}$ | 542 |
| $\frac{d\mathbb{G}_{22}}{dt} = -\left((a'_{22})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G^*_{22}\mathbb{T}_{21}$ | 543 |
| $\frac{d\mathbb{T}_{20}}{dt} = -\left((b'_{20})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T^*_{20}\mathbb{G}_j\right)$ | 544 |
| $\frac{d\mathbb{T}_{21}}{dt} = -\left((b'_{21})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T^*_{21}\mathbb{G}_j\right)$ | 545 |
| $\frac{d\mathbb{T}_{22}}{dt} = -\left((b'_{22})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T^*_{22}\mathbb{G}_j\right)$ | 546 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(4)}$ *and* $(b''_i)^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 547 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- | 548 |

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| $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br>$\frac{\partial (a_{25}^{''})^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \ , \frac{\partial (b_i^{''})^{(4)}}{\partial G_j}((G_{27})^* \ ) = s_{ij}$ | |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}^{'})^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$ | 549 |
| $\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}^{'})^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$ | 550 |
| $\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}^{'})^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$ | 551 |
| $\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}^{'})^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$ | 552 |
| $\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}^{'})^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$ | 553 |
| $\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}^{'})^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$ | 554 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(5)}$ *and* $(b_i^{''})^{(5)}$ Belong to $C^{(5)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 555 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{29}^{''})^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \ , \frac{\partial (b_i^{''})^{(5)}}{\partial G_j}(\ (G_{31})^* \ ) = s_{ij}$ | 556 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}^{'})^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$ | 557 |
| $\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}^{'})^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$ | 558 |
| $\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}^{'})^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$ | 559 |
| $\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}^{'})^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$ | 560 |
| $\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}^{'})^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$ | 561 |
| $\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}^{'})^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$ | 562 |

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| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(6)}$ *and* $(b_i^{''})^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 563 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{33}^{''})^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)}$ , $\frac{\partial (b_i^{''})^{(6)}}{\partial G_j}((G_{35})^*) = s_{ij}$ | 564 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}^{'})^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$ | 565 |
| $\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}^{'})^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$ | 566 |
| $\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}^{'})^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$ | 567 |
| $\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}^{'})^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$ | 568 |
| $\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}^{'})^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$ | 569 |
| $\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}^{'})^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$ | 570 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(7)}$ *and* $(b_i^{''})^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | 571 |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial (a_{37}^{''})^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)}$ , $\frac{\partial (b_i^{''})^{(7)}}{\partial G_j}\left((G_{39})^{**}\right) = s_{ij}$ | 572 |
| Then taking into account equations and neglecting the terms of power 2, we obtain from | |
| $\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}^{'})^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$ | 573 |
| $\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}^{'})^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$ | 574 |

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| $\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ | 575 |
| $\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ | 576 |
| $\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ | 578 |
| $\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ | 579 |
| Obviously, these values represent an equilibrium solution<br><br>**ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(8)}$ *and* $(b_i^{''})^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.<br><br>Proof: Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-<br><br>$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$<br><br>$\frac{\partial(a_{41}^{''})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)} \quad , \frac{\partial(b_i^{''})^{(8)}}{\partial G_j}\left((G_{43})^*\right) = s_{ij}$ | 580 |
| Then taking into account equations and neglecting the terms of power 2, we obtain | |
| $\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ | 581 |
| $\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ | 582 |
| $\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ | 583 |
| $\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ | 584 |
| $\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ | 585 |
| $\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ | 586 |
| **ASYMPTOTIC STABILITY ANALYSIS**<br><br>**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a_i^{''})^{(9)}$ *and* $(b_i^{''})^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable. | 586 A |

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| **Proof:** Denote | |
| **Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- $G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$ $\frac{\partial (a_{45}^{''})^{(9)}}{\partial T_{45}}(T_{45}^*) = (q_{45})^{(9)} \quad , \frac{\partial (b_i^{''})^{(9)}}{\partial G_j}((G_{47})^*) = s_{ij}$ | |
| Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44 | |
| $\frac{d\mathbb{G}_{44}}{dt} = -\left((a_{44}^{'})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G_{44}^*\mathbb{T}_{45}$ | 586 B |
| $\frac{d\mathbb{G}_{45}}{dt} = -\left((a_{45}^{'})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G_{45}^*\mathbb{T}_{45}$ | 586 C |
| $\frac{d\mathbb{G}_{46}}{dt} = -\left((a_{46}^{'})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G_{46}^*\mathbb{T}_{45}$ | 586 D |
| $\frac{d\mathbb{T}_{44}}{dt} = -\left((b_{44}^{'})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T_{44}^*\mathbb{G}_j\right)$ | 586 E |
| $\frac{d\mathbb{T}_{45}}{dt} = -\left((b_{45}^{'})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T_{45}^*\mathbb{G}_j\right)$ | 586 F |
| $\frac{d\mathbb{T}_{46}}{dt} = -\left((b_{46}^{'})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T_{46}^*\mathbb{G}_j\right)$ | 586 G |
| **The characteristic equation of this system is** | 587 |
| $\left((\lambda)^{(1)} + (b_{15}^{'})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a_{15}^{'})^{(1)} + (p_{15})^{(1)}\right)$ $\left[\left(\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(q_{13})^{(1)}G_{13}^*\right)\right]$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T_{14}^* + (b_{14})^{(1)}s_{(13),(14)}T_{14}^*\right)$ $+\left(\left((\lambda)^{(1)} + (a_{14}^{'})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G_{13}^* + (a_{13})^{(1)}(q_{14})^{(1)}G_{14}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T_{14}^* + (b_{14})^{(1)}s_{(13),(13)}T_{13}^*\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $\left(\left((\lambda)^{(1)}\right)^2 + \left((b_{13}^{'})^{(1)} + (b_{14}^{'})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$ $+\left(\left((\lambda)^{(1)}\right)^2 + \left((a_{13}^{'})^{(1)} + (a_{14}^{'})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$ $+\left((\lambda)^{(1)} + (a_{13}^{'})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G_{14}^* + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G_{13}^*\right)$ $\left(\left((\lambda)^{(1)} + (b_{13}^{'})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T_{14}^* + (b_{14})^{(1)}s_{(13),(15)}T_{13}^*\right)\} = 0$ $+$ | |

$$(\lambda)^{(2)} + (b_{18}')^{(2)} - (r_{18})^{(2)})\{((\lambda)^{(2)} + (a_{18}')^{(2)} + (p_{18})^{(2)})$$

$$\left[\left((\lambda)^{(2)} + (a_{16}')^{(2)} + (p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*} + (a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$$

$$\left((\lambda)^{(2)} + (b_{16}')^{(2)} - (r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*} + (b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\Big)$$

$$+\left((\lambda)^{(2)} + (a_{17}')^{(2)} + (p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*} + (a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\Big)$$

$$\left((\lambda)^{(2)} + (b_{16}')^{(2)} - (r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*} + (b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((a_{16}')^{(2)} + (a_{17}')^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$\left((\lambda)^{(2)}\right)^{2} + \left((b_{16}')^{(2)} + (b_{17}')^{(2)} - (r_{16})^{(2)} + (r_{17})^{(2)}\right)(\lambda)^{(2)}\Big)$$

$$+\left((\lambda)^{(2)}\right)^{2} + \left((a_{16}')^{(2)} + (a_{17}')^{(2)} + (p_{16})^{(2)} + (p_{17})^{(2)}\right)(\lambda)^{(2)}\Big)(q_{18})^{(2)}\mathrm{G}_{18}$$

$$+\left((\lambda)^{(2)} + (a_{16}')^{(2)} + (p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*} + (a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$$

$$\left((\lambda)^{(2)} + (b_{16}')^{(2)} - (r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*} + (b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\Big)\} = 0$$

$$+$$

$$(\lambda)^{(3)} + (b_{22}')^{(3)} - (r_{22})^{(3)})\{((\lambda)^{(3)} + (a_{22}')^{(3)} + (p_{22})^{(3)})$$

$$\left[\left((\lambda)^{(3)} + (a_{20}')^{(3)} + (p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*} + (a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$$

$$\left((\lambda)^{(3)} + (b_{20}')^{(3)} - (r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*} + (b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\Big)$$

$$+\left((\lambda)^{(3)} + (a_{21}')^{(3)} + (p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*} + (a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\Big)$$

$$\left((\lambda)^{(3)} + (b_{20}')^{(3)} - (r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*} + (b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((a_{20}')^{(3)} + (a_{21}')^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$\left((\lambda)^{(3)}\right)^{2} + \left((b_{20}')^{(3)} + (b_{21}')^{(3)} - (r_{20})^{(3)} + (r_{21})^{(3)}\right)(\lambda)^{(3)}\Big)$$

$$+\left((\lambda)^{(3)}\right)^{2} + \left((a_{20}')^{(3)} + (a_{21}')^{(3)} + (p_{20})^{(3)} + (p_{21})^{(3)}\right)(\lambda)^{(3)}\Big)(q_{22})^{(3)}G_{22}$$

$$+\left((\lambda)^{(3)} + (a_{20}')^{(3)} + (p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*} + (a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$$

$$\left((\lambda)^{(3)} + (b_{20}')^{(3)} - (r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*} + (b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\Big)\} = 0$$

$$+$$

$$\left((\lambda)^{(4)}+(b_{26}^{'})^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}^{'})^{(4)}+(p_{26})^{(4)}\right)$$

$$\left[\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\Big]$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\Big)$$

$$+\left((\lambda)^{(4)}+(a_{25}^{'})^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\Big)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}^{'})^{(4)}+(b_{25}^{'})^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\Big)$$

$$+\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}^{'})^{(4)}+(a_{25}^{'})^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\Big)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}^{'})^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left((\lambda)^{(4)}+(b_{24}^{'})^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(5)}+(b_{30}^{'})^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}^{'})^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\Big]$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\Big)$$

$$+\left((\lambda)^{(5)}+(a_{29}^{'})^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\Big)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}^{'})^{(5)}+(b_{29}^{'})^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\Big)$$

$$+\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}^{'})^{(5)}+(a_{29}^{'})^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\Big)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}^{'})^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left((\lambda)^{(5)}+(b_{28}^{'})^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(6)}+(b'_{34})^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a'_{34})^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)\Big]$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(33)}T^{*}_{33}\Big)$$

$$+\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^{*}_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^{*}_{33}\Big)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(32)}T^{*}_{32}\Big)$$

$$\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$\left((\lambda)^{(6)}\right)^{2}+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\Big)$$

$$+\left((\lambda)^{(6)}\right)^{2}+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\Big)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^{*}_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^{*}_{32}\right)$$

$$\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^{*}_{33}+(b_{33})^{(6)}s_{(32),(34)}T^{*}_{32}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)\Big]$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(37)}T^{*}_{37}\Big)$$

$$+\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^{*}_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^{*}_{37}\Big)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(36)}T^{*}_{36}\Big)$$

$$\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$\left((\lambda)^{(7)}\right)^{2}+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\Big)$$

$$+\left((\lambda)^{(7)}\right)^{2}+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\Big)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^{*}_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^{*}_{36}\right)$$

$$\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^{*}_{37}+(b_{37})^{(7)}s_{(36),(38)}T^{*}_{36}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)\Big]$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(41)}T^{*}_{41}\Big)$$

$$+\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^{*}_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^{*}_{41}\Big)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(40)}T^{*}_{40}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$\left((\lambda)^{(8)}\right)^{2}+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\Big)$$

$$+\left((\lambda)^{(8)}\right)^{2}+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\Big)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^{*}_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^{*}_{40}\right)$$

$$\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^{*}_{41}+(b_{41})^{(8)}s_{(40),(42)}T^{*}_{40}\Big)\}=0$$

$$+$$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)\Big]$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(45)}T^{*}_{45}\Big)$$

$$+\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^{*}_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^{*}_{45}\Big)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(44)}T^{*}_{44}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$\left((\lambda)^{(9)}\right)^{2}+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\Big)$$

$$+\left((\lambda)^{(9)}\right)^{2}+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\Big)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^{*}_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^{*}_{44}\right)$$

$$\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^{*}_{45}+(b_{45})^{(9)}s_{(44),(46)}T^{*}_{44}\Big)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and**

| | |
|---|---|
| **this proves the theorem.** | |
| Note: **Same dovetailing explanation holds good for d/dt (partial differential with respect to t), d/dt,d^2/dt^2 (acceralation: double dot). Such an exposition is helpful in optimisation problems, duality of motion, wave motion a homogeneous functions studies, and control theory which we intend to incorporate in future**. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk. | |

## Persistence, Coexistence And Thrusty Permanence For Interacting Structured Particles, Transactions In Finance, Econometric Productivity Functions, Environment, Interfaces, Cosmological Interactions Et Al

## Josef Hofbauer And Sebastian J. Schreiber Model Thereof: A Functional Appraisal

ABSTRACT. **JOSEF HOFBAUER AND SEBASTIAN J. SCHREIBER** studied the dynamics of interacting structured systems (in their case population dynamics) can be modeled by $\frac{dxi}{dt} = A_i\ (x)\ xi$ where $x_i\ \epsilon\ R^n_{i,}\ x = (x_1, \ldots, x_k)$, and $A_i(x)$ are matrices with non-negative off- diagonal entries. These models are permanent if there exists a positive global attractor and are robustly permanent if they remain permanent following perturbations of $A_i\ (x)$. Necessary and sufficient conditions for robust permanence are derived using dominant Lyapunov exponents $X_i\ (\mu)$ of the $A_i\ (x)$ with respect to invariant measures $\mu$. The necessary conditions requires $\max_i\ X_i(\mu)>0$ for all ergodic measures with support in the boundary of the non-negative cone. The sufficient condition requires that the boundary admits a Morse decomposition such that $\max_i$ Xi$(\mu)>0$ for all invariant measures $\mu$ supported by a component of the Morse decomposition. When the Morse components are Axiom A, uniquely ergodic, or support all but one population, the necessary and sufficient conditions are equivalent. Applications to spatial ecology, epidemiology, and gene networks are given. Appeared in *Journal of Differential Equations*, 248, 1955-1971 (2010) It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and

corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## INTRODUCTION

A fundamental issue in population biology is what are the minimal conditions to ensure the long-term survivorship for interacting populations whether they be viral particles, bio-chemicals, plants, or animals. When these conditions are met the interacting populations are said to persist or coexist. Since the pioneering work of Lotka and Volterra on competitive and predator prey interactions, Thompson, Nicholson, and Bailey on host-parasite interactions, and Kermack and McKendrick on disease outbreaks, nonlinear difference and differential equations have been used to understand conditions for population persistence [1-5]. One particularly important form of persistence for deterministic models is *permanence* or *uniform persistence* which corresponds to the existence of a global attractor bounded away from extinction of one or more species. When such an attractor exists, interacting populations are able to recover from "vigorous shake ups" of the of the population state. Permanence has been characterized from a topological perspective [6-8] and with average Lyapunov functions [9-12] for differential equation models of the form

$$\frac{dxi}{dt} = x_i f_i(x) \qquad i = 1, \ldots, k \qquad (12)$$

Where $x_i$ is the abundance of population $i$, $x = (x_1, \ldots, x_k)$, and $f_i(x)$ is the per-capita growth of population $i$.

An important extension of the concept of the permanence is *robust permanence* [13-16] which requires that (1) remain permanent after sufficiently small perturbations of the per-capita growth rates $f_i$. The importance of robust permanence as one crude form of structural stability for population models are approximations to reality. Consequently, if nearby models (e.g. more realistic models) are not permanent despite the focal model being permanent, and then one can draw few conclusions about the persistence of the biological system being approximated by the model. One can view robust permanence as one crude form of structural stability for debits-credits particles, propositions, transactions in the celestial subterranean realm and ceratoid dualism, finance like debiting say a current account and crediting to another account, or recrediting to the account which had been earlier debited, further conduction of the cash credit account, black hole formation, guzzling of inventories for the output,

creation of income from output, reinvestment of income to investment,gormadizing of matter by black holes, and emission of radiation therefore, gamma ray bursts, in fact most of the transactions and interactions in the universe are conservative and quantum mechanical universe, something like a account acting as predator and prey. More often than not as we have seen from Haimovici model such models could be applied to the problems, propositions, which are not subsumed by the concomitant problems in the bearing. the population models. For this perspective, it is not unexpected that there are permanent systems that cannot be approximated by robustly permanent systems [17]. In [14, 16], criteria for robust permanence were developed with respect to the average per-capita growth rates $\int f_i(x)d\mu(x)$ with respect to invariant probability measures $\mu$ supported on the boundary of the positive orthant (i.e. where one or more Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates are extinct). Roughly, these criteria for robust permanence require that the average per-capita growth rate (i.e. invasion rate) is positive for some missing species.

While (1) can account for many types of Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates. Interactions, it assumes that all individual within populations are exactly the same. However, theoretical biologists have long recognized that different individuals within a population may be in different states (e.g. different sizes or ages, living in different spatial locations) and these differences can have important consequences for population dynamics [18-24]. To account for how these differences influence persistence of interacting populations, we develop criteria for robust permanence for the dynamics of $k$ interacting structured populations. Roughly, these dynamics correspond to replacing $x_i$ in (1) by vectors and $f_i(x)$ by matrices. In section 2, we describe these models in greater detail and review some basic concepts from dynamical systems theory. In section 3, we introduce the structured analog of average per-capita growth rates and define the concept of robustly unsaturated invariant sets. Necessary and sufficient conditions for an invariant set to be robustly unsaturated are proven. For Axiom A or uniquely ergodic invariant sets, the necessary and sufficient criteria are shown to be equivalent. In section 4, we use Morse decompositions and the criteria for robustly unsaturated invariant sets to develop necessary and sufficient conditions for robust permanence. In sections 5 and 6, we provide several applications of our results to spatially structured ecological models, structured epidemiological models, and models of gene networks. We apply the model to econometrics. To give an example investment needs inventory to be used towards the consummation of output, part of output is to be sold for consolidation of income, and some income might be replenished in the investment and so

on. In finance a cash credit account could be debited or credited when there is say purchase of demand bills or discount of usance bills. Cash credit account is debited whenever cheques are issued towards the purchase of raw materials and so on. In fact some of the models have been applied to Organic psychiatry and we have shown how such models are useful in dealing with the propositions of Husserl, Derrida, Deleuze and others.

## 12. MODELS AND ASSUMPTIONS

Let $x_i$ denote the state of the $i$-th population that lies in the non-negative cone $C_i$ of $R^{ni.}$ Define $x=(x_1,\ldots x_k)\epsilon C$ to be the non-negative cone of $R^n$ where $n=\sum_{i=1}^{k} n_i$. Let $C_i^+$ denote positive cone { $x_i \epsilon C_i$: $\prod_i$i noitalupop rof$^j$

$i$ *and* $C^+=\prod_i C^+_i$ , the positive cone for the interacting populations. If $x \mapsto A_i\,(x)$ is a map into $n_i x n_i$ matrices that describes the growth of population I, then the dynamics of the interacting populations are given by

$$(2)\frac{dx_i}{dt} = A_i(x)x_i \quad i=1,\ldots,k$$

Let $x.t$ denote the solution of (2), with the initial condition x, and more generally the semi flow generated by (2). Before stating our assumption on (2), recall a few definitions from dynamical systems. Given sets $I \subseteq R$ and $K \subseteq C$, let $K.I = \{x.t : t \epsilon I, x \epsilon K\}$. A set $K \subseteq C$ *is invariant* if K.t = K for all t>0, and *forward invariant* if $K.t \subset K$ for all $t > 0$. The *omega limit* set of a set $K \subseteq C$ equals $\omega(K) = \bigcap_{t\geq 0,} \overline{K.[t,\infty)}$. The *alpha limit* set of $K \subseteq C$ is $a(K) = \bigcap_{t\leq 0} \overline{K.(-\infty,t]}$. Given a forward invariant set K, $B \subset K$ is an attractor for the semi flow $x.t$ *restricted* to K provided there exists an open neighborhood $U \subseteq K$ of B such that $\omega(U) = B$. The stable set of a *compact invariant* set $K$ is defined by

$$W^s(K) = \{x \epsilon C : \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq K\}.$$

The semi flow generated by (2) is *dissipative* if there exists an attractor B with $W^s(B) = C$.

Throughout this paper, we make the following assumptions:

**A1**: $x \mapsto A_i\,(x)$ is continuous

**A2:** $A_i\,(x)$ is irreducible and has non-negative off- diagonal entries,

**A3:** $x.t$ is defined for all $t \geq 0$

**A4:** $x.t$ is dissipative with global attractor $\Gamma\,(A)$.

A1 is a basic regularity assumption. The non-negative of off- diagonal entries in A2 implies that there are no negative feedbacks between individuals of different states in population *i*. This assumption is meet for many types of structured models as discussed in sections 5 and 6. The irreducibility assumption A2 is generally meet and implies that all individuals within a population can pass through all states. In Remark1, we discuss how this irreducibility assumption can be relaxed. A3 ensures that the population dynamics are defined for all future time. A4 requires that population densities/ abundances eventually are uniformly bounded, a condition that should be meet for an biologically realistic model.

We say that the semi flow of (2) is *permanent* if this semi flow is dissipative and there is a positive attractor $B \subset C^+$ such that $W^s(B) \subset C^+$. We will denote this positive attractor by $\Lambda(A)$. Permanence ensures that populations can recover from rare large perturbations and allows for a diversity of dynamical behaviors.

For K ⊂ C and $\delta > 0$, define the $\delta$- *neighborhood* of K as

$$N\delta(K) = \{x \in C : \backslash x - y < \delta \text{ for some } y \in K\}.$$

We define a $\delta$- *perturbation of (2)* to be a system of the form

$$\frac{dx}{dt} = \tilde{A}(x)x$$

that satisfies assumptions A1-A4, $\Gamma(\tilde{A}) \subset N$ *( Γ(A)), and* $||A(x) - \tilde{A}(x)|| \leq \delta$ *for all* $x \in N$ *(Γ(A)).* (2) is *robustly permanent* if there exist >0 and $\epsilon > 0$ such that all $\delta$- perturbations of (2) are permanent, and $d(\Lambda(\tilde{A}), C\backslash C^+) > \epsilon$ for all $\delta$- perturbations i.e., there is a common/ uniform region of repulsion around the boundary.

## INUNDATION OF COMPACT SETS

We begin by studying the linear skew product flows on $\Gamma(A) \times R^n_i$ defined by $(x,y).t = (x.t, B_i(t,x)y)$ where $Y(t) = B_i(t,x)$ is the solution to $\frac{dY}{dt}(t) = A_i(x.t)Y(t)$ with y(0) equal to the identify matrix. Our assumption that $A_i$ is irreducible with non- negative off diagonal entries implies that $B_i$ (t,x)$C_i \subset C_i^+$ for all $x$ and t > 0 (see, e.g., [25] ). A result of Ruelle [26, Prop, 3.2] implies that there exist continuous maps $u_i\, v_i : \Gamma(A) \rightarrow C_i^+$ with $u_i(x)| = |v_i(x)| = 1$ such that The line bundle $E_i(x)$ spanned by $u_i(x)$ is invariant i.e. $E_i(x.t) = B_i(x,t)E_i(x)$ for all t ≥ 0.The vector bundle $F_i(x)$ perpendicular to $v_i(x)$ is invariant i.e. $F_i(x.t) = B_i(x,t)F_i(x)$ for all $t \geq 0$.There exist constants $a > 0$ *and* $\beta > 0$ such that(3) $||B_i(t,x)F_i(x)|| \leq a\, exp(-\beta t)||B_i(t,x)E_i(x)||$ For all $x \in K$ *and* $t \geq 0$.**Interaction rates.** Given any x $\in \Gamma$(A), define the *interaction and inundation rate* of *species i at particle/proposition/problem/ state x as* $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ *in*

$||B_i(t,x)||$ when $x_i = 0$, $\lambda_i(x)$ provides an upper bound to the rate of growth of population $i$ when introduced at infinitesimally small densities. Important properties of this invasion rate are summarized in the following two propositions. For instance, the second proposition implies if $(x.t)_i$ stays bounded away from zero (i.e. population $i$ persist), then

$\lambda_i(x)=0$. Alternatively, if $\lambda_i(x)<0$,then population $i$ is doomed to extinction. **Proposition 1.** *$\lambda_i(x)$ satisfies the following properties:* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) In \ |B_i \ (t,x)v|$ *for any* $v>0$ *in* $R^{ni}$, *and* $\lambda_i(x)= limsup_{t\to\infty}\left(\frac{1}{t}\right) \int^t_0 (A_i(x.s)u_i(x.s),u_i(x.s))ds.$

**Proof:** To prove the first property, we first show that

$$\lambda i(x) = limsup_{t\to\infty} \frac{1}{t} In\ |B_i(t,x)u_i(x)|.$$

To this end, let $v \in R^{ni}$ be any non-zero vector. Since $R^{ni}= E_{i(x)} \ F_i(x)$, we can write $v=au_i(x)+w$ with $a \in R$ and $w \in F_i(x)$. Equation (3) implies

$|B_i(t,x)v| \ \leq \ a| \ B_i(t,x)u_i(x)|+ \ |B_i(t,x)w| \ \leq \ | \ B_i(t,x)u_i(x)|(a+ \alpha+ exp\ (-\beta t)| \ w|)$. Hence, $limsup_{t\to\infty} \frac{1}{t} In\ B_i(t,x)\ v| \leq$

$$limsup_{t\to\infty} \frac{1}{t} In\ B_i\ (t,x)\ u_i\ (x) \leq \lambda_i(x)$$

for all non zero vectors $v \in R^{ni}$. Since $||B_i(t,x)|| = sup|v|=1B_i(t,x)v|$, this inequality implies that $\lambda i(x) = \limsup_{t\to\infty} \frac{1}{t}$ In $|B_i(t,x)v|$, this inequality implies that $\lambda_i(x) = limsup_{t\to\infty}\left(\frac{1}{t}\right)$ In $|t,x)u_i(x)|$. Now let v $\in C_i^+$. Then, we can write $v = au_i(x)+ w$ with $a >0$ and $w \in F_i$.

Equation (3) implies

$$|B_i(t,x)v| \ \geq \ a|B_i(t,x)u_i(x)|-|B_i(t,x)w|$$

$$\geq | \ B_i(t,x)u_i(x)|(a- \alpha+ exp\ (-\beta t)| \ w|).$$

Since a>0, this inequality implies that

$$\lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)v| \ \geq \ \lambda i(x) \geq \limsup_{t\to\infty} \frac{1}{t} \text{In } |B_i(t,x)u_i(x)| = \lambda i(x)$$

To prove the second assertion, let $b_i(t,x) = In\ |B_i(t,x)u_i(x)|$. Invariance of $u_i(x)$ implies that $b_i(t,x)$ is additive:

$$B_i(t+s,x) = In|B_i(t+s,x)\, u_i(x)| = In\, |B_i(s,x.t)B_i(t,x)u_i(x)$$

$$= \text{In } |B_i(s,x.t)\, u_i(x.t)||\, B_i(t,x)\, u_i(x)|$$

$$= \text{In } b_i(s,t.x)\, b_i(t,x)$$

Additivity of $b_i(t,x)$ and the fact that $b_i(0,x)$ =0 implies

$$\frac{d}{dt}b_i(t,x) = \lim_{s\to 0}\frac{bi(t+s,x-bi(t,x)}{S-}$$

$$= \lim_{s\to 0}\frac{bi(s,x.t)}{S} = \frac{d}{ds}|_{s=0}\, b_i(s,x.t)$$

=

$$=(A_i(x.t)u_i(x.s),u_i(x.s),$$

The Fundamental Theorem of Calculus implies

$$B_i(t,x)= \int_0^t (A_i(x.s)\, u_i(x.s),\, u_i(x.s))ds\backslash$$

and the second assertion follows.

**Proposition 2:** *For the solutions of (2) we have:* if $x_i>0$, then $\lambda i(x) \le 0$. If $\lambda i(x)<0$, then $\lim_{t\to\infty} (x.t)_i=0$ If $limsup_{t\to\infty}$ $(x.t)_i|>0$, then $\lambda i(x)=0$.

**Proof:** First, assume that x satisfies $x_i>0$. Since the semi flow is dissipative, there exists $\gamma >0$ such that $|x.t| \le \gamma$ for all $t \ge 0$. Proposition 1 and the definition of the skew product flow imply

$$\lambda i(x) = \limsup_{t\to\infty}\frac{1}{t}\,\text{In } |B_i(t,x)x_i| = \limsup_{t\to\infty}\frac{1}{t}\,\text{In } |\,(x.t)_i\,|$$

$$\le \limsup_{t\to\infty}\frac{In\,\gamma}{t} = 0$$

To prove the second assertion, assume that $\lambda i(x) < 0. If\ x_i=0$, then the invariance of the faces of C imply that $(x.t)_i=0$ for all $t\ge 0$. Alternatively, if $x_i> 0$, then Proposition I and the skew product flow imply

$$\limsup_{t\to\infty}\frac{1}{t}In|(x.t)_i| = \lambda i(x) < 0.$$

The final assertion follows from the first two assertions. 5

**3.2. Invariant measures.** We review some definitions from ergodic theory. Given a Borel probability measure $\mu$ on $\boldsymbol{C}$, the *support* of $\mu$, denoted supp( $\mu$),is the smallest closed set whose complement has measure zero. A Borel probability measure $\mu$ is called *invariant* for (2) provided that Integral $h(x)d\mu(x)$ = Integral $h(x.t)d\,\mu(x)$for all $t \geq 0$ and for all bounded continuous functions $h$:C→R. An invariant measure $\mu$ is called *ergodic* provided that $\mu(B)$=0 or 1 for any invariant Borel set *B.*

For an invariant measure $\mu$ define the invasion rate species $i$ with respect to $\mu$ as

$$\lambda_i(\mu) = \int_c \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle d\mu(x)$$

**Proposition 3:** Let $\mu$ be an *invariant measure.* $\lambda_i(\mu)$ *Satisfies the following properties:*$\lambda_i(\mathrm{x}) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle \mathrm{A}_i(x.s)\mathrm{u}_i(x.s), \mathrm{u}_i(x.s)\rangle, ds$ exists for $\mu$ - *almost every* $x$. *Moreover, if* $\mu$ *is ergodic,* $\lambda_i(\mathrm{x}) = \lambda_i(\mu)$ for $\mu$-*almost every* $x$.*If* $\mu$ *is ergodic, then there exists* $I \subset \{1, \dots, k\}$such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1 and $\lambda_i(\mu) = 0\ for\ all\ \ i \in I$.

**Proof:** Let $\mu$ be an invariant measure. Define $\mathrm{h}_i(x) = \langle \mathrm{A}_i(x)\mathrm{u}_i(x), \mathrm{u}_i(x)\rangle$. The Birkhoff's' ergodic theorem and Proposition 1 imply

$$\lambda_i(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{h}_i(x.s)\mathrm{ds}$$

Exists for $\mu$- almost every $x$. Moreover, $\lambda_i(x) = \lambda_i(\mu)$ $\mu$- almost surely if $\mu$ is ergodic.

Assume $\mu$ is ergodic. By ergodicity and invariance of the faces of **C**, there exists $I \subset \{1, \dots, k\}$ such that $\mu$ ( $\Pi_{i\in I}\boldsymbol{C}_i^+$)=1. Let K $\subset \Pi_{i\in I}\boldsymbol{C}_i^+$ be a compact set such that $\mu(K) > 0$. The Poincare recurrence theorem and the Birkhoff ergodic theorem imply that there is $x \in \Pi_{i\in I}\boldsymbol{C}_i^+$such that $x.\mathrm{t}_n \in K$ for some $\mathrm{t}_n \uparrow \infty$ and $\lambda_i(x) = \lambda_i(\mu)$ for all $i$. The third assertion of Proposition 2 implies $\lambda_i(\mu) = 0$ for all $i \in I$.

**3.3. Robustly unsaturated sets**. Let K $\subset \partial C$ be a compact isolated invariant set for the flow $x.t$ restricted to $\partial C$.*K* is unsaturated if Ws (K) $\subset \partial C$ and K is isolated for $x.t$. If *K* is not unsaturated, then *K* is saturated. *K* is robustly unsaturated for (2) if there exists $\delta > 0$ such that the continuation of K for any $\delta -$ perturbation of (2) is unsaturated.**Theorem 1:** Let *K* be a compact isolated invariant set for $x.t$ restricted to$\partial C$. If one of the following

equivalent conditions hold For all invariant measures $\mu$ supported by $K$ (4) $\max_{1\le i\le k} \lambda_i(\mu) = 0$

There exist $p_i > 0$ such that (5) $\sum_{1\le i\le k} p_i \lambda_i(\mu) = 0$

For all ergodic probability measures supported by $K$.

then $K$ is robustly unsaturated for (2). Alternatively, if $x \to A(x)$ is twice continuously differentiable and $K$ is robustly unsaturated, then (4) holds for all ergodic measures $\mu$ supported by $K$.

**Remark 1**: For some applications (e.g., the disease model considered in section 6.1) it useful to relax the irreducibility assumption A2. For instance, if there exists an open neighborhood U of K such that for each $i$, $A_i(x)$ has a fixed off-diagonal sign pattern for all $x \in U$,then $A_i(x)$ can be decomposed into a finite number, say $m_i$, of irreducible components. For each of these irreducible components, one can define $\lambda_i^j(\mu) = \int_C \langle A_i^j(x) u_i^j(x), u_i^j(x)\rangle \, d\mu(x)$where $A_i^j(x)$ is the submatrix of $A_i(x)$ corresponding to the $j$-th irreducible component of $A_i(x)$ $and$ $u_i^j(x)$ is the continuous invariant sub bundle for the irreducible linear cocycle determined by $A_i^j(x)$. If we define $\lambda_i(\mu) = \max_{1\le j\le m_i} \lambda_i^j(\mu)$, then all of the assertions of Theorem 1 still hold.

**Proof:** To prove the first assertion of the theorem, we need the following lemma. We call an invariant measure $\mu$ for (2) saturated if $\lambda_i(\mu) \le 0$ for all $i$

**Lemma 1**: Let $\delta_n$ be a non –negative sequence that converges to zero as n$\to\infty$. If $\mu_n$ are saturated invariant measures for $\delta_n -$ perturbations of (2), then the weak$^*$ limit points of $\{\mu_n\}_{n=1}^{\infty}$ is a non-empty set consisting of saturated invariant measures for (2).

**Proof:** Let $\delta_n$ be a non-negative sequence that coverages to zero. Let $\mu_n$ be saturated invariant measures for $\delta_n$ perturbations, $\dot{x} = A^n(x)x$, of $\dot{x} = A(x)x$. By weak$^*$ compactness of Borel probability measures supported on $N_1(\Gamma(A))$, there exist weak$^*$ limit points of $\mu_n$. Let $\mu$ be such a weak$^*$ limit point. Since $\mu_n$ are supported by $N_1(\Gamma(A))$, for all $n$ sufficiently large, $\mu$ is supported by $N_1(\Gamma(A))$.To verify that $\mu$ is an invariant measure for (2),let $h: C \to R$ be a bounded continuous function. Let $t > 0$ and $\in > 0$ be given. Choose $n$ sufficiently large

so that $|\int_c h(x)\,d\mu(x) - \int_c h(x)d\mu_n(x)| \leq \in$, $|\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)d\mu_n(x)| \leq \in$, and $|h(x.t) - h(x^n.t)| \leq \in$ for all $x \in N_1(\Gamma(A))$.Then $|\int_c h(x.t) - h(x)d\mu(x)|$ is

$$\leq \quad |\int_c h(x.t)\,d\mu(x) - \int_c h(x.t)\,d\mu_n(x)| + |\int_c h(x.t)\,d\mu_n(x) - \int_c h(x)\,d\mu(x)|$$

$$\leq \quad \in + \int_c |h(x.t) - h(x^n.t)|\,d\mu_n(x)| + |\int_c h(x^n.t)|\,d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq \quad 2 \in + |\int_c h(x^n.t) - h(x)\,d\mu_n(x)| + |\int_c h(x)d\mu_n(x) - \int_c h(x)d\mu(x)|$$

$$\leq \quad 3 \in$$

Where the last line follows from the invariance of $\mu_n$ for $x^n.t$. Since $\in > 0$ and $t > 0$ are arbitrary, we have $\int_c h(x.t)d\mu(x) = \int_c h(x)d\mu(x)$ for all $t > 0$ and all bounded continuous functions $h: C \to R$. It follows that $\mu$ is an invariant measure for (2). To see that $\mu$ is saturated, define $h_i^n(x) = \langle A_i^n(x)u_i^n(x), u_i^n(x)\rangle$ where $u_i^n(x)$ spans the invariant one-dimensional bundle given by $[26, Prop.3.2]$ for $\frac{dx}{dt} = A^n(x)x$. Since $A_i^n(x) \to A_i(x)$ uniformly for $x \in N_1((A))$ as $n \to \infty$, $[26, Prop.3.2]$ implies that $u_i^n(x) \to u_i(x)$ converges uniformly for $x \in N_1(\Lambda(A))$. Given $\in > 0$, choose $n$ sufficiently large so that $|h_i^n(x) - h_i(x)| \leq \in$ for all $x \in N_1(\Lambda(A))$ and $|\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)| \leq \in$.

Then

$$\lambda_i(\mu) \leq \int_c |h_i(x) - h_i^n(x)|d\mu(x) + |\int_c h_i^n(x)\,d\mu(x) - \int_c h_i^n(x)\,d\mu_n(x)|$$

$$+ \int_c h_i^n(x)\,d\mu_n(x)$$

$$\leq 2\in$$

Since $\in > 0$ is arbitrary, $\lambda_i(\mu) \leq 0$ for all $i$ and $\mu$ is saturated.

Using this lemma, we prove that if K is saturated, then there exists a saturated invariant measure $\mu$ supported by K. Assume K is saturated. Work of Hofbaue and So $[7, Thm.2.1]$ implies that either there exists $y \in W^s(K) \cap C^+$ or K is not isolated for the unrestricted flow $x.t$ If there exists $y \in W^s(K) \cap C^+$, then for all $t \leq 0$ define $v_t = \frac{1}{t}\int_0^t \delta_{y.s}\,ds$ where $\delta_{y.s}$ denotes a Dirac measure based at the point $y.s$. Dissipativeness of (2) and weak*

compactness of the Borel probability measures supported by (A) imply there exist $t_k \to \infty$ such that the sequence $v_{tk}$ converges in the weak* topology to a Borel probability measure $\mu$ with support in K.A standard argument implies that $\mu$ is $x.t$ invariant. Define $h_i: C \to R$ by $h_i(x) = (A_i(x)u_i(x), u_i(x))$.Proposition 2 and weak* convergence imply that

$$0 \geq (\lambda i(y) = \limsup_{t\to\infty} \frac{I}{t}\int_0^t h_i(y.s)ds$$

$$= \limsup_{t\to\infty} \int_c h_i \, dv_t$$

$$\geq \limsup_{t\to\infty} \int_p h_i \, d\, v_{tk} = \lambda i(\mu)$$

For all $i$. Alternatively suppose that K is not isolated for the semi flow Then there exists a sequence of positive $\omega - limit$ sets that accumulate on K. Let $\mu_n$ be a sequence of ergodic probability measures supported by these $\omega -$ limit sets. Propositions 1 and 3 imply that $\mu_n$ are saturated for all $n$ Applying Lemma I with $\delta_n$=0 for all $n$ implies that there exists a saturated invariant measure $\mu$ suppoorted by *K*. Hence we have shown that if *K* is saturated then there exists a saturated invariant probability measure $\mu$ supported by *K* Equivalently, (4) holding for all invariant probability measures with support in *K* implies that *K* is unsaturated.

Next we show that if *K* not robustly saturated, then there exists a saturated invariant measure supported by *K* Indeed suppose *K* is not robustly saturated, Then there exists a non-negative sequence $\delta_n$ converging to zero and a sequence of saturated measures $\mu_n$ for $\delta_n$-perturbations of (2)with support in the continuation of *K*. Let $\mu$ be a weak* limit point of $\{\mu_n\}\infty_{n=1}$ Lemma 1 implies that $\mu$ is saturated. Moreover, since the continuation of *K* converges to *K* as $\delta_n \to 0, \mu$ is a saturated invariant measure for (2) supported by *K* Hence we have shown that if *K* is not robustly saturated then there exists a saturated invariant measure supported by *K* Equivalently if (4) holds for all invariant measures with support in *K* then *K* is robustly unsaturated.

To see the equivalence of the conditions given by (4) and (5) let $\Delta = \{p \in R_+^k : \sum_i P_i = 1\}$ and notice that

$$\text{Min max } \lambda i \, (\mu = \min_{\mu} \max_{p\in\Delta} \sum_i Pi\lambda i(\mu)$$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K* The Minimax theorem (see, e. g.,[27] implies that

(6) $\min \max \lambda i(\mu) = \min_{\mu} \max_{p \epsilon \Delta} \sum_{i} Pi\lambda i(\mu)$

Where the minimum is taken over invariant probability measures $\mu$ with support in *K*. Since min$\mu$

$\sum_{i} P_i \lambda_i(\mu)$ is attained at an ergodic probability measure with support in *K*, the equivalence of the conditions given by (4) and (5) is established.

To prove the final assertion of the Theorem assume $x \to A(x)$ is twice continuously differentiable and there exists a saturated ergodic measure $\mu$ supported by *K* Proposition 3 implies that there exists *I*C {1,…,k} such that ($\sum_{i \in} iC_i^+$)=1and $\lambda i(\mu) = 0$ For all $i \in I. SinceK$c. $\partial$[1, …. $k\}\backslash I$ is non-empty. We will show that (2) is not robustly permanent by proving that for all $\delta > 0$ there exists a $\delta$ −perturbation of (2) for which the continuation of *K* is saturated Let $\delta > 0$ be given Choose V c N$\delta$ ($\Gamma$(A)) to be a compact neighborhood of Γ(A) such that V.t c int *V* for all t > 0.Let $\eta > 0$ be such that *W*=:*N* $\eta$(Γ(A))c *V* Let p:c→ $[0,1] be$ a$C^{\infty}$function such that $p(x) = 1 \quad for\ all\ x \in \Gamma(A) and\ p(x) = 0 for\ all\ x \in C\backslash W\ Define\ \tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_k)$ by

$$\tilde{A}_i(x) = \begin{cases} \frac{A(x)}{A(x) - \frac{\delta}{2Idp(x)}} & if\ i \epsilon I \\ if\ i \notin I \end{cases}$$

Where Id denotes the identity matrix of appropriate dimension Let x. t denote the semi flow of $x = \tilde{A}(x)x$.Since$x.t = x.t\ \ whenever\ x.[0,t] \in C\backslash W, it\ follows\ that\ \ \tilde{V}.t \subset intV\ for\ all\ t >$ $0. Hence, \Gamma(\tilde{A})$c *V* c $N\delta$ ( $(A)$)). $We\ \ also\ have\ ||A(x) - \tilde{A}(x)|| \leq \frac{\delta}{2}$Therefore $x = \tilde{A}(x) is\ a\ \delta -$ perturbation of (2) By construcation,$x.t = x.t\ for\ all\ x \in K and\ t \geq 0. Hence, \mu$ is ergodic for the semiflow of $\dot{x} = \tilde{A}(x)x\ and\ \lambda_i(\mu) \leq -\frac{\delta}{2}$ for this semiflow and $i \in \{1, \ldots, k\}\backslash I$.Let $\mathcal{L}$ and O be the Lyapunov exponents and Oseledec regular points supported by $\mu$ for $x = \tilde{A}(x)x(see, e.g.,$

[28] for definitions).At each point $x \in O$, the splitting of $R^n$ determines three subspaces: the stable subspace $E^s(x)$, the center subspace $E^c(x)$, and the unstable subspace $E^u(x)$ Proposition 3 and our choice of Ã imply that $E^s(x) \cap C^+ \neq \emptyset$. The Pesin stable manifold theorem [28, Corollaries 3.17 and 3.18] implies that tangent to $E^s(x)$ $E^c(x)$ and $E^u(x)$ are locally $x.t$ −invariant families of $C^1$ discs $W^s_x, W^c_x$ and $W^u_x$ corresponding to the stable ,center and unstable manifolds The family of stable manifolds $W^s_x$ is contained in $W^s$ (Ã,supp $(\mu)$). *Since* $E^s(x) \cap C^+ \neq \emptyset$, $W^s_x \cap C^+ \neq \emptyset$ for some $x \in K$. Consequently $K$ is saturated for $x = \tilde{A}(x)x$.

The difference between the sufficient and necessary condition is that for the sufficient condition (4) has to hold for all invariant measures supported by *K*, while for the necessary condition (4) has to hold only for ergodic measures supported by *K*. since the invariant measures lie in the convex hull of the ergodic measures the sufficient condition can be more restrictive then the necessary condition.

For three Classes of invariant sets the necessary and sufficient conditions coincide Recall a compact invariant set *K* for (2) is uniquely ergodic if K only supports one invariant measure Recall a compact invariant set *K* is Axiom A if the flow of (2) restricted to *K* is transitive and hyperbolic (see, e.g.,[29] for definitions.)

**Corollary 1:** Assume $x \to A(x)$ is twice continuously differentiable It K⊂ ∂ is a compact invariant set and either is uniquely ergodic or Axiom A, then *K* is robustly unsaturated if and only if (4) holds for all ergodic, measures $\mu$ with support in *K*

Proof. If K is uniquely ergodic then the assertion follows immediately For an Axiom A invariant set Sigmund [30, thm.1] has proven that invariant measures supported by periodic orbits of *K* are dense in the set of invariant probability measures supported by *K* In particular any invariant measure can be approximated by an ergodic measure and consequently the result follows.

Another special case where the necessary and sufficient conditions coincide is when *K* supports all populations except one for discrete time models this case was considered by [22].

**Corollary2:** Assume $x \to A(x)$is twice continuously differentiable Let K⊂ $II_i \geq 2C_i^+$ be a compact invariant set. Then K is robustly unsaturated if and only if $\lambda_1(\mu) > 0$ for all invariant measures supported by *K*

## 4. MORES DECOMPOSITIONS AND ROBUST PERMANENCE

To state the sufficient condition for robust permanence, we use a characterization of permanence due to Garay [8] and Hofbauer and So [7] that involves Morse decompositions of the boundary flow Conley[31] defined a collection of sets$\{M_{1,\ldots,}M_m\}$ to be a Morse decomposition for a compact invariant set $K$ if $M_1\ldots,M_m$ are pairwise disjoint, compact isolated invariant sent for the flow of (2) restricted to K. For each $x \in K$ there are integers r=r$(x) \leq s = s(x)$ such that $a(x) \subseteq M_r$and $\omega(x) \subseteq M_{s.}$ If $r(x) = s(x), then\ x \in M_{r.}$ Garay, Hofbauer and so [7,8] proved the following characterization of permamnce

**Theorem 2** (Garay, Hofbauer-so) if $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$then (2) is permanent if and only if each of the components $M_1$ are unsaturated.

Theorem 1 and 2 imply the following result: As stated in the above all these results are true for most of problems, interactions, transactions in cosmology, econometrics, finance and other subjects, which we shall illustrate as we go on constructing models for the same.

**Theorem 3:** If $\{M_1,\ldots..,M_m\}$ is a Morse decomposition for $\partial$and (4) holds for each of the components of the More decomposition, then (2) is robustly permanent Conversely if $x \to A(x) is\ twice\ continuously\ differentiable$

*and (4) is violated by an ergodic measure supported by one of the components of the Morse decomposition, then (2) is not robustly permanent*. Theorem 3 in conjunction with Corollaries 1 and 2 yield a characterization of robust permanence for a class of structured models.

**Corollary 3:** *Let $\{M_1,\ldots..,M_m\}$ os a Morse decomposition for* $\partial$C. *Assume $x \mapsto A(x)$ is twise continuously differentiable and for each Morse component $M_i$ one of the following assertions hold $M_i$ is Axiom A, $M_i$ is uniquely ergodic, or There exists $j \in \{1,\ldots,k\}$ such that $M_i$ ----- $\{x \in C : x_l \gg 0 \text{ for } l \neq j\}$ Then (2) is robustly permanent if and only if (4) holds for all ergodic measures supported by $U_i M_i$.*

Our result also provided a structured analogue for characterizing totally permanent systems[32]. For an ergodic probability measure$\mu$, define sup $(\mu)$ to be the subset $I$ -----$\{1,\ldots,k\}$ such that $\mu(---_{i \in I} C_+) = 1$

**Corollary 4:** *The following statements are equivalent: For all ergodic probability measures μ with support in ∂C, $\lambda_i(\mu)>0$ for all $I \in \{1,\ldots,k\} \setminus supp(\mu)$ (2) and all of its subsystems are robustly permanent.*

*Proof. Suppose the first statement holds. Then unsaturated condition given by (5) with p= (1,…,1) and Theorem 3 implies that (2) and all its subsystems are robustly permanent.. The other direction follows form the necessary condition for robust permanence in Theorem 3 and the second assertion of Proposition 3.*

*To illustrate the broad applicability of theorem 3, we develop applications to spatially structured models in section 5 and to a disease model and a gene network model in section 6.*

## 5. APPLICATIONS: PATCH MODELS

A fundamental application of our result are to spatially structured models, with *k* species dispersing between *m* patches {33-35}:

(7) $$\frac{dx_i^j}{dt} = x_i^j f_i^j(x^j) + \sum d_i^{jl} x_i^l - e_i^j x_i^j \qquad i = 1,..,k \qquad j=1,\ldots,m$$

*l*

Where $x_i^j$ denotes the density of species I in patch *j*, $x^j = (x^j_1,.....,x^j_k)$ is the vector of species densities in patch *j*, $f^j_i$ is the per-capita growth rate of species *i* in patch *j*, $d_i^{jl} \geq 0$ is the dispersal rate for species *i* from patch *l* into patch *j*, and $e_i^j$ is the emigration rate of species *i* out of patch *j*. Hence $e_i^j \geq \sum_l d_i^{lj}$. We assume that the matrices $(d_i^{lj})_{j,l}$ are irreducible for each species *i* . it is then easy to write (7) in the form (2), with $x_i = (x^l_i, \ldots, x_i^m)$ 'where' denotes transpose.

For a single species (7) generates a monotone flow. Under mild assumption, e.g., each $f^j_i$ *is* decreasing and negative for large densities, there is a globally stable equilibrium, see, e.g.,[36] and [37, sect. 5.4]. This equilibrium will be positive if the invasion rate at the origin λ (0), which is given by the leading eigenvalue (stability modulus) of the matrix $(d^{jl} + f^j(0) - e^j)$, is positive.

5.1 Two types of debitable and creditable systems:. For *two competing species*, e.g., $f^j_i$ is decreasing with respect to both species and is negative at large densities, the dynamics are still monotone, and hence almost all orbits converge

to an equilibrium. Robust permanence requires two conditions. First, $\lambda_i$ (0) >0 for both species in which case there are two single species equilibria $E_1$ and $E_2$. Second, the invasion rates $\lambda_2$ ($E_1$ ) and $\lambda_2$ ($E_2$) are positive. Whether there is a unique positive globally stable equilibrium for the robustly permanent system depends in a delicate way on the system parameters, see [38]. For slowly dispersing populations (i.e. $0< e^j_i \ll 1$ for all *i, j* ), the robust permanence condition is particularly

Straight forward to verify. Let $E^j_i$ be the largest solution to $x^j_i \quad f_i{}^j$ ($x^j_i$) = 0 i.e. the equilibrium attained by species *I* in patch *j* when there is no dispersal and no competitors. For slowly dispersing populations, $E_1$ is close to the product ($E_1^1$ ,…,$E_1^m$), and the matrix $A^2$ ($E_1$) is close to the diagonal matrix diag($f^i{}_2(E_1{}^j)$)$^m_{j=1.}$ Hence the invasion rate $\lambda_2$ ($E_1$) is close to $\max_j f_2^j (E^j_1)>0$. So we obtain: The system with two slowly dispersing competing species is robustly permanent if and only if system refers to system of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

(8) $\max_j f_2^j (E^j_1)>0$ and $\max_j f_2^j (E^j_2)>0$

Intuitively, robust permanence requires for each species there is at least one patch where it can persist.

For *systems, particle interactions, financial transactions, cosmological interactions and reactions, as we shall show model by model the* situation are even easier. Since (under mild assumption on the prey dynamics) the Morse decomposition of $\partial$C is given by two equilibria {0,$E_1$}, the spatial predator-prey system is robustly permanent, if the predator can invade at the prey equilibrium: $\lambda_2$ ($E_1$) > 0. However, the global dynamic are likely to be more complicated. For instance, if in each patch there is a globally stable limit circle (e.g.,[39-41]) and dispersal is sufficiently slow (i.e. $0< e^j{}_i \ll 1$for all *i, j*), our result about robust permanence and the theory of normally hyperbolic manifolds [42] imply that there is a positive *m* dimensional torus, which attracts almost all orbits in C, whenever $\max_j f_2^j (E^j_1)>0$

**5.2 Rock-Paper-Scissors Dynamics.** The Lotka-Volterra model of rock-paper-scissor dynamics is a simple model that is used as property for understanding intransitive ecological outcomes [43,44]. Here, a simple spatial version of this dynamic is given by

$\frac{dx_1^j}{dt} = x_1^j(1 - x_1^j - \beta^j\ x_2^j\ - \alpha^j\ x_3^j\ ) + \sum d_1^{jk}\ \ x_1^k\ - e_1^j\ x_1^j$

$\frac{dx_2^j}{dt} = x_2^j(1 - \alpha^j\ x_1^j\ - x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_2^{jk}\ \ x_2^k\ - e_2^j\ x_2^j$

$\frac{dx_3^j}{dt} = x_i^j(1 - \beta^j\ x_1^j\ - \alpha^j x_2^j - \beta^j\ \ x_3^j\ ) + \sum d_3^{jk}\ \ x_3^k\ - e_3^j\ x_3^j$

Where $a^j \in (0,1)$ and $\beta^j > 1$ for all $j$. A more general version of this dynamic is presented in [44]. Under the assumption that there is no cost to dispersal (i.e. $e^j_i = \sum_l d^{l,j}_i$ for all $i, j$) , the maximal invariant set in $\partial C$ consist of the origin 0 and a heteroclinic cycle connecting positive single species equilibria$E_1$, $E_2$ and $E_3$. For slowly dispersing population ($0 < e^j_i \ll 1$ for all $i$, j), $E_1$ is close to $(x_1, x_2, x_3) = (1,\ldots,1,0,\ldots,0,0,\ldots,0)$ and the invasion rates $\lambda_2$ ($E_1$)and $\lambda_3$ ($E_1$)are close to $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. For the other equilibria $E_i$, the debitable rates of the missing species are also given by $\max_j 1 - a^j$ and $\max_j 1 - \beta^j$. Consider the Morse decomposition of $\partial C$ given by 0 and the heteroclinic cycle. Since $\lambda i\ (0) = 1 > 0$ for $i = 1, 2, 3, 0$ is robustly insaturated. An algebraic computation reveals that (4) holds for all invariant supported by the heteroclinic cycle (i.e. all convex combination of the Dirac measures supported by the equilibria) if and only if

$$\max_j(1 - a^j) > -\max_j(1 - \beta^j)$$

Equivalently,

$$2 > \min_j a^j + \min_j \beta^j$$

In particular, even if the heteroclinic cycle is attracting for each patch when the system is uncoupled (i.e. $2 < a^j + \beta^j$ for all $j$ ), it can be repelling for the weakly coupled system.

5.3. Three species **Lotka Volterra in spatially homogenous environments.** Consider Lotka-Volterra dynamics in a spatially homogenous environment: $(x + a)^n =$

(9) $\frac{dx_i^j}{dt} = x_i^j (r_i - \sum_s a_{is} x_s^j) + \sum_l d_i^{jl} x_i^l - e_i^j x_i^j , \quad i = 1, \dots, k; \; j = 1, \dots, m$

We assume that dissipative can be shown by a linear Liapunov function: There are $c_i > 0$ such that $\sum_i c_i a_{is} x_i x_s < 0$ holds for all $x \in R_+^k \setminus \{0\}$. Then the weighted sum of densities across all patches $\sum_{i,j} c_i x_i^j$ is decreasing for large densitities and hence (9) generates a dissipative semiflow with global attractor $\mathcal{A}$. We further assume that for each patch *j* and each species *i* the immigration rate equals the emigration rate , i. e,

(10) $e_i^j = \sum_l d_i^{jl}$

This guarantees that the set $H = \{x \in C: x_i^j = x_i^l \; \forall i, j, l\}$ of spatially homogenous states is forward invariant under (9).

Now consider the Lotka-Volterra dynamics without spatial structure

(11) $\frac{dx_i}{dt} = x_i(r_i - \sum_s a_{is} x_s), \quad i = 1, \dots, k$

(11) $\frac{dx_i}{dt} = x_i(r_i - a_{is} x_s), \quad i = 1, \dots, k$

It has been shown [12,ch.16.1,16.2] That for $k = 3$,(11) is robustly permanent (robust meaning here within the class of Lotka –Volterra systems) if and only if all equilibria on the boundary of $R_+^3$ are unsaturated and whenever there is a heteroclinic cycle connecting the one species equilibria then this cycle is repelling. Equivalently the following four conditions hold. There exists an interior equilibrium $\dot{x}$ (i.e $A\dot{x} = r \;\; with \; \dot{x} \gg 0$); Det (-A)>0; the 2 species subsystems are not bistable competition systems if there is a heteroclinic cycle between the one species equilibria, say $\mathbf{E_1} \to \mathbf{E_2} \to \mathbf{E_3} \to \mathbf{E_1}$ then the following inequality holds

12) $\lambda_2(E_1)\lambda_3(E_2)\lambda_1(E_3) > |\lambda_3(E_1)\lambda_1(E_2)\lambda_2(E_3)|$

Furthermore, the proof in [12, ch.16.1, 16.2] shows that the boundary flow of (11) has a simple Morse decomposition: if (iv) applies then there are two Morse sets, the origin 0 is a repeller, and the heteroclinic cycle is

the dual attractor (within$\partial R^3_+$); if (iv) does not apply then the finitely many boundary equilibria form a Morse decomposition of the boundary flow.

**Species, population we mean types of and population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.**

**Theorem 4:** Under the above assumptions,(9) for k=3 is permanent if and only if *(i),(ii),(iii)* and *(iv)* hold.

Proof: Suppose (9) is robustly permanent. Since the nonspatial system (11) is the restriction of (9) to the invariant subspace **H**, (11) must be robustly permanent and hence*(i)-(iv)* hold

Conversely, suppose that (i)-(iv) hold. We show that the above Morse decomposition of the boundary flow of (11) (now in **H**) is also Morse decomposition for (9) restricted to the maximal invariant subset of $\partial C$ which is contained in the spatial two species subsystems of (9). For this we show that in each 2 species system the orbits of (9) converge to **H** (For the one species subsystems this is obvious.) If a two species subsystem of (11) has an internal equilibrium, say $\mathbf{E_{12}}$ then by assumption (ii),this is not a saddle but a sink, and hence $a_{11}a_{22} > a_{12}a_{21}$. This implies that the 1-2 submatrix of A is VL-stable [12,ch.15.3,],and [45] implies that $\mathbf{E_{12}}$ is the global attractor for the spatially structured system (9) restricted to the first two species.

If a two species subsystem of (11) has no internal equilibrium, then a one species equilibrium say $\mathbf{E_1}$ is the global attractor. Then Lemma 2 (see below) show the corresponding result for (9).A similar proof applies to the case where the origin **0** is the global attractor of a two species subsystem.

Finally observe that under the above assumptions (9)-(10), the invasion rates $\lambda_i$ at spatially homogenous boundary equilibria are the same for (9) and (11) Since all boundary equilibria are unsaturated for (11), they are for (9) as well. The same applies in case (iv) to the heteroclinic cycle. Note that the condition (12) on the eigenvalues is equivalent to the fact that all invariant measures supported on the heteroclinic cycle are unsaturated, see [16,Ex.4.5]

Species we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

**Lemma 2:** Suppose, in a two species Lotka –Volterra system (11) (k=2) all interior orbits converge to the one species equilibrium $\mathbf{E_1}$ .Then all orbits of the spatial version (9) (k=2) converge to the spatially homogeneous one species equibrium $\mathbf{E_1} \in \mathbf{H}$.

**Proof:** If the system is of predator-prey or mutualistic type then Hasting's [45] result applies again. So let us assume that the local interaction is given by the competition system

(13) $\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - ax_2)$ $\frac{dx_2}{dt} = r_2 x_2(1 - \beta x_1 - x_2)$

With 0<a <1< $beta$.In this case, Kazuo Kishimoto (letter to JH, Oct 1987) has given the following argument. There exists a family of forward invariant rectangles contracting to $\mathbf{E_1}$. This shows that all interior solutions of (9) (k=2) converge to $\mathbf{E_1}$.

Theorem 4 shows that a spatial network of identical patches is permanent, if the within patch dynamics is permanent, and three species Lotka-Volterra dynamics. It is not clear how to extend this result to more than three species. The crucial step in the three species case is that permanence precludes bistable two species subsystems. The spatial version of a bistable two species system allows plenty of stable equilibria outside **H,** see [33].However, permanent four species Lotka Volterra systems may have two species bistable subsystems [12,ch.16.4].So there may be invariant sets on the boundary outside **H** which need to be unsaturated to make (9) permanent.

## 6. MORE APPLICATIONS

6.1. **Disease, particle, product, marketing, financial, econometric dynamics with density-dependent demography**. GAO and Hethcote [46] introduced a model of disease dynamics with density-dependent demography. Here we consider a variation of their model in which a population of size P has a per-capita birth rate b(P) and per –capita death rate d. To ensure that the population persists in the absence of the disease, we assume that there exists K>0 such that b(K)=d. To describe the disease dynamics, let *S* be the number of individuals susceptible to the disease, *I* the Number of infected individuals, and R the number of individuals that have recovered from the disease. We assume that P=S+*I* +R. On the same lines we can think of sick and non performance accounts, and those which are conducting themselves well, and those which are satisfactory and find a parallel formula. The same thing holds true in case of a black hole eating matter and black hole radiating radiation.. If the disease transmission is asymptotic, β is the contact rate between susceptible and infected individuals, γ is the rate at which individuals recover from the disease, and m the mortality rate due to disease, then the population dynamics is given by

$$\frac{dp}{dt} = b(P)P - dP - mI$$

$$\frac{dI}{dt} = \beta(P - I - R)\frac{I}{\epsilon + P} - (\gamma + d + m)I$$

$$\frac{dR}{dt} = \gamma I - dR$$

Where $\epsilon>0$ is a constant It is useful to introduce a change of coordinates in which $y=\frac{I}{P}$ and $z=\frac{R}{P}$:

$$\frac{dP}{dt} = (b(P)-d-my)P$$

$$\frac{dy}{dt} = \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(P)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(P) - my)z$$

Setting $x_1$=P and $x_2$=(y,z) 'where' denotes transpose yields a structured model where $A_1(x)$=b-d-my and

$$A_2(x) = \begin{pmatrix} \beta(1-y-z)\frac{P}{\epsilon+P} - \gamma - m(1-y) - b(p) & \gamma \\ O & my-b(p) \end{pmatrix}$$

Since {0,(*K*,0,0)}is a Morse decomposition of the boundary dynamics Theorem 3 and Remark 1 imply that this model is robustly permanent if and only if b(0)>d and $\beta\frac{K}{\epsilon+K} > \gamma + m + d$. In particular, for $\epsilon > 0$ sufficiently small, one requires that the basic reproductive number $\frac{\beta}{r+m+d}$ is greater then one.

Remarkably this same criterion determines robust permanence of model with significantly more complicated boundary dynamics. For example, suppose the focal population is a predator species. If the prey has abundance *N* and exhibits logistic dynamics f(*N*)is the per-capita predator consumption rate of the prey and b(N)is the per-capita reproductive rate of the predator then the dynamics become

$$\frac{dN}{dt} = rN(1 - N/K - f(N)P$$

$$\frac{dP}{dt} = (b(N) - d - my)P$$

$$\frac{dy}{dt} = \beta(1 - y - z)\frac{P}{\epsilon+P} - r - m(1-y) - b(N)\ y$$

$$\frac{dz}{dt} = \gamma y - (b(N) - my)z$$

Where r>0 and *K*>0, and *f*(*N*),b(N)satisfy f(0)=b(0)Define $x_1$=*N*, $x_2$=*p*, $x_3$=*(y,z)*' Let us assume that the disease-free community (i.e.y=z=0)is permanent (i.e.b(K)>d) and let A be the global attractor in the interior of the P-N plane Under these assumption a Morse decomposition of the boundary dynamics is given by $(x_1, x_2, x_3) = (0,0,0)$, $(x_1, x_2, x_3) = (K, 0,0)$and A. Since the equilibria (0,0,0) and (k,0,0) are unsaturated, it remains to characterize $\Lambda_3(\mu)$for any invariant measure μ supported by A. Proposition 3 and Remark 1 imply

$$\Lambda_2(\mu) = \int_c (bx_1)\text{-}d)d\mu(x) = 0$$

*Hence* $= \int_c bx_1)d\mu(x) = d$ *and*

$$\Lambda_3(\mu) = \beta \int \frac{x_2}{\epsilon + x_2} d\mu(x) - r - m - d$$

**Theorem 3**: and Remark 1 implies that this structured model is robustly permanent if $\frac{\beta}{r+m+d} > 1 and\ \epsilon > 0$ is sufficiently small.

6.2. Gene networks. The repressilator is an oscillatory gene network based on three (or more generally an odd number of transcriptional repressors. Such a system has been genetically engineered in E-coli by Elowitz and Leibler [47] Mathematical model go back to [48.49] they all involve concentrations of proteins and mRNs. A modified model with auto-activation was suggested in [50]

$$(14)\ x_i = \beta(y_i - x_i)$$

$$(15)\ y_i = aF(x_{)i} \text{-} y_i)$$

With

$$(16)\quad F(x)_i = x_i g\ (x_i, x_i - 1) = \frac{x_i}{1 + x_i + p x_{i-1} + Kp\, x_i x_i - 1}$$

Here $x_i$ and $y_i$are normalized concentrations of proteins and mRN As belonging to gene $i$,p is the strength of repression and k a cooperativity parameter this is a structured system of type(2)

$$(17)\ \begin{pmatrix} x_i \\ y_i \end{pmatrix} dot = \begin{pmatrix} -\beta & \beta \\ ag(x_i,x_i)-1 & \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

In [50,Thm 3] it was shown that for n=3,a>1 and p>1,system(14)has a heteroclinic cycle connecting the 3 single gene equilibria (similar to the rock –scissors-paper dynamics of section5.2)Moreover

- the system is permanent i.e. the boundary of $R^6$+is repelling, if $\lambda + \mu > 0$
- this heteroclinic cycle is asymptotically stable if $\lambda + \mu < 0$

where $\lambda > 0$ and $\mu < 0$ are the invasion rates of the two missing genes i.e. the leading eignvalues of the 2X2 matrix in (17)evaluated at the appropriate single gene equilibria (for explicit expressions see[50,eqs.(127,128)].)

Theorem 3 implies the permanence condition implies the stronger conclusion of robust permanence A similar result holds for n odd, but the heteroclinic cycle is then actually a heteroclinic network.

Note: For references please see the original paper.

One of the examples for which we apply the model are as follows: **Progress in Particle and Nuclear Physics Volume 88, May 2016, Pages 1–28 Chiral magnetic and vortical effects in high-energy nuclear collisions—A status report D.E. Kharzeeva, b, J. Liaoc, d, , , S.A. Voloshine, G. Wangf**

The interplay of quantum anomalies with magnetic field and vorticity results in a variety of novel non-dissipative transport phenomena in systems with chiral fermions, including the quark–gluon plasma. Among them is the Chiral Magnetic Effect (CME)—the generation of electric current along an external magnetic field induced by chirality imbalance. Because the chirality imbalance is related to the global topology of gauge fields, the CME current is topologically protected and hence non-dissipative even in the presence of strong interactions. As a result, the CME and related quantum phenomena affect the hydrodynamical and transport behavior of strongly coupled quark–gluon plasma, and can be studied in relativistic heavy ion collisions where strong magnetic fields are created by the colliding ions. Evidence for the CME and related phenomena has been reported by the STAR Collaboration at Relativistic Heavy Ion Collider at BNL, and by the ALICE Collaboration at the Large Hadron Collider at CERN. The goal of the present review is to provide an elementary introduction into the physics of anomalous chiral effects, to describe the current status of experimental studies in heavy ion physics, and to outline the future work, both in experiment and theory, needed to eliminate the existing uncertainties in the interpretation of the data. Keywords Chiral magnetic effect; Chiral vortical effect; Chiral anomaly; Quark-gluon plasma; Heavy ion collisions Corresponding author at: Physics Department and Center for Exploration of Energy and Matter, Indiana University, 727 E Third Street, Bloomington, IN 47405, USA. Copyright © 2016 Elsevier B.V. All rights reserved.

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# Sufficient Conditions For The Existence And Linear Stability Of Time Independent Age Distributions In Respect Of Particles, Propositions, Problems, Hegelian Structures, Deleuzean Treatises, Interactions And Transactions In Cosmos And Others

**Abstract:** Gopalaswamy studied sufficient conditions are obtained for the existence and linear stability of time independent age distributions in two species competition with age time lagged density dependent mortality and fertility functions.. Work was published in Bulletin Australian Mathematical Society Volume 25 (1982). We show how the model could be applied in case of finance, econometrics, psycho economics, interactions between particles and transactions in nature. Basically the competition is for resources, notwithstanding such experiments could be conducted in laboratory like a rat kept in front of a cat, such incidents could be represented mathematically, and the corresponding linear stability conditions ca be obtained in case of interactions in cosmos. Take for example the case of a cash credit account of a borrower. Many inward bills waiting for paid, or inland or foreign bills payment all are in line for the debit of the account, which also acts as a resource. In addition there are recurrent cases where in the credit is issued to the cash credit account in case of demand and inland bills. In case of economics, investment has inventories and it has to be utilized towards the end of consummation of output. And this is case of output gobbling part of investment which is inventories. Output has to be sold to the customer and the amount received is diverted partly to savings and consumption and part could be employed towards the consolidation of the system. So this is a case of investment itself acting as a both receiver and payer. It shall eat and also produce the finance. Savings as stated above could be deployed for the further diversification and proliferation and that is towards the end of putting the company on terra firma. Another example in cosmology is the case of black holes guzzling matter. There are lot of black holes and lot of matter. Result of such a guzzling the production of Hawking radiation. Horava gravity is another example. How the conformal field theory could be changed in the case of Horava gravity. Still another case is the case of Vonneumann algebra in the cae of Horava gravity. In this case like cash credit account, the Horava gravity and Von Neumann algebra act as both receivers and takers. Such incidents are legion in cosmology and even in philosophy. We have considered self as individual general ledger, and the debits and credits are raised to the same account thus anointing as both a receiver and payer. We give plethora of such cases and apply the model. Due to spatial constants, same module is used for all the modules and we rewrite to state that the model is applicable to all modules. This is due to spatial restrictions. What is to be done is that each and every parameter is to be changed with a subscript or a superscript and finally it is to be concatenated towards the end of obtention of prediction model or optimizing model, for Duality of motion model for further analysis. Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification

doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module. In essence the paper is to be read as holistic one with the sole intention, primary objective and raison d' etre to build a TOE. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

**Introduction**

Let $P_1$ *(t)* and $P_2$ *(t)* denote the total particles, interactions, transactions, etc, which we dilate upon further investigation. Perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration

dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. **Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems** sizes (or biomasses) at time $t \geq 0$ of two interacting species living in a common habitat and competing for a common pool of resources. The competition for resources will be implicit in our model similar to that in the two species Lotka- Volterra competition system. Assuming constant sex ratios in the two species we can consider $P_1(t)$ and $P_2(t)$to be the Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

population of f; immigration, emigration and internal dispersion in the habitat are assumed to play no significant role in the dynamics of the community.Popoulation we mean Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

We suppose that the two species contain respectively $p_1(a, t)\,da$ and $p_2(a, t)\,da$ individuals with ages between $a$ and $a + da\ (a \geq 0)$ at time $t$ so that we have

(1.34) $P_1(t) = \int_0^\infty \rho_1(a, t)\,da,\ P_2(t) = \int_0^\infty \rho_2(a, t)\,da,\ t \geq 0.$

The rates of change of the two species densities are by definition $D\rho_1(a, t)$ and $D\rho_2(a, t)$ where

(1.35) $$D\rho_i(a, t) = \limsup_{h \to 0} \frac{\rho_i(a+h, t+h) - \rho_i(a, t)}{h},\ i = 1, 2.$$

Assuming the existence of the limits in (1.2) we consider the following time lagged model system

(1.36) $$\begin{cases} D\rho_1(a,t) + f_1(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_1(a, t) = 0, & a>0,\ t>0 \\ D\rho_2(a,t) + f_2(a, P_1(t), (a, P_1(t), P_2(t-\tau))\rho_2(a, t) = 0, & \text{(1.4)} \\ \rho_1(0, t) = \int_0^\infty b_1(a, P_1(t), P_2(t-\tau))\rho_2(a, t)da, & t>0, \\ \rho_2(0, t) = \int_0^\infty b_2(a, P_1(t), P_2(t-\tau))\rho_1(a, t)da, & \end{cases}$$

(1.5) $\rho_i(a, s) = \varphi_i(a, s),\ a.0,\ s \in [-\tau, 0],\quad i = 1, 2,$

(1.6) $P_i(s) = \int_0^\infty \varphi_i(a, s)\,da,\ s \in [-\tau, 0],$

Where $\tau$some is fixed nonnegative constant; $f_1$ and $f_2$ denote the age and density dependent mortality rates with time lags in interspecific interactions, $b_1$ and $b_2$ denote the age and density dependent fertility functions again having time lags in the interspecific interactions; $\varphi_1$ and $\varphi_2$ denote the initial age distributions needed for the formulation of the model. Note we mean the population of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

The model system (1.1)-(1.6) will be meaningful only if $\varphi_1$, $\varphi_2$, $f_1$, $f_2$, $b_1$, $b2$ are nonnegative functions of the respective arguments. Also since the total initial populations have o be finite, $\varphi_1(\cdot, s)$ should belong to $L_1(R^+)$ for each $s\epsilon$ [$-\tau$, 0]. That $f_i$ and $b_i$ $(i = 1, 2)$ are independent of t explicitly and this assumption can be interpreted to represent the temporal uniformity of the environment.

One of the fundamental questions for (1.1)-(1.6) is the following; under what conditions on $f_i$, $b_i$ and $\tau$ does the system (1.1)-(1.4) have time independent nontrivial solutions, and when such solutions, and when such solutions exist, are they stable with respect to some suitable stability criterion? When such stable time invariant solutions exist for (1.1)-(1.4) we say that the two types has a stable age distribution. Existence of a stable age distribution is the analogue of the existence of a stable steady state for the corresponding age independent system. Types we mean types of Particles, predications, postulations, ions, quarks, molecules, interactions, transact=ions, production rates, and destruction rates or mortality rates.

In this article we investigate the above question for the case of a competitive interaction with some additional constraints on the vital rates $f_i$ and $b_i$ $(i = 1, 2)$. Age dependent population systems without time delays have been considered by several authors (Gurt in and MacCamy [3, 4, 5], Gurt in and Levine [6], Haimovici [7] and Rotenberg [8]). The question of the relation between an age independent system and a corresponding age dependent system has been considered by Gurt in and MacCamy [4] and Gurt in and Levine [6] who have established an asymptotic ($as$ $t\rightarrow\infty$ ) relation between such models by constructing a higher dimensional 1 umped parameter system (in terms of ordinary differential equations) to represent the age dependent distributed parameter system. Assuming the existence of stationary age distributions Haimovici [7] considers their stability in a system of two interacting populations explicitly taking into consideration the dynamical nature of the encironments resources and pollution.

In this article we add another realistic feature namely time lags in the age dependent models and obtain sufficient conditions for the existence of stationary age distributions in (1.1)-(1.4) and show that the time lags in interspecific interactions have no effect on the linear stability of the age distributions although the decay rates of perturbations will depend on the magnitudes of the time delays. Such “harmless” natures of time lags in interspecific interactions have been noted by the author for the age independent models in [1, 2].

## 12. Existence and uniqueness of solutions

We first narrate our assumptions on the vital rates of $f_i$ and $b_i$ $(i = 1, 2)$ so that the system (1.1)-(1.6) will denote a competition system. Let $BC\ (0, \infty)$ denote the linear space of bounded continuous real functions on $(0, \infty)$ with the norm defined by

$$||f|| = \{max\ |\ f(t)\ |,\ t\ \in (0, \infty)\}.$$

Let $K_1$ be the cone of nonnegative functions of $BC\ [0, \infty]$. Let $C\ [-\tau, 0]$ denote the space of real continuous functions with a norm defined by $||g|| = \{\ max\ |g(s)\ |, s \in [-\tau, 0]\}$; let $K_2$ be the nonnegative cone of functions in $C\ [-\tau, 0]$ . We present our first set of assumptions on $f_i$, $b_i$ and $\varphi_i$ for $i = 1, 2$.

$A_1$. $f_1 \in C\ (R^+ \times K_1 \times K_2, R^+)$, $f_1 \geq d_1^*$ on $R^+ \times K_1 \times K_2$

$f_2 \in C\ (R^+ \times K_2 \times K_1, R^+)$, $f_2 \geq d_2^*$ on $R^+ \times K_2 \times K_1$ ,

( $d_1^*$ , $d_2^*$ are positive constants);

$f_1$ and $f_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ ; that is, *for* $i = 1, 2$,

$$|f_i\ (a, P_1^{'}\ , P_2^{'}) - f_i(a, P_1^{"}\ , P_2^{"})| \leq d_i\ [||P_1^{'} - P_1^{"}, P_2^{'} - P_2^{"}||]$$

($d_1$, $d_2$ are positive constants),

$$\partial f_1/\ \partial P_1, \partial f_1/\ \partial P_2 \in C(R^+ \times K_1 \times K_2, R^+)\ ,$$

$$\partial f_2/\ \partial P_1, \partial f_2/\ \partial P_2 \in C(R^+ \times K_2 \times K_1, R^+),$$

$A_2$. $b_1 \in C\ (R^+ K_1 \times\ K_2, R^+)$, $b_1 \leq \beta_1^* on\ R^+ \times K_1 \times K_2$ ,

$b_2 \in C\ (R^+ K_2 \times\ K_1, R^+)$, $b_2 \leq \beta_2^* on\ R^+ \times K_2 \times K_1$ ,

*($\beta_1^*$ , $\beta_2^*$ are positive constants);*

$b_1$ , $b_2$ are Lipschitzian with respect to their last two arguments uniformly in $a \in R^+$ *; that is for* $i = 1, 2$,

$|b_i\ (a, P_1^{'}, P_2^{'}) - b_i\ (a, P_1^{"}, P_2^{"})| \leq \beta_i\ [||P_1^{'} - P_1^{"}|| + ||P_2^{'} - P_2^{"}]$

$B_1$ , $B_2$ Are positive constants),

$b_i(\circ, (P_1, P_2) \epsilon L^{\infty}(R^+, R^+)$ For fixed $P_1, P_2, i = 1,2$,

$$\frac{\partial b_1}{\partial P_2}, \frac{\partial b_1}{\partial P_2} \epsilon\ C\ R^+ x\ k_1 x\ k_2, R^-),$$

$$\frac{\partial b_2}{\partial P_1}, \frac{\partial b_2}{\partial P_2} \epsilon\ C\ R^+ x\ k_2 x\ k_1, R^-),$$

$A_3$. $\varphi_i(\cdot, s) \in L^2(R^+, R^+) \cap C(R^+, R^+)$ for each $s \in [-\tau, 0]$,

$i, = 1, 2,$

By a solution of $(1.1) - (1.6)$ on $[0, T]$, we mean

$P_i: R^+ \times [-\tau, T] \to R^+$ $(i, = 1, 2,)$ With the following properties:

$S_1$. $P_i(\cdot, t) \in L^1(R^+, R^+)$ for each $t \in [-\tau, T]$;

$S_2$. $P_i(a, \cdot) \in C([-\tau, T], R^+)$ for each $a \in R^+$;

$S_3$. $DP_i$ Exists along the characteristics $t - a =$ constant

On $R^+ \times \{R^+ \cap [0, T]\}$ And is continuous $(i, = 1, 2,)$;

$S_4$. $P_i$ Satisfies the system (1.1)- (1.6) for

$(a, t,) \in R^+ \times [-\tau, T]$ .

Let us first convert the system (1.1)- (1.6) into an equivalent system of integral equations. For brevity we denote by P the pair ( $P_1, P_2$) If we let

$$P_i(a + x, t + x) = P_i(x), (i, = 1, 2,),$$

Then (1.3) considered at $(1.3)$ considered at $(a + x, t + x)$ becomes

$$\frac{dp_i(x)}{dx} + f_i(a + x, P(t + x)) P_i(x) = 0$$

Which has a solution of the form?

$$P_i(x) = P_i(0) \exp\left[ - \int_0^\infty f_i(a + s, P(t + S)) ds \right]$$

And hence

(2.1) $p_1(a + x, t + x) = p_1(a, t) exp\left[ - \int_o^x f_1(a + s, p(t + s)) ds \right]$

For $a \geq t$ *we choose* $(a,t)=(a-t,0)$ in (1.3) and $x=t$ in (2.1) so that

(2.2) $p_1(a, t) = \varphi_1(a - t, 0) \exp\left[ - \int_o^t f_1(a - t + s, p_1(s)) ds \right]$

$$O \leq t \leq a, i = 1,2,$$

With

$P_1(s)=(p_1(s),p_2(s-t))$,

$P_2(s)=(p_1(s-t),p_2(s))$,

For $0 \leq a < t$ we choose $(a,t)=(0,t-a)$ in (1.3) and $x=a$ in (2.1) so that

(2.3) $p_i(a,t)= B_i(t-a)\exp[-\int_o^a f_1(s, p_i(t-a+s))ds]$

$t > a \geq, i = 1,2,$

Where

(2.4) $B_i(t) = p_i(0, t), i = 1,2, t > 0.$

Define $M_i$ and $L_i$ $(i = 1, 2)$ as follows:

$M_i(a,t, p_i(t)) = exp[-\int_o^{t-a} f_1(s, p_i(a-a+s))ds]$

(2.5) $L_i(a, t, p_i(t)) = \exp[-\int_o^t f_1(a+s, p_i(s))ds]$

It will now follow from (1.1),(1.3),(2.2)-2.5)that

(2.6) $p_i(t) = \int_o^t B_1(a) M_1(a, t, p_i(t))da + \int_o^\infty \varphi_i(a, 0) L_i(a, t, p_i(t)da,$

$t > 0, i = 1,2,$

(2.7) $p_i(s) = \Phi_i(s) = \int_o^\infty \varphi_i(a, s)da, s \in [-\tau, 0], i = 1,2,$

(2.8) $\beta_i(t) = \int_0^t \beta_i(a) b_i(t-a, P_i(t)) M_i.(a, t, P_i(t))da$

$+\int_0^\infty b_i(a + t, P_i(t))\varphi_i(a, 0)L_i(a, t, P_i(t))da$

$i = 1, 2, t > 0$

the equivalence of (2.6)-(2.8) with (1.1)-(1.6) is established by the following whose proof is identical to a similar result of Gurtin and MacCamy [4]; hence we omit the details of the proof of the following.

THEOREM 1 : *Suppose the system* (1.1)-(1.6) *satisfies the conditions* $A_1, A_2, A_3$ . *If* $(P_1, P_2)$ *is a solution of*

*(1.1)-(1.6) then the total population* $P_1, P_2$ *and the birth rates* $B_1, B_2$ *satisfy the integral equation (2.6)-(2.8).*

*conversely if* $P_1, P_2, B_1, B_2$ *are nonnegative continuous solution of (2.6)-(2.8) on [ 0, T] and if* $P_1, P_2$ *are*

*defined by (2.2)-(2.4) on* $R^+$ x [0, T] *then such* $P_1, P_2$ *provide a solution of (1.1)-(1.6) on* $R^+$ x [0, T] .

The following a *priori* estimates are useful to prove our existence theorem below. It will immediately follow from the bounds for $f_i$ and $b_i$ that

$$M_i(a,t,P) \le \exp[-d_i^*(t-a)] \le 1, if \quad t \ge a,$$

*(2.9)*

$$L_i(a,t,P) \le \exp[-d_i^* t]$$

From (2.7)-(2.8) we derive that

$$\beta_i(t) \le \int_o^t \beta_i^* \beta_i(a) \exp[-d_i^*(t-a)]\, da + \beta_i^* \Phi_i(0) \exp[-d_i^* t], \quad i = 1,2,$$

And hence by Gronwall's inequality,

(2.10) $$\beta_i(t) \le \beta_i^* \Phi_i(0) \exp[\delta_i t], \ \delta_i = \beta_i^* - d_i^*$$

(2.5), (2.6) and (2.10) lead to

(2.11) $$|P_i(t) - 0_i(0)| \le \beta(0)\, exp[\delta_i\, a - d_i^*(t-a)]\, da$$

$$+\int_0^\infty \varphi_i(a,0)|1 - exp(-d_i^* t)|da$$

$$\le \beta_i^* \Phi_i(0)\, exp(\beta_i^* t) \left\{\frac{1-exp[-(d_i^*+\beta_i^*)t]}{d_i^*+\beta_i^*}\right\} + \phi_i(0) d_i^* t$$

$$\le [\beta_i^* \phi_i(0)\, exp(\beta_i^* t)\, \phi_i(0) d_i^*]t \quad for \ t > 0.$$

We can now prove the following result on the existence and uniqueness of solution of the integral equations (2.)-(2.8)

**THEOREM 2:** *Assume that the system* (1.1)-(1.6) *satisfies the hypotheses*

$A_1, A_2, A_3$. *Then there exists a unique set of continuous function* $(P_1, P_2, B_1, B_2)$ such *that*

$$P_i : R^+ \to R^+, \qquad B_i : R^+ \to R^+ \ (i = 1,2),$$

*Which satisfy the integral equations* (2.6)-(2.8)

Proof. We will first show the existence of local solution on same (possibly small) interval [0, T] $\subset R^+$ such that

$$P_i \in C([0,T], R^+), \quad B_i \in C([0,T], R^+), i = 1,2,$$

Satisfying (2.6)-(2.8) and then show the $P_i$ and $B_i$ can be continued as solution for any finite interval in $R^+$ for some fixed

Positive number $T$ we let

$$C^+[-\tau, T] = \{f \in C([-\tau, T], R) | f \geq 0\}$$

Let $||.||_T$ denote the supremum norm in $C^+[-\tau, T]$. Then for any

$(x, y) \in C^+[-\tau, T] \times C^+[-\tau, T]$ we define

$$||(x,y)|| = ||x||_T + ||y||_T,$$

$$C_T = C^+[-\tau, T], \quad C_T^2 = C^+[-\tau, T] \times C^+[-\tau, T].$$

Members of $C_T$ will be denoted by $Z_t$ with the meaning that

$Z(t+s) = z_t(s)$ *for* $s \in [-\tau, T]$ *and* $t \in [0, T]$. Now for any fixed

$P_t = (P_{1t}, P_{2t}) \in C_T^2$, (2.8) is a linear system of uncoupled Volterra integral equations in B = $(B_1, B_2)$ and hence (2.8) will

have a unique continuous solution under the assumption $A_1 - A_3$. let us denote such a solution by

$$B(t) = B^T(P_t), \quad t \in [0, T]$$

Or

$$B_1(t) = B_1^T(P_{1t}, P_{2t}), \quad B_2(t) = B_1^T(P_{1t}, P_{2t}), \quad t \in [0, T]$$

If we supply this solution $(B_1, B_2)$ in (2.6) we find that (2.) will be satisfied if and only if $(P_{1t}, P_{2t})$ is a fixed point of the

operator

$$\Pi(P_t) = \{\Pi_1(P_{1t}, P_{2t}), \Pi_2(P_{1t}, P_{2t})\}$$

Where $\pi_1$ *and* $\pi_2$ respectively denote the right sides of (2.6) for $i = 1,2$.

It is not difficult to see from the nature of $M_i$, $L_i$ and the hypotheses on $f_i, b_i, \varphi_i$ that

$$B: C_T^2 \to C_T^2 \text{ and } \Pi: C_T^2 \to C_T^2$$

(which guarantees the nonnegativity of $P_i$ and $B_i$). We will first show that

There exist positive constants $r^*, t^*$ such that $\pi$ is a mapping of a sphere

$$S_{t*}(\Phi, r^*) = \left\{ P_t \in (P_{1t}, P_{2t}) \in C^2_{t*} \mid \|P_t - \Phi_t\|_{t*} \le r^* \right\}$$

Into itself and also is a contraction where $\Phi_t$ is defined on $[-\tau, t^*]$

By the following:

$$\Phi_t = (\Phi_{1t}, \Phi_{2t});$$

(2.12)

$$\Phi_{it} = \begin{cases} \int_0^\infty \varphi_i(a,s)da \ for\ s \in [-\tau, 0]. \\ \int_0^\infty \varphi_i(a,0)da \ for\ s \in [0, t^*]. \end{cases}$$

We note that similar to (2.11) one gets the estimates

$$|\Pi(P)(t) - \Phi_i(0)| \le \{B_i^* \Phi_i(0) \exp[B_i^* t] + \Phi_i(0) d_i^*\} t$$

Which together with (2.7) and (2.12) lead to

(2.13) $$\|\Pi(P_t) - \Phi_t\|_{t*} \le \left( \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* t) + d_i^*] \right) t^*$$

Thus if we choose,

$$r^* = T \sum_{j=1}^2 \|\Phi_{jt}\|_{t*} [B_j^* \exp(B_j^* T) + d_i^*]$$

Then $\Pi$ is a mapping of $S_{t*}(\Phi, r^*)$ into itself for all $t^* \le T$. we then will have to show that for small $t^* > 0$, $\Pi$ is a contraction.

Let $P_t^{(1)}$ *and* $P_t^{(2)}$ be arbitrarily chosen in $S_{t*}(\Phi, r^*)$ and consider $|\Pi(P_t^{(1)}) - \Pi(P_t^{(2)})|$ for $t \le t^*$.

Corresponding to the chosen $P_t^{(1)}$, $P_t^{(2)}$ we let $B^{(1)}(a) = B(P_T^{(1)})$ and $B^{(2)}(a) = B(P_T^{(2)})$. Then from

(2.6),

$$|\Pi(P^{(1)}(t)) - \Pi(P^{(2)}(t)| \le \left| \int_0^t B_1^{(1)}(a) \left\{ M_1\left(a, t, P_t^{(1)}\right) - M_1\left(a, t, P_t^{(2)}\right) \right\} da \right|$$

$$+ \left| \int_0^t [B_1^{(1)}(a) - B_2^{(2)}(a)] M_1(a, t, P_t^{(2)}) da \right|$$

$$+\left| \int_0^t \varphi_1(a,0)\left[L_1\left(a,t,P_t^{(1)}\right) - L_1\left(a,t,P_t^{(2)}\right)\right]da \right|$$

$$= J_1 + J_2 + J_3 \text{ (say)}$$

using the elementary inequality

$$|e^x - e^y| \leq |x-y| \ for \ x,y \leq 0$$

We derive from (2.5) that

$$\left| M_1(a,t,\ p_t^{(1)}) - M_1(a,t,\ p_t^{(2)}) \right|$$

$$\leq \int_o^{t-a} \left| f_1(s,\ p_t^{(1)}(a+s)) - f_1(s,\ p_t^{(2)}(a+s)) \right| ds$$

$$\leq d_1 /\!/ p_t^{(1)} - p_t^{(2)} ||_{t^*} (using\ A_1)$$

*Now,using (2.10),*

$$\left| J_1 \right| \leq \int_o^a \beta_1^* \Phi_1(0)\, e^{\delta 1^a} \left(\int_o^{t-a} d_1 /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} ds\right) da$$

$$\leq d_1 \beta_1^* \Phi_1(0)\, t^2 /\!/ P_t^{(1)} - P_t^{(2)} /\!/ e^{\delta 1^t/2}$$

In a similar way

$$\left| J_3 \right| \leq \int_o^\infty \varphi(a,o)\left[\int_o^t \left| f_1(a+s, P_t^{(1)}(s) - f_1(a+s, P_t^{(2)}(s)) \right| ds\right] da$$

$$\leq d_1 \Phi_1(0) /\!/ P_t^{(1)} - P_t^{(2)} /\!/_{t^*} t$$

Now we have to estimate $\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right|$ in order to estimate $J_2$, from (2.8),

$$B_1^{(1)}(a) - B_1^{(2)}(a)$$

$$= \int_o^a \left[B_1^{(1)}(\sigma) - B_1^{(2)}(\sigma) b_1\left(a-\sigma, P_t^{(1)}(a)\right) M_1(\sigma, a, P_t^{(1)}(a) d\sigma + \int_o^a \left[B_1^{(2)}(\sigma)\right.\right.$$

$$\left. X\left[b_1(a-\sigma, P_t^{(1)}(a) M_1(\sigma, a, P_t^{(1)}(a) - b_1\left(a-\sigma, P_t^{(1)}(a) M_1\left(\sigma, a, P_t^{(2)}(a)\right)\right)\right] d\sigma\right.$$

$$+\int_o^\infty \varphi(\sigma,o)\left[b_1\left(\sigma+a,P_t^{(1)}(a)\right)-b_1\left(\sigma+aP_t^{(2)}(a)\right)\right],L_1\left(\sigma,a,P_t^{(1)}(a)\right)d\sigma$$

$$+\int_o^\infty \varphi(\sigma,o)b_1\left(\sigma+a,P_t^{(2)}(a)\right)\left[L_1\left(\sigma,a,P_t^{(1)}(a)\right)-L_1\left(\sigma,a,P_t^{(2)}(a)\right)\right]d\sigma$$

$$=R_1+R_2+R_3+R_4 \text{ (say)}.$$

By the uniform Lipschitz continuity of $b_1$ in $A_2$ we get

$$\left|b_1\left(a+\sigma,P_t^{(1)}(a)\right)-b_1\left(\sigma+a,P_t^{(2)}(a)\right)\right|\le B_1\ \|P_t^{(1)}-P_t^{(2)}\|_{t^*} \text{ for } a\ \varepsilon[o,t^*]$$

And hence we have

$$|R_3(a)+R_4(a)|\le\beta_1\Phi_1(0)||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*}+\Phi_1(0)\beta_1 d_1||P_t^{(1)}\text{-}P_t^{(2)}||_t^{*a},$$

$$|R_2(a)|=[\int_0^a[B_1^{(2)}(\sigma)\{b_1(a\text{-}\sigma,P_t^{(1)}(a))\text{-}b_1(a\text{-}\sigma,P_t^{(2)}(a))\}M_1(\sigma,a,P_t^{(1)}(a))$$

$$+b_1(a,\text{-}\sigma,P_t^{(2)}(a))\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$+b_1\left(a-0,P_t^{(2)}(a)\right)\{M_1(\sigma,a,P_t^{(1)}(a))\text{-}M_1(a\text{-}\sigma,P_t^{(2)}(a))\}]d\sigma]$$

$$\le\int_0^a[\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\beta_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}+\beta_1 d_1\|P_t^{(1)}-P_t^{(2)}\|_{t^*}(a\text{-}o)]do$$

$$\le\beta_1^*\Phi_1(0)\,e^{\delta 1^a}\|P_t^{(1)}-P_t^{(2)}\|_{t^*}\{\beta_1 a+d_1 a^2/2\}$$

Thus

$$\left|B_1^{(1)}(a)\text{-}B_1^{(2)}(a)\right|$$

$$\le\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|B_1^{*d}\sigma$$

$$+\Phi_1(0)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}\left\{B_1+B_2 d_1 a+B_1^*\left(B_2 a+d_1 a^2/2\right)e^{\delta 1^a}\right\}$$

$$\le\beta_1^*\int_o^a\left|B_1^{(1)}(\sigma)\text{-}B_1^{(2)}(\sigma)\right|do+\left(r^*+\|\Phi_t\|_{t*}\right)\|P_t^{(1)}\text{-}P_t^{(2)}\|_{t^*}g_1^{(a)}$$

Where $g_1(a)\to\beta_1$ as $a\to 0$. By Gronwall's inequality it will follow that

$$\left| B_1^{(1)}(a)-B_1^{(2)}(a) \right| \leq \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} \{ g_1(a)+0(a) \}$$

Using this estimate for $R_1(a)$ we get

$$|R_2(a)| \leq \left| \int_0^a [B_1^{(2)}(\sigma) B_1^{(2)}(\sigma),] b_1(a))-b_1(a-\sigma, P_t^{(1)}(a)) M_1(\sigma, a, P_t^{(1)}(a))do \right|$$

$$\left| B_1^* \left( r^* + \|\Phi_t\|_{t*} \right) \| P_t^{(1)}-P_t^{(2)} \|_{t*} 0(a) as\ a \to 0 \right.$$

*Thus*

$$\left| B_1^{(1)}(a) - B_1^{(2)}(a) \right| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(1) as \quad a \to o$$

And hence

$$\left| \pi_1(P_t^{(1)} - \pi_1\left(P_t^{(2)}\right) \right| \leq ((r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_t o(t) as \quad t \to o$$

Which together with a similar bound for $\left| \pi_2(P_t^{(1)} - \pi_2\left(P_t^{(2)}\right) \right|$ leads to

$$\|\pi\ (P_t^{(1)} - \pi\ \left(P_t^{(2)}\right) \| \leq (r^* + \|\Phi_t\|_{t*}) \| P_t^{(1)} - P_t^{(2)} \|_{t*} o(t^*)$$

As $t^* \to o$ showing that if $t^*$ is small enough, the mapping $\pi$ on

$s_{t*}(\Phi, r^*)$ is a contraction. Hence the system (2.6)-(2,8) has a unique solution for $o \leq t \leq t^* for\ t^*$ sufficiently small and positive

Now if [0,T] is any finite interval on $R^+$, we can consider [0,T] as a finite union of intervals of length less than or equal to $t^*$ Since the estimate (2.11)is valid for any finite interval we can extend the solution of (2,6)-(2.8) form $[0, t^*]$ to [0,T] and this completes the proof,

3. Stationary age distributions and their linear stability we will now establish the existence of time independent solution

$P_t^*, P_t^*$ Of (1.1)-(1.6): such solutions satisfy

$$(3.1)\ \frac{dp\, P_2^*(a)}{da} + f_i(a, P_1^*, P_2^*)\ P_i^*(a) = 0, \quad i = 1, 2, a > 0,$$

$$(3.2)\ P_i^* = \int_o^\infty P_i^*(a) da, i = 1,\ 2,$$

$$(3,3)\ P_i^* = \int_o^\infty b_i(a,\ P_1^*,\ P_2^*) P_i^*(a) da,\ i = 1.\ \ 2$$

All solutions of (3.1)are of the from

$$(3.4)\ P_i^*(a) = P_i^*(o) \exp\left[ - \int_o^\infty f_i(s, P_1^*, P_2^*) ds \right], i = 1, \quad 2, a \geq 0,$$

Which together with (3.3) lead to

$$1=\int_o^{\infty} b_1(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_1(s,P_1^*,P_2^*)ds\right]da,$$

(3.5) $$1=\int_o^{\infty} b_2(a,P_1^*,P_2^*)\exp\left[-\int_o^a f_2(s,P_1^*,P_2^*)ds\right]da$$

Thus the existence of stationary solutions of (1.1)-(1.6) reduces to the existence of a pair of positive constants $P_1^*, P_2^*$ satisfying (3.5). We can now prove the following,

THEOREM 3. Assume that $f_1$ and $b_i$ satisfy ( $A_1$ ) and ( $A_2$ )and let

(3.6) $$F_{i.}(P_1,P_2) = \int_o^{\infty} b_i(a,P_1,P_2)\exp\left[-\int_o^{\infty} f_i(s,P_1,P_2)ds\right]da,$$

$$i=1,\ 2,\quad P_1 \geq 0,\quad P_2 \geq 0,$$

**And suppose the following hold:**

(3.7) $A_{4.}$ $F_{i.}(0,\ 0) > 1,\ i = 1,2$:

$A_{5.}$ *there exists a positive constant c for which*

$$F_1(C,0) < F_{2.}(C,0),$$

(3.8) $$F_1(0.C) < F_{2.}(0,C),$$

Then there exists a unique pair $(P_1^*, P_2^*)$ of real numbers $P_1^* > o$,

$P_2^* > 0$ such that

(3,9) $$F_1(P_1^*,P_2^*) = 1 = F_2(P_1^*,P_2^*)$$

And the unique nonegative solution of (3.1)-(3.3) is given by

(3.10) $$P_2^*(a) = [P_2^*\exp[-f_i(s,P_1^*,P_2^*)ds])/(\int_o^{\infty}\exp[-\int_o^a f_i(s,P_1^*,P_2^*)ds]da),$$

$$i=1,2,\quad a \geq 0,$$

Proof Consider the elements of the surfaces defined by

$$Z= F_1(P_1,P_2), z = F_2(P_1,P_2),\ P_1 \geq 0, P_2 \geq 0,$$

In the ($P_1$,$P_2$,z) space. By hypothesis ($A_1$) we have $\partial F_i/\partial P_j < 0$ *and* , by ($A_2$), $\partial b_i/\partial P_j \leq 0$ for all

$$P_1 \geq 0, P_2 \geq 0,\quad i=1,2,$$

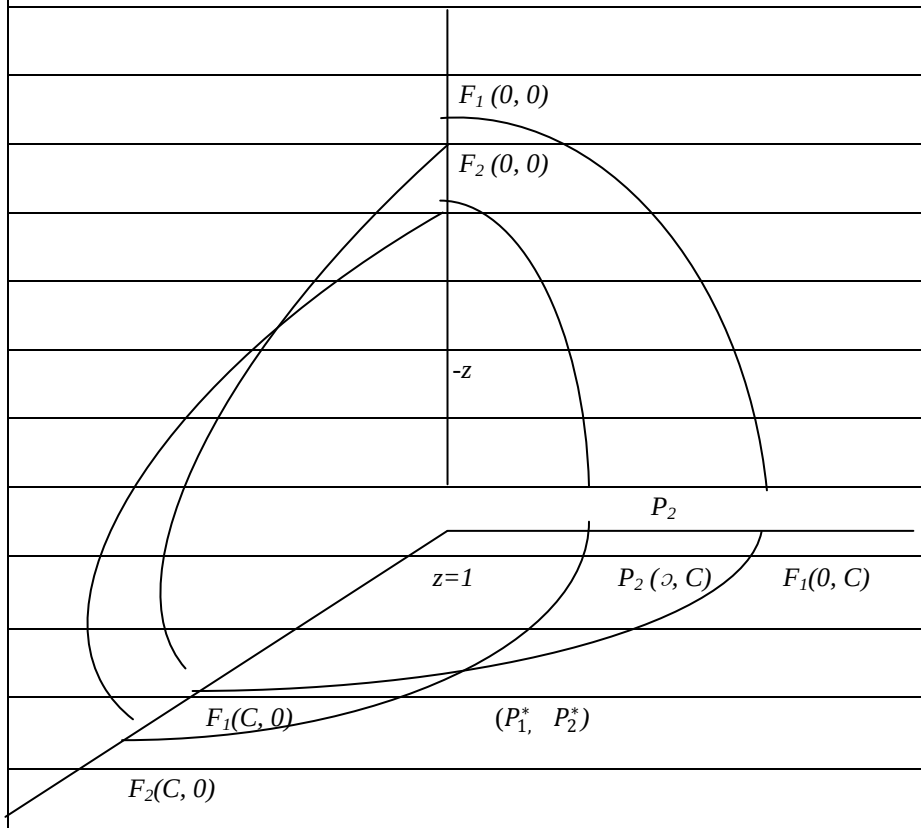
$j=1,2,$ Hence by ($A_4$) the intersections of these surfaces with the plane z=1 define two curves on the z=1 plane connecting the I ines {z=1, $P_1 \geq 0, P_2 \geq 0,$} and {z= 1, $P_2 \geq 0, P_1 \geq 0$, These two curves are defined by

$$F_1(P_1, P_2 = 1 \; and \; F_2(P_1, P_2) = 1, \; F_1(P_1, P_2 \geq 0,$$

Which by ($A_s$) will intersect at a unique point say $((P_1^*, P_2^*)$ with $(P_1^* > 0, P_2^* > 0$ (see figure). The uniquess of the point $((P_1^*, P_2^*)$ *is a* consequence of the smoothness of the surfaces $z = F_1(P_1, P_2)$ and $z = F_2(P_1, P_2)$

Following is a hypothetical case for Particles, predications, postulations, ions, quarks, molecules, interactions, transactions, production rates, and destruction rates or mortality rates attributable or ascribable in the transactions and reactionary potentialities.




(The figure corresponds to the case $F_1(0,0) > F_2(0,0) > 0$. The other possibility $1 < F_1(0,0)$ is treated similarly) It will now

Follow that for such a pair $(p_1^*, p_2^*)$ the unique nonnegative solution of (3.1) - (3.3) is given by (3.10).

Now to examine the linear stability of the stationary age distributions $p_1^*, p_2^*$, we let

$$(3.11) \quad \begin{cases} p_i(a,t) = p_i^*(a) + u_i(a,t) \\ p_i(t) = p_i^* + p_i(t), \; i = 1, 2, \quad a > 0, \quad t > 0, \\ p_i(t) = \int_0^\infty u_i(a,t)da, \end{cases}$$

In (1.1)-(1.5) and derive the following variational system after neglecting the nonlinear terms in the perturbations $\boldsymbol{u}_i$ and $\boldsymbol{p}_i$ $(i = 1, 2,)$ (such a procedure of linear stability analysis can be justified As has been done in Gurtin and MacCamy [4].)

(3.12)
$$\begin{cases} Du_i(a,t) = -f_i\left(a, p^*_1, p^*_2\right) u_i(a,t) - p^*_i(a)\left\{\sum_{i=1}^{2} \frac{\partial f_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\}, \\ u_i(0,t) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right) u_i(a,t)da + \int_0^\infty p^*_i(a)\left\{\sum_{\square 1}^{2} \frac{\partial b_i}{\partial p^*_{\square}}\, p_{i\square}(t)\right\} da \end{cases}$$

Where $i$ = 1,2 and (3.13)
$$p_{11}(t) = p_1(t), \quad p_{12}(t) = p_2(t-\tau), \quad p_{21}(t) = p_1(t-\tau), \quad p_{22}(t) = p_2(t)$$

To consider the asymptotic bahaviour as $t \to \infty$ as of solutions of (3.12) we let

(3.14)
$$\begin{cases} u_i(a,t) = u_{\square}(a)\exp[\lambda t], \quad {}_{\square}(a) \to 0 \text{ as } a \to \infty, \\ p_i(t) = p^*_i \exp[\lambda t], \\ \qquad\qquad i = 1, 2, \quad a > 0, \quad t > 0, \\ p^*_i = \int_0^\infty {}_{\square}(a)da, \end{cases}$$

Using (3.14) in (3.12) - (3.13),

$$\frac{d_{\square}}{da} + \lambda_{\square} = -f_i\left(a, p^*_1, p^*_2\right)_{\square}(a) - p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial f_i}{\partial p^*_j}\, p^*_{i\square}\right\},$$

(3.15)

$$_{\square}(0) = \int_0^\infty b_i\left(a, p^*_1, p^*_2\right)_{\square}(a)da + \int_0^\infty p^*_i(a)\left\{\sum_{j=1}^{2} \frac{\partial b_i}{\partial p^*_j}\, p^*_{i\square}\right\} da,$$

Where $i = 1, 2,$ and

$$p^*_{11} = p^*_1, \quad p^*_{12} = p^*_2 \exp\{-\lambda\tau\},$$

$$p^*_{21} = p^*_1, \exp\{-\lambda\tau\}, \; p^*_{22} = p^*_2.$$

If we integrate the first of (3.15) and use the fact that $\mu_i(a) \to 0$ as $a \to \infty$ we can derive that $p_1^*$ and $p_2^*$ are governed by

(3.16) $$\lambda p_1^* - \int_0^\infty p_1^*(a)\left[\left(\frac{\partial b_1}{\partial p_j^*} - \frac{\partial f_1}{\partial p_1^*}\right)p_1^* + \left(\frac{\partial b_1}{\partial p_2^*} - \frac{\partial f_1}{\partial p_2^*}\right)p_2^* e^{-\lambda\tau}\right]da$$

$$= \int_0^\infty \left[b_1(a, p_1^*, p_2^*) - f_1(a, p_1^*, p_2^*)\right]\mu_i(a)da,$$

(3.17) $$\lambda p_2^* - \int_0^\infty p_2(a)\left[\left(\frac{\partial b_2}{\partial p_1^*} - \frac{\partial f_2}{\partial p_1^*}\right)p_1^* e^{-\lambda\tau} + \left(\frac{\partial b_2}{\partial p_2^*} - \frac{\partial f_2}{\partial p_2^*}\right)p_2^*\right]da$$

$$= \int_0^\infty \left[b_2(a, p_1^*, p_2^*) - f_2(a, p_1^*, p_2^*)\right]\mu_2(a)da,$$

From the first of (3.15) we get

(3.18) $$\mu_i(a) = \{\mu_i(0)p_i^*(a)/p_i^*(0)\exp[\lambda a]\}$$
$$- p_i^*(a)\exp[-\lambda a]\int_0^a \left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, \quad i = 1, 2,$$

Which by the second of (3.15) becomes,

(3.19) $$\mu_i(a)\left[p_i^*(a)/p_i^*(0)c_i\exp(\lambda a)\right]\left[\int_0^a p_i^*(a)\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}da\right.$$

$$\left. - \int_0^\infty b_i(a, p_1^*, p_2^*)\, p_i^*(a)\, e^{-\lambda a}\left\{\int_0^a\left(\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right)e^{\lambda s}ds\right\}da\right]$$

$$- p_i^*(a)\, e^{-\lambda a}\int_0^a\left\{\sum_{j=1}^{2}\frac{\partial f_i}{\partial p_j^*}\, p_i^*\mu_i\right\}e^{\lambda s}ds, i = 1, 2,$$

where

(3.20) $$c_i = 11 - \frac{1}{p_i^*(0)}\int_0^\infty b_i(a, p_1^*, p_2^*)p_i^*(a)e^{-\lambda a}da, \quad i = 1, 2,$$

Now supplying $\mu_1(a)$ and $\mu_2(a)$ from (3.19) in (3.16)-(3.17) we derive that

$$(\lambda - A_{11})p_1^* - A_{12}p_2^*\exp(-\lambda\tau) = g_{11}(\lambda)p_1^* + g_{12}(\lambda)p_2^*\exp(-\lambda\tau),$$

(3.21)

$$-A_{21}p_1^*\exp(-\lambda\tau) + (\lambda - A_{22})p_2^* = g_{21}(\lambda)p_1^*\exp(-\lambda\tau) + g_{22}(\lambda)p_2^*$$

Where

(3.22) $$A_{i} = \int_0^{\infty} p_i^*(a)\left[\frac{\partial b_i}{\partial p_{\square}^*} - \frac{\partial f_i}{\partial p_{\square}^*}\right]da,$$

(3.23) $$g_{i}(\lambda) = \left[p_i^*(0)c_i\right]^{-1}\left\{\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\,da\right\}\left\{\int_0^{\infty}\left[p_i^*(a)\frac{\partial b_i}{\partial p_{\square}^*} - b_i\left(a, p_1^*, p_2^*\right)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)\right]da\right\}$$

$$-\int_0^{\infty} \sigma_i(a)p_i^*(a)\,e^{-\lambda a}\left(\int_0^a \frac{\partial f_i}{\partial p_{\square}^*}\,e^{\lambda s}\,ds\right)da,\ i, \square = 1,\ 2,$$

(3.24) $$\sigma_i(a) = b_i\left(a,\ p_1^*, p_2^*\right) - f_i\left(a,\ p_1^*, p_2^*\right),\ i,\ = 1,\ 2.$$

To solve the linear perturbational system (3.12) it is enough to find the constants $\lambda, p_1^*, p_2^*$ satisfying (3.21) since one can then use (3.19) to find the perturbations from (3.14). A nontrivial solution $\left(p_1^*, p_2^*\right)$ for (3.21) will exist if and only if $\lambda$ is a root of the equation $et\begin{bmatrix}\lambda - \left(A_{11} + g_{11}\langle\lambda\rangle\right) & -\left(A_{12} + g_{12}\right)e^{-\lambda\tau} \\ -\left(A_{21} + g_{21}\right)e^{-\lambda\tau} & \lambda - \left(A_{22} + g_{22}\langle\lambda\rangle\right)\end{bmatrix} = 0$

Or equivalently

(3.25) $$\lambda^2 - \lambda\left(A_{11} + A_{22}\right) + A_{11}A_{22} - A_{12}A_{21}\exp\left[-2\lambda\tau\right] = \lambda s_1(\lambda) + s_2(\lambda,\tau)$$

Where $S_1(\lambda) = g_{11}(\lambda) + g_{22}(\lambda)$ ,

$S_2(\lambda,\tau) = \left[A_{21}\,g_{12}(\lambda) + A_{12}\,g_{21}(\lambda) + g_{12}(\lambda)\,g_{21}(\lambda)\right]\exp\left[-2\lambda\tau\right]$

$$-A_{11}\,g_{22}(\lambda) - A_{22}\,g_{11}(\lambda) - g_{11}(\lambda)\,g_{22}(\lambda)$$

The following result establishes the linear local asymptotic Stability of the stationary age distributions $p_1^*(a)$ and $p_2^*(a)$

THEOREM 4 Assume the following:

$A_6.\ 0 < \beta_i^* < d_i^*\ (i,\ = 1,\ 2.)\ \left(see(A_1)\ and\ (A_2)\right)$;

$A_7.\ \left\{A_{11} + A_{22} + \| S_1(0) \|\right\}^2 < 4\left\{(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0) \|\right\}$,

$$\| S_1(0) \| = |g_{11}(0)| + |g_{22}(0)|,$$

$$\| S_2(0) \| = |A_{11}\,g_{12}(0)| + |A_{12}\,g_{21}(0)| + |g_{12}(0)\,g_{21}(0)|$$

$$+ |A_{11}\,g_{22}(0)| + |A_{22}\,g_{11}(0)| + |g_{11}(0)\,g_{22}(0)|$$

Then all the roots of (3.25) have negative real parts and hence the

perturbations $u_i(a,\ t) = \square(a)\exp[\lambda t] \to 0\ as\ \ t \to \infty$ for $i = 1, 2\ and\ a \geq 0$

Proof . Define $F_1(\lambda,\tau)$ and $F_2(\lambda,\tau)$ as follows:

$F_1(\lambda,\tau) = \lambda^2 - \lambda(A_{11} + A_{22}) + A_{11}A_{22} - A_{12}A_{21}\exp[-2\lambda\tau];$

$F_2(\lambda,\tau) = \lambda S_1(\lambda) + S_2(\lambda,\tau),$

Using $(A_6)$ one can show that for $\lambda$ with re $(\lambda) \geq 0$, $c_1$ and $c_2$ are positive and bounded away from zero, Now since $A_{ij} > 0$ and $\tau \geq 0$ we find from the nature of the dependence of $g_{ij}$ on $\lambda$ that for all $\lambda$ $\mu \pm iw$ with $\mu \geq 0$ and $w \geq 0$ we have

(3.26) $$\left| F_1(\lambda,\tau) - P_2(\lambda,\tau) \right|$$

$$\geq \left| F_1(\lambda,0) \right| - F_2(\lambda,\tau) \left|\right.$$

$$\geq |\lambda|^2 - |\lambda|\{A_{11} + A_{22} + \| S_1(0) \|\} + \{A_{11}A_{22} - A_{12}A_{21} - \| S_2(0,\tau) \|\}$$

$$\geq \left[(|\lambda| - P)^2 + (q)\right]^2$$

Where

$p = \{A_{11} + A_{22} + \| S_1(0) \|\}/2,$

$q = \{4|(A_{11}A_{22} - A_{12}A_{21}) - \| S_2(0,T) \| | - |A_{11} + A_{22} + \| S_1(0) \| |^2\}/2,$

It will now follow, from (A) and (3.26), $F_1(\lambda,\tau) = F_2(\lambda,\tau)$ connot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net creation rates of each system is greater than utility; a condition of this type is well known in age wise classification of particles, atoms, molecules, cosmological phenomenon. (3.8) can be interpreted to mean that the intraspecific inhibition is higher than that of the interspecific interaction. The conditions of Theorem 4 are analytical and do not lend themselves for any worthwhile interpretation in terms of the model parameters. So also an interpretation can be recorded as a remark in respect of destruction rates of particles, properties, elements, or projections.

Note: The model is applicable to each and every module. As stated in the foregoing, there are possibilities of situation where an account itself be a receiver and it is also the one to which debit is raises. Such examples are legion in all the fields of finance, economics, political science and physics and chemistry, and of course mathematics. Classification based on time is the preeminently and predominantly suited premise for such model. We repeat that the model is to be read with each module We will expatiate the importance and implicational ramifications as we proceed with the work. Assortment-blend, collection-combinations, combo-gallimaufry, goulash-hash, hodgepodge-medley, mishmash-mixture, motley-pastiche patchwork-salmagundi, soup-stew, mixed bag-mélange, olio- olio podrida, assortment variety, Array choice, collection combination, combo diversity, hodgepodge jumble, kind medley, miscellany mishmash, mixed bag mixture, mélange potpourri, selection sort, brew concoction, beverage blend, broth compound, distillation drink, fermentation hash, hodgepodge infusion, instillation liquor, mélange miscellany, mishmash mixture, mélange potpourri, borscht bouillon, bowl brew, chowder concoction, consommé decoction, gumbo hodge-podge, olio porridge, potage potpourri, pottage puree, splash stock, vichyssoise hodgepodge, mixture- mess, collection combination, goulash hash, jumble medley, miscellany

mishmash, mixed bag mélange, olio patchwork, potpourri salmagundi, jumble clutter, disarrangement disarray, disorder farrago, gallimaufry garbage, goulash hash, litter medley, mess miscellany, mishmash mixture, muddle mélange, olio pastiche, patchwork potpourri, salmagundi scramble, shuffle snarl, tangle tumble, medley miscellany, assortment brew, collection combo, conglomeration farrago, hodgepodge jumble, melee mingling, mishmash mixed bag, mixture mélange, pasticcio pastiche, patchwork potpourri, salmagundi variety, mélange mixture, assortment combo, farrago gallimaufry, pasticcio pastiche, patchwork potpourri, salmagundi soup, stew miscellany, accumulation aggregation, anthology assortment, brew collectanea, combination combo, compilation conglomeration, accumulation diversity, farrago gallimaufry, melee mishmash, mix mixed bag, mixture muddle, mélange odds and ends, olio pasticcio, pastiche patchwork, potpourri salad, salmagundi smorgasbord, stew variety, mishmash jumble, medley mess, miscellany mixed bag, mixture mélange, olio patchwork, potpourri salmagundi, mixed bag array, assortment choice, collection combination, combo conglomerate, diversity hodgepodge, jumble medley, mélange miscellany, mishmash mixture, montage mélange, odds and ends, patchwork potpourri, smorgasbord sundries, mix/mixture assortment, combination, admixture adulteration, alloy amalgam, amalgamation assimilation, association batter, blend brew, combine combo, commixture composite, compound concoction, confection conglomeration, cross crossing dough fusion, goulash grab bag, hodgepodge hybrid, hybridization incorporation, infiltration interfusion, jumble mash, medley merger, mingling miscellany, mishmash mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, odds and ends assortment, bits and pieces, debris etcetera, hodgepodge jumble, knickknacks leavings, leftovers litter, medley melee, miscellaneous paraphernalia, miscellany mishmash, mixed bag motley, mélange notions, novelties oddments, olio particles, patchwork potpourri, rags refuse, remainder remains, remnants rest, rummage scraps, sundries sundry items, compilation mishmash, paste-up patchwork, potpourri reappropriation, reproduction synthesis, conglomeration cross, crossing dough, fusion goulash, interfusion jumble, mash medley, merger mingling, miscellany mishmash, mix mixed bag, mosaic mélange, package patchwork, potpourri salmagundi, saturation soup, stew transfusion, union variety, colluviums gallimaufry, hash mixture, pasticcio pastiche, conglomeration cross section, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multiformity multiplicity, mélange potpourri, range shift, variegation variousness, divergencies diverseness, departure discrepancy, disparateness divergencies, diverseness diversification, diversity fluctuation, heterogeneity heterogeneousness, incongruity intermixture, many-sidedness medley, miscellaneousness miscellany, mishmash mixed bag, mixture modification, multifariousness multiformity, mélange potpourri, range shift, soup stew, variance variation, variegation variousness, oddment odds and ends, assortment bits, polymorphism assortment, change collection, combo conglomeration, cross section departure, discrepancy disparateness, divergencies diverseness, diversification diversity, fluctuation heterogeneity, heterogeneousness incongruity, intermixture many-sidedness, medley miscellaneousness, miscellany mishmash, mixed bag mixture, notions novelties, oddment oddments, olio particles, patchwork potpourri of the particles, interactions, accounts, transactions in the cosmos. The wisdom and understanding of our own divine essence is like a beaconing begorra, bureaucratic burdock, befuddling binary and digital digamma, that invites us inward toward ever deeper connection with all around us. Yet rather than look within, most in this world prefer to look outward for divine wisdom, thus sustaining a dependence upon a vast hierarchy that stretches between the individual and the Divine. In all our wanderings away from the Divine, humankind has obscured, obfuscated, obliterated, adjured, abnegated and abjured its most compelling features through a persistent belief systems in limitations arising out of the controlling mechanisms of this hierarchy of sub ordinates, superordinates and coordinates. Has my consciousness changed? Why that doubt? Twist and shake, ponderous novelty themes, pathos, poignancy in writing in different styles, accord-coetaneousness, coevality-coincidence, concurrence-conformity, conjunction-contemporaneousness, harmony, order, peace, simultaneousness- synchronicity exciting optimism of youth, power obmbudsman,pulsations and throbbing of pop, hard luck stories of people, modernity and mischief, grimalkins of rurality, wobbling boats of lovers, saxophone in mouths all point to newer value orientations. The Divine dances and the “Grand design “is perpetrated outside of the confines of any hierarchical structures, be it feudal, organizational.

It is complete within itself, and has a singular purpose of demonstrating the collective potential of all life within the universe. It is the archetype of perfection. It is the standard bearer of each soul's innate design and ultimate destiny. The essence of the Divine is far beyond mental conception, yet humanity's tendency is to resort to the limiting language of the hierarchical paradigm acceding- accession, accommodation- accord, accordance, acknowledging, adjustment-affiliation, affinity-alliance, amity-approving, arbitration-arrangement, assenting, authorizing, bargaining-compatibility, compliance--compromise, concert-concession, conformity-congruity, consistency-correspondence, endorsing, granting-harmony in conformity to pen chance ,predilections of the protagonist to define and understand it. Searching for Connection and Wholeness calls for not lumping elements like me made of clay feet and weak knees, owing allegiance to the fringe, without courante and courage of conviction and will for vindication, visual restlessness and exaggerated narratives of self to its own, exciting potentialities available, terrestrial passion, and skew of scams all bear an infallible demonstration for new order of consciousness. Theorists have posited a virtual smorgasbord of objects to account for dark matter, although many of them have fallen prey to observational constraints. As leading possible candidates for baryonic dark matter, there are black holes (large and small), brown dwarfs (stars too cold and faint to radiate), sun-size MACHOs, cold gas, dark galaxies and dark clusters, to name only a few. The range of particles that could constitute nonbaryonic dark matter is limited only slightly by theorists' imaginations. The particles include photinos, neutrinos, gravitinos, axions and magnetic monopoles, among many others. Of these, researchers have detected only neutrinos--and whether neutrinos have any mass remains unknown. Experiments are under way to detect other exotic particles. If they exist, and if one has a mass in the correct range, then that particle might pervade the universe and constitute dark matter. But these are very large "ifs." To a great extent, the details of the evolution of galaxies and clusters depend on properties of dark matter. Without knowing those properties, it is difficult to explain how galaxies evolved into the structures observed today. As knowledge of the early universe deepens, I remain optimistic that we will soon know much more about both galaxy formation and dark matter. What we fail to see with our eyes, or detectors, we can occasionally see with our minds, aided by computer graphics. Computers now play a key role in the search for dark matter. Historically, astronomers have focused on observations; now the field has evolved into an experimental science. Today's astronomical experimenters sit neither at lab benches nor at telescopes but at computer terminals. They scrutinize cosmic simulations in which tens of thousands of points, representing stars, gas and dark matter; interact gravitationally over a galaxy's lifetime. A cosmologist can tweak a simulation by adjusting the parameters of dark matter and then watch what happens as virtual galaxies evolve in isolation or in a more realistic, crowded universe. Old consciousness will not do. Newer one has to be imbibed and cultivated. When people are unaware of their inner wisdom and wholeness, they tend to search for order and security outside themselves. That is my personal experience. There is always search for sanctity of emotion, aesthetic expression, pursuit of harmony, unraveling realms, syncopated ending and stillness of mind and shiver of heart at the mirror broken somewhere are the strains of newer wisdom if it can be called one. Same questions shall be asked and same shall be answer in the similar vein. Uncertain of their place within the hierarchical order of the world, they define themselves based upon their insecurities. Individuals thus become only pieces of their wholeness and like shards of glass from a beautiful vase; they bear little resemblance to their aggregate beauty. Within the hierarchy, many in high positions of power have taken advantage of our collective insecurities in attempting to guide the development of all humankind. They have obscured the direct connection between the individual and the Divine through a variety of means designed to intercede between our inner essence and our divine source. Each individual can come to know them to be free of all forms of institutional and organisational and societal hierarchical control. This is not to imply that we should not trust others or join together in bonds of friendship and community. It is simply a reminder that relative truth is constantly shifting in the hands of those who desire to control. And even when the motive for their controlling behaviors may be of good will, it is still a form of control. When the revealers of "truth" within the hierarchy withhold and suppress information, they are usually positioning themselves to maintain or acquire more power rather than to disseminate empowerment to all. The desire for connection and wholeness is a fuel that drives us to seek out and explore the hierarchical paradigm. This inner longing provides us with the motivation to seek help and guidance from a specific group within the hierarchy, and in so doing, cultivate a sense of belonging and connection.

Furthermore, the hierarchical paradigm is a stage whereby we develop a sense of connection to some grand, encompassing vision. This is why the hierarchy nurtures prophets who point toward a greater vision. Spiritual leaders are able to peer deeply beneath the surface reality of life and experience how intricately connected every life form is, and how the composite of all life is intelligent far beyond measure. These visionary leaders can thus interpret reality through their personal abilities to perceive and express life's dimensional depth and limitless intelligence. Yet no one is able to articulate life's full dimensional depth and breadth with the tools of language. They can only, at best, describe their interpretation or their impressions. In actuality, all of us are able to peer beneath the surface reality of life and perceive a unique vision of the universe. We require only time and intention to develop our own interpretations. And this is precisely what many great spiritual leaders have taught. Life's deeper essence is not an absolute to be experienced by the chosen few, but an evolving, dynamic intelligence that wears as many faces as there are life forms. No group or people have the exclusive portal into the universe by which the Divine expresses itself in all its majesty. The portal is open and available to all, because the Divine is within all things. Those recognized as great prophets each produced a vision of the universe beyond what was currently defined by the hierarchy. Because their interpretations were articulated with authority and depth of insight, they became a target of debate among various groups in the hierarchy. This debate then created a polarity of belief. A sympathetic constituency emerged to defend and embellish their leader's interpretation, while established groups held it in contempt of previously held beliefs. Invariably, the leader's vision became confined and shaped into dogma by followers who desired to create a new religion or sect. Thus, this infusion of fresh insight quietly receded into the hands of the hierarchy, where its deeper meaning was obscured by the very fact that it was incorporated into a rigid structure that both protected and promoted it. A new paradigm is emerging that promotes a clear connection of individual consciousness to the compelling features of the Divine without the intervention of a hierarchy. This is when the fables and myths of history step into the light and become known as they were originally intended. This is the time when language will be transformed into a new form of communication that breaks down all barriers of control. Personal transformation, through the awakening of inner wisdom and understanding, is the pathway into wholeness. The transformation paradigm is initiated simply by the recognition that rather than the dependency-inducing ways of the hierarchical paradigm, there are accelerated, independent pathways that bypass the hierarchy and lead to self-mastery. These new pathways lead to the divine wisdom and understanding that is present within all of us. This wisdom can be accessed through the practice of three principles of transformation: seeing the Divine in all, nurturance of life, and gratitude. The application of these life principles disengages individuals from the controlling elements of the hierarchy, thus initiating the transformation experience. As there are relative truths, there are relative freedoms. As individuals evolve through the hierarchical process, an ever-increasing sense of freedom is gained, yet external forces continue to exert control through limiting language, confining belief systems, and more. These controlling influences lead to anatomical architecture, body, build, cagey carcass, construction, enclosure, fabric, flounce, form, framework, fringe, groundwork, hem, mount, mounting outline, physique, scaffold, scaffolding scheme, setting shell, stage, structure, support, system, trim, trimming, truss- valance continued reliance upon the hierarchy as it unceasingly attempts to impose a sense of inequality between us. The underlying equation of the hierarchical paradigm is: individual + hierarchy = God connection. In the case of personal transformation it is: individual + inner wisdom and understanding divine equality with all. The time has come to integrate the dominant paradigm of the hierarchy with the liberating transformation paradigm. That shall redeem all promises made to oneself. There shall be a Matrix force or balancing force that shall remain UN satisfies until conservativeness is reached. This integration occurs naturally once we have fully explored the two paradigms and develop a synthesis paradigm whereby transformation is attained by searching for connection and wholeness through our own inner wisdom and understanding. It is this combination of release from dependency on the hierarchy and transformation from within that initiates the synthesis paradigm. Once we take responsibility for our own transformation and integration, it does not mean that the hierarchy is to be shunned or avoided. The hierarchy is quite benign as a manipulative force and merely represents one important stage in the journey toward wholeness. What is being set in motion now is the initial preparation for these paradigm shifts. More specifically, these paradigms will be simultaneously played out over the coming years. As always, it will be the choice of each one of us as to which

paradigm we embrace in our journey. All of the highest imaginings of humanity are yet unaware of our deepest foundation. We have sought the upper reaches of the building, yet remain unaware of the foundation's design. It is here, at the very core of existence that the Divine is bursting forth with its creative energy while simultaneously reintegrating with its invitation to wholeness. It is here that equality is realized, not in the lofty places of relative truth lodged in the hierarchy, but rather in the deepest part of the foundational plan of life's original source and ultimate destiny. The origin and destiny of existence is the tone of equality in life. Listen for this tone – this frequency of vibration – and follow it back into the very foundation from whence all things arise and return. Consider these words as symbols only. Feel the truth that stands behind these words, and tap into this empowering energy force that reaches out for you. Know it as a tone or vibration – a resonance that waits for you around every corner in which your life will turn. It is a beacon of the Divine gathering itself into the form of language in order to usher you to a place from which you can experience the formless tone of equality – the bypass of limitation. It is the primal language of our divine source that bestows to you the freedom to generate your own deepest beauty in the expression of your highest truth.(Reference: Wikipedia)

# Gopalasamy Model For Procrastinated Conducive, Felicitous Counter Sallies In Two Specified Systems

Abstract

Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure

has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 23. Preliminary remarks

Almost all the equations of population ecology have been derived starting from the following simple format (Note this can be used for the development of the creed and type of systemic types in case of cosmology and astrophysics and fiancé and econometrics. To give an example of finance, we can think of a account of the company giving rise to various accounts of its employees, and that is the contribution of the single account for the development of the total system types of the given configuration). It may be account, particle, atom, molecule, proposition, problem or any such thing. We shall delineate and disseminate information and particulars as and when the model shall be applied to the problem in question.

$$\frac{dN(t)}{dt} = \{\text{an individual's contribution to population growth}\} N(t) \quad (1.1)$$

where $N(t)$ denotes the size of a population of a single species (or biomass) with overlapping generations at time $t$, and $dN(t)/dt$ is the rate of change of the population size at time $t$. Subsequently one makes an assumption regarding the factor in the brackets in (1.1); in particular if one assumes that an individual entity's contribution to the population growth is denoted by a function say, f(t, N), defined suitably for all t>0, N≥0 then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f(t, N(t))\, N(t). \quad (1.2)$$

Various choices of $f$ together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in $f$ lead to various equations. For instance if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation in the form

$$\frac{dN(t)}{dt} = rN(t)$$

And if one assumes $f(t, N) \equiv r - (r/K) N$ for some positive constants r and K one gets the familiar logistic equation of population biology

$$\frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t)}{K} \right\}$$

In order to device models with time delayed effects, such effects are introduced in the average growth rate $F(t, N)$ of (1.2); for instance if $f(t, N) \equiv r - (r/K) N(t - \tau)$ for some constant $\tau > 0$ we obtain Hutchinson's[19]delay loistic equation

$$\left\{ \frac{dN(t)}{dt} = rN(t) \left\{ 1 - \frac{N(t-\tau)}{K} \right\} \right. . \quad (1.3)$$

If for instance

$$F(t, N) = r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) F(N(s)) \, ds$$

Where $k$ is a piecewise continuous nonnegative valued function such that $\int_0^{\infty} k(s) \, ds = 1$ with some appropriate function $F$, we will get Volterra's model of a population which pollutes its environment and pollution has an accumulative and toxic effect; such an equation is of the form

$$\frac{dN(t)}{dt} = N(t) = \left\{ r - (r/K) N(t) - \int_{-\infty}^{t} k(t - s) \right\} F(N(s)) \, ds \quad (1.4)$$

Which has been studied in different formats by several authors (Cushing [12], Cohen *et al.* [10], and Gopalasamy [16])

One of the significant advantages of deriving various models from the prototype (1.2) is that if the initial; conditions (initial population sizes) are nonnegative then the nonnegativity of the population size $N(t)$ for $t > 0$ follows from the fact that solutions of (1.2) satisfy

$$N(t) = N(0) \exp\left[\int_0^1 f\left(s, N(s)\right) ds\right]$$

Which implies that $N(t) \geq 0$ if $N(0) \geq 0$.

While the time delayed logistic equation (1.3) explains some of the observed oscillations of certain populations, its derivation has not been entirely satisfactory. For instance it has been pointed out by Ricklefs ([27], page 488) that

"time lags might be expected to occur primarily in stabilizing processes involving production rather than death because death is an immediate response to environmental change". Recently Cushing [13] has examined the effects of response and maturation delays in a number of single and multi system models; as it has been noted by Cushing [13], the models involving delayed reproduction or not necessarily derivable from the Kolmogorov type systems of the form (1.2) by suitably defining the average growth rate $(1/N)\ (dN/dt) = f(t, N(t))$. While Cushing [13] has been mostly concerned with the possibility of delay induced oscillations we will consider a class of delays in reproduction and interspecific interaction which maintain the stability of the systems.

Our starting point to drive models with delayed reproduction is the following balance equation which assumes that there is no immigration or emigration. Account, particle, atom, molecule, proposition, problem et al

$$\frac{dN(t)}{dt} = \text{birth rate} - \text{death rate.} \tag{1.5}$$

Now for instance if we consider a population of adult flies then the birth rate at time $t$ depends on the adult population at time $t - \tau$ where $\tau$ is the time required for the larvae to become adult flies. If the birth and death rates are governed by density dependent factors then we have from (1.5) the following time delayed model

$$\frac{dN(t)}{dt} = b\big(N(t-\tau)\big) - m(N(t)) \tag{1.6}$$

Where the functions $b$ and $m$ denote the density dependent birth and death rates of Account, particle, atom, molecule, proposition, problem et al respectively. If the time delay is continuously distributed then one can consider in the place of (1.6) an equation of the form

$$\frac{dN(t)}{dt} = b\int_{-\infty}^{t} k\{(t-s)N(s)ds - m(N(t))$$

Or equivalently

$$\left\{\frac{dN(t)}{dt} = b\int_{0}^{\infty} k(s)\right\}N(t-s)ds - m(N(t)) \tag{1.7}$$

Where the delay kernel $k$ denotes the weight of the delayed of the delayed effects.

It is not always obvious that suitable nonnegative initial conditions for (1.6) and (1.7) will imply the nonnegativity of the solutions of (1.6) and (1.7). The purpose of this article is to show that in two species systems of competitive

and prey- predator communities, delayed reproduction and delayed interactions donot destabilise the systems if the self- regulating negative feedback effects are sufficiently strong compared to interspecific interactive effects and if the self regulating effects are realised with no time delays. When we say that time delays do not destabilise, we will mean that delays do not render an asymptotically (locally) stable steady state unstable in such a manner that the loss of stability either leads

To a delay – induced bifurcation to persistent and undamped oscillations (as in the case of Hopf- bifurcation) or the relevant linear variational system has unbounded solutions; that is a mathematical instability is not induced by delays in the sense, that the roots of the characteristic equation are pushed to the imaginary axis or to the right of it on the complex plane. However delays can make an otherwise stable system less stable which means that the rate of convergence to zero of solutions of the variational system can decrease with an increase of delay and the variational system can even oscillate (perhaps violently) before converging to the trivial state. Such a reduction in stability can happen even when all the roots of the characteristic equation have negative real parts. For our purposes such a reduced stability is still stability. In order to see the effects of delays on such less stable systems we will propose below a method of estimating the rate of decay of local perturbations based on the variational system and indirectly relying on the characteristic equation. For a special class of two species models with a single delay we have obtained (Gopalsamy [16]) estimates of decay rates (or return times) usingthe characteristic equation; such a procedure is almost impossible in the case of many species systems with several delays. The method we propose in Section 6 below for the estimate of decay rates can be adapted to *n*- species systems with $n^2$ discrete delays.

Let $x_1(t)$ and $x_2(t)$ denote the population sizes (or biomasses) of two systems competing for a common pool of resources with common pool of resources.. Let $b_i$ and $m_i$ $(i = 1, 2)$ denote the respective density dependent birth (potential) and death rates so that in the absence of any time delays the population sizes are governed by (of Account, particle, atom, molecule, proposition, problem et al). In addition as usual, we assume that the system studied follow Haimovici's equations, for which the dove tailing explanation has been appended

$$\frac{dx_1(t)}{dt} = b_1\left(x_1(t)\right) - m_1(x_1(t)), x_2(t))$$

$$\frac{dx_2(t)}{dt} = b_2\left(x_2(t)\right) - m_2(x_1(t)), x_2(t))$$

(2.1)

In order to make the system (2.1) represent a competition system we will make the following assumptions on the birth and death rates;$_1$, $m_1$ *(i* = 1, 2) are continuous with continuous partial derivatives for all $x_1$ *(I = 1, 2);* also we assume (see for instance Rescigno and Richardson [26]) $\frac{\partial b_i}{\partial x_1} > 0, \frac{\partial m_1}{\partial x_j} > 0$ for $x_i > 0$ *(i, j* = 1, 2), (2.2)

(xxxiv) $b_1(0) = 0, \quad m_1(0, x_2) \equiv 0, b_2(0) = 0, \quad m_2(x_1, 0) \equiv 0,$

(2.3)For some $x_{1,}^* > 0, x_{2,}^* > 0$ we have

$$b_1(x_1^*, -m_1(x_1^*, 0) = 0,$$

$$b_2(x_2^*, -m_1(0, x_2^*) = 0,$$

(2.4)

(xxxv) There exist positive constants $\delta_1$, $\delta_2$ such that

$b_1(\delta_1) - m_1(\delta_1, x_2) < 0$ ,

$b_2 \quad (\delta_2) - m_2(x_1, \delta_2) < 0,$ for $x_1 \geq 0, \quad x_2 \geq 0,$

(2.5)

(xxxvi) There exists a point $(\alpha, \beta), \alpha > 0, \beta > 0$ such that

$$b_1(\alpha) - m_1(\alpha, \beta) = 0 = b_2(\beta) - m_2(\alpha, \beta).$$

(2.6)

The conditions on $b_i$ in (2.2) mean that the potential (not average) birth rates are density dependent and any negative density dependent feedbacks to the opened or birth rate of Account, particle, atom, molecule, proposition, problem et al rates are included in the destruction rates; the conditions on $m_i$ indicate intraspecific and interspecific confederation needs and necessities. (2.3) implies that (0, 0) is a (trivial) steady state of the system (2.1) as it is in the case of all ecological models; (2.4) means that in the absence of either of the two species the other one has a nontrivial steady state; (2.5) implies that each of the two populations cannot grow unboundedly since when $x_1(t) > \delta_i, \frac{dx_1}{dt} < 0$. (2.6) means that there exists a nontrivial steady state $(\alpha, \beta)$ in the interior of the positive quadrant of the ($x_1$ , $x_2$)- plane. These assumptions are standard and are minimal in the opinion of the author.

An immediate example of (2.1) –(2.6) is the familiar Volterra –Lotka competition system described by

$$\frac{dx_1(t)}{dt} = r_1 x_1(t) - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t),$$

$$\frac{dx_2(t)}{dt} = r_2 x_2(t) - a_{21}x_1(t)x_2(t) - a_{22}x_2^2(t),$$

(2.7)

Where $r$, $a_{ij}$ , $(i, j = 1,2)$ are positive constants satisfying the condition

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

One of the first questions to be resolved for (2.1) is the following: if $x_i$ (0) > 0, $x_2$ (0)> 0 does it follow that the corresponding solutions of (2.1) satisfy the condition $x_1(t) \geq 0$, $x_2(t) \geq 0$ for those $t$ for which a solution of (2.1) exists? In the $(x_1, x_2)$ phase plane of (2.1) we have $(0, 0), (x_1^*, 0), (0, x_2^*), (\alpha, \beta)$ as the steady states of (2.1) and also it will follow from (2.3) that the co ordinate axes

$x_1 = 0$ and $x_2 = 0$ are invariant sets f (2.1) ; such invariance of the coordinate axes will imply that if $x_1(0) > 0$, $x_2(0) > 0$ then the corresponding population trajectory $(x_1(t), x_2(t))$ of (2.1) cannot reach the outside of the nonnegative quadrant $x_1 \geq 0$, $x_2 \geq 0$.

Our plan is to derive first a set of sufficient conditions for the coexistence of the two competing species and then in the next section show that under the same conditions the two species can coexist if there are time delays in the reproduction and interspecific interactions. We recall that the two competing species described by (2.1) are said to coexist if and only if the nontrivial steady state $(\alpha, \beta)$ is (at least locally) asymptotically stable. To examine such a stability of $(\alpha, \beta)$ we derive the corresponding linear variational system for the perturbations $X_1, X_2$ where

$$x_1(t) \equiv \alpha + X_1(t), \qquad x_2(t) \equiv \beta + X_2(t)$$

Such a variational system is found to be

$$\frac{dX_1(t)}{dt} = \beta_{11} X_1(t) + \beta_{12} X_2(t),$$

$$\frac{dX_2(t)}{dt} = \beta_{21} X_1(t) + \beta_{22} X_2(t), \qquad (2.8)$$

Where

$$\beta_{1J} = \left(\frac{\partial b_1}{\partial x_J} - \frac{\partial m_1}{\partial x_j}\right) \text{ evaluated at } (\alpha, \beta) \qquad (i, j = 1,2).$$

(2.9)

The steady state $(\alpha, \beta)$ of (2.1) is locally asymptotically stable if and only if the trivial solution (0, 0) of (2.8) is asymptotically stable. We can derive the following result from elementary considerations.

In the system (2.1)-(2.6) assume the following:

$$\left.\begin{aligned} \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \\ \frac{\partial m_1}{\partial x_1} &> \frac{\partial b_1}{\partial x_1} + \frac{\partial m_2}{\partial x_1}, \end{aligned}\right\} \text{ at } (\alpha, \beta)$$

(2.10)

Then the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (2.1) – (2.6).

The proof follows easily if we note that the characteristic equation associated with the linear system (2.8) is given by

Det $\begin{pmatrix} \lambda-\beta_{11} & -\beta_{12} \\ -\beta_{21} & \lambda-\beta_{22} \end{pmatrix} = 0$

Or equivalently,

$$\lambda^2 - \lambda\,(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 0. \qquad (2.11)$$

It follows from (2.2) and (2.10) that,

$$\beta_{11} < 0, \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

And hence the roots of (2.11) have negative real parts implying the asymptotic stability of the trivial solution of (2.8)

If apply the condition (2.10) to the Volterra-Lotka model (2.7) we note that when a nontrivial steady state say $(\alpha, \beta)$ exists for (2.7), (2.10) leads to

$$a_{11}\alpha > a_{21}\,\beta \text{ and } a_{22}\,\beta > a_{12}\,\alpha,$$

Which together imply $a_{11}\,a_{22} > a_{12}a_{21}$. If $(\alpha, \beta)$ exists it is known that $a_{11}\,a_{22} > a_{12}a_{21} > 0$ is a sufficient condition for the (local) asymptotic stability of $(\alpha, \beta)$.

The condition (2.10) which guarantees the coexistence of competing populations in (2.1)-(2.6) simply means that the intraspecific self – regulating feedback effects $\partial m_i / \partial x_1\ (i = 1, 2)$ bare higher than its own positive feedback $\partial b_1 / \partial x_1\ (i = 1, 2)$ together with its negative (competitive) effects on its competitor $\partial m_I / \partial x_J\ (i,\ j = 1, 2; i \neq j)$.

## 24. Coexistence under delayed production and interactions, transactions

We will now establish that two competing species with a sufficiently strong intraspecific regulation can coexist in the presence of delayed reproduction and delayed interspecific interactions. It is well known that if there are delays in the self – regulating intraspecific effects, stationary coexistence may not always be possible (Gopalsamy [16], Gopalsamy and aggarwala [15], Shibata and Saito [28]). Delayed reproduction is more general and widespread in most biological populations. To build a competition model with such delays we proceed as follows.

Let $\tau_{ij}$ $(i,j=1,2)$ be a set of nonnegative constants with $\tau = \max\ \tau_{ij}\ , i,j=1,2,$ and suppose that the two ompeting species display delayed reproduction and delayed interspecific interactions with no delays in the self – regulating negative feedback effects. Such a competition system in a constant environment can be modeled by the autonomous delay differential system

$$\frac{dx_1(t)}{dt} = b_1\big(x_1(t-\tau_{11})\big) - m_1\big(x_1(t), x_2(t-\tau_{12})\big),$$

$$\frac{dx_2(t)}{dt} = b_2\big(x_2(t-\tau_{22})\big) - m\big(x_1(t-\tau_{21}), x_2(t)\big), \qquad (3.1)$$

Where both the creation and destruction rates and the morality rates $b_1$ and $m_1$ $(i=1,2)$ satisfy the same conditions (2.2)-(2.6) as in the previous section. Along with (3.1) we will

Suppose that the initial population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) sizes are provided in the form

$$x_i(s) = \phi_i(s)\ \ 0, \qquad s\ \in [-\tau, 0], \qquad i=1,2. \qquad (3.2)$$

(Note that the specification of $\phi_i$ need not be on the entire interval$[-\tau, 0]$; but such a hypothesis is convenient.) Since (3.1) is not of the standard Kolmogorov-type, we have to establish that the solutions of (3.1) will remain nonnegative for nonnegative initial conditions.

To prove the nonnegativity of solutions of (3.1)-(3.2), let $\{\psi_1^{(n)}(s)\}, s\in[-\tau,0], i=1,2; (n=1,2,\dots)$ be a sequence of strictly positive continuous functions such that

$$\lim_{n\to\infty} \psi_1^{(n)}(s), = \psi_1\ (s), \qquad s\in[-\tau,0], \qquad i=1,2 \qquad (3.3)$$

Let $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ be the solution of (3.1) corresponding to the initial condition

$$\{x_1^{(n)}(t), x_2^{(n)}\ (t)\} = \{\psi_1^{(n)}(s), \psi_2^{(n)}(s)\}, \quad s\in[-\tau,0],. \qquad (3.4)$$

Consider the solution $\{x_1^{(n)}(t), x_2^{(n)}\ (t)\}$ for $t\ \in [\ 0,\ \tau]$ (assuming it exists on the whole of $[\ 0,\ \tau]$). Suppose $x_1^{(n)}$ does not remain nonnegative for all $[\ \ 0,\ \ \tau^*]$; then there exists a $t^* = [0,\tau^*](\tau^* =$ the positive minimum of $\tau_{ij}\ (i,j=1,2))$ for which

$$x_1^{(n)}(t^*) = 0 \ and \ x_1^{(n)}(t)(t^* - \tau_{11})) > 0.$$

It will follow from (3.1) and the positivity of the initial condition that

$$\frac{dx_1^{(n)}(t^*)}{dt} = b_2\left(x_1^{(n)}(t^* - \tau_{11})\right) > 0 \qquad (3.5)$$

Where we have used the properties of $b_i$ and $m_i$ from (2.2)-(2.6); it is found that (3.5) contradicts the definition of $t^*$. Thus $x_1^{(n)}(t) > 0$ on $[0, \tau^*]$; similarly $x_2^{(n)}(t) > 0 \ on \ [0, \tau^*]$. We can repeat this procedure for intervals of the form $[\tau^*, 2\tau^*], [2\tau^*, 3\tau^*] \ etc$. Thus it will follow that so long as $(x_1^{(n)}(t), \ x_2^{(n)}(t))$ exists, we have

$$x_1^{(n)}(t) \geq 0 \ x_2^{(n)}(t) \geq 0.$$

If we now consider the limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{x_1^{(n)}(t), \ x_2^{(n)}(t)\right\} = \{x_1(t), x_2(t)\} \qquad (3.2)$$

With $x_1(t)$ , $x_2(t)$ where $(x_1(t), x_2(t))$ is the solution of (3.1)-(3.2) and this is a consequence of the continuous dependence of solutions of (3.1) on initial conditions (Hale [17], page 41).

A second question for (3.1) with (2.2)-(2.6) is whether solutions of (3.1) exist for all $t > 0$? suppose now that solutions of (3.1) do not exist for all $t$ ; then there exists a $t_1 > 0$ such that at least for one of either of $x_1 \ or \ x_2$ we have

$$\lim_{t\to t_1-} x_i(t) = \infty. \qquad (3.7)$$

To be specific let us suppose that $\lim_{t\to t_1-} x_i(t) = \infty$; let $t_2$ be the first instant of time for which $x_1(t_2) = \delta_1, t_2 < t_1$. It will follow from (3.1) that

$$\frac{dx_1(t_2)}{dt} = b_1(x_1(t_2 - \tau_{11})) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(x_1(t_2)) - m_1(x_1(t_2), x_2(t_2 - \tau_{12}))$$

$$< b_1(\delta_1) - m_1(\delta_1, x(t_2 - \tau_{12})) < 0 \qquad (3.8)$$

Which contradicts the definition of $t_2$. Thus solutions of (3.1), (2.2) – (2.6) exist for all $t$ .

We are now ready to consider the effects of time delays in reproduction and interspecific interactions on the coexistence of the two competing species. The steady state $(\alpha, \beta)$ of (2.1) is again a steady state of (3.1). To examine the (local) asymptotic stability of $(\alpha, \beta)$ in (3.1) we let

$$x_1(t) = \alpha + Y_1(t), \quad x_2(t) = \beta + Y_2(t)$$

In (3.1) and derive that the linear variational system governing the perturbations $Y_1, Y_2$ is given by

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + b_{21}Y_1(t-\tau_{21}) - m_{22}Y_2(t-\tau_{22}),$$

Where

$$m_{ij} = \frac{\partial m_1}{\partial x_j}, \quad b_{ii} = \frac{\partial b_i}{\partial x_i} \; at \; (\alpha, \beta). \tag{3.10}$$

It is well known (Bellman and Cooke [3], page 336) that the steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) if the trivial solution of the variational system (3.9) is asymptotically stable. We formulate our first result in the following form whose proof is given in the appendix.

*THEOREM 3.1. Assume that in the two systemic types competition model system (3.1) with delayed density dependent birth and death rates $b_i$ and $m_i$ $(i = 1, 2)$ satisfy the conditions (2.2)-(2.6). Suppose that the conditions (2.10) hold. Then the nontrivial steady state $(\alpha, \beta)$ is (locally) asymptotically stable for (3.1) whatever the delays be $\tau_{ij}$ 0 $(i = 1, 2)$.*

Let $y_1(t) and\ y_2(t)$ respectively denote the population sizes (or biomasses) of a prey species and a predator species which feeds exclusively on the single prey species in a temporally constant environment. In the format of (1.5) the following system will describe the rates of change of the two population sizes:

$$\left.\begin{aligned} \frac{dy_1(t)}{dt} &= f_1(y_1(t)) - g_1(y_1(t), y_2(t)), \\ \frac{dy_2(t)}{dt} &= f_2(y_1(t)) - y_2(t) - g_2(y_2(t)), \end{aligned}\right\} \tag{4.1}$$

Where $f_i \; and \; g_i \; i = 1,2)$ respectively denote the birth and death rates of the two species; in order to make (4.1) denote the dynamics of a prey – predator species we will assume that $f_i \; and \; g_i$ satisfy the following conditions (see for instance Rescigno and Richardson [26]); $f_i \; and \; g_i$ are continuous with continuous partial derivatives for all $y_i \geq 0 \; (i = 1, 2); also \; we \; require$

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial y_1} > 0; \quad \frac{\partial g_1}{\partial y_1} \; 0; \quad \frac{\partial g_1}{\partial y_2} \; 0, \\ &\frac{\partial f_2}{\partial y_1} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0; \quad \frac{\partial g_2}{\partial y_2} \; 0, \end{aligned}\right\} \text{Where } y_1 \; 0, y_1 \; 0, \tag{4.2}$$

(xlv)

$$\left.\begin{aligned} f_1(0) &= 0, \quad g_1(0, y_2 \equiv 0, \\ f_2(0, y_2) &= 0 = f_2(y_1, 0); \quad g_2(0) = 0, \end{aligned}\right\} \tag{4.3}$$

(xlvi) For some constant $y_1^* > 0$ we have

$$f_1(y_1^*) - g_1(y_1^* 0) = 0, \tag{4.4}$$

(xlvii) There exists a point $(\alpha^* \beta^*), \alpha^* > 0, \beta^* > 0$ such that

$$\left.\begin{aligned} f_1(\alpha^*) - g_1(\alpha^*, \beta^*) &= 0, \\ f_2(\alpha^*, \beta^*) - g_2(\beta^*) &= 0, \end{aligned}\right\} \tag{4.5}$$

(xlviii) There exist positive constants $n_1, n_2$ such that

$$f_1(\eta_1) - g_1(\eta_1, y_2) < 0, y_2 \geq 0,$$

$$f_2(\eta_1, \eta_2) - g_2(\eta_2) < 0 \quad (4.6)$$

The conditions (4.2) mean that the system (4.1) is of the oppugnacy systems type; (4.3) will imply that (0,0) is a trivial steady state of (4.1); (4.4) means that in the absence of the predator, the prey species has a nontrivial (positive) steady state;

(4.5) implies that (4.1) has a nontrivial steady state in the interior of the positive quadrant of the state species of (4.1); (4.6) can be shown (see below) to imply that no credit type and debit type system types can grow unboundedly.

An example of (4.1)-(4.6) is the familiar Volterra-Lotka prey-predator system

$$\frac{dy_l(t)}{dt} = r_l y_l(t) - [a_{ll} y_l^2(t) + a_{12} y_l(t) y_2(t)],$$

$$\frac{dy_2(t)}{dt} = a_{12} y_1(t) y_2(t) - [r_2 y_2(t) + a_{22} y_2^2(t)],$$

Where $r_l, a_{ij}\ (i, j = 1,2)$ are positive costants such that

$$a_{21} r_1 - r_2 a_{11} > 0.$$

As in the previous case of competition we have to verify that when $y_1(0) > 0, y_2(0) > 0$ the corresponding solution of (4.1) will be nonnegative for $t \geq 0$ for which the solution exist. Since the co-ordinate axes $y_1 = 0, y_2 = 0$ are invariant sets for (401) by virtue of (4.3), it will follow that the solution of (4.1) starting from the interior of the positive quadrant cannot reach the outside of that quadrant. This is a consequence of the invariance of the co-ordinate axes for (4.1)

To prove that solutions of (4.1)-(4.6) exist for all $t \geq 0$ let us suppose that $y_l 0\ and\ y_2(0) satisfy\ 0 < y_l(0)$ satisfy $0 < y_l(0) < \eta_l, 0 < y_2(0) < \eta_2$ ( on the other hand if either one of $y_l(0) \geq \eta_l and\ y_2(0) \geq \eta_l$ or both hold, it will follow that the corresponding $y_l(t)$ will decrese by virtue of (4.6) at least for small $t > 0,$ ). Suppose $y_l(t)$ does not exist for all $t \geq 0$; then for some $t_l < \infty\ lim_{t \to t_l -} y_l(t) = \infty$ and let $t_2 > 0$ be the first instant for which $y_1(t_2) = \eta_1$; then we have from (4.1) and (4.6) that

$$\frac{dy_1(t_2)}{dt} = f_1(\eta_1) - g_1(\eta_1, y_2) < 0\ (assumimg\ y_1(0) < \eta_1)$$

Which contradicts the definition of $t_2$ showing that $y_1(t)$ exists for all $t \geq 0$; if $y_2(t)$ is not defined for all $t \geq 0$, then for some $t_3 \geq \infty$ we will have $lim_{t - t_3 -} y_2(t) = \infty$;now let $t_4$ be the first instant for which $y_2(t_4) = \eta_2$; then from the properties of $f_2$ we have

$$\frac{dy_2(t_4)}{dt} = f_2(y_1(t_4), \eta_2) - g_2(\eta_2) \leq f_2(\eta_1, \eta_2) - g_2(\eta_2)$$

$$< 0 (by\ (4.6) assuming\ 0 < y_1(0) < \eta_1, 0 < y_2(0) < \eta_2)$$

Which again contradicts the definition of $t_4$. Thus $y_2(t)$ is defined for all $t \geq 0$.

We will now derive a set of sufficient condition for the (local) asymptotic stability of the steady state $(\alpha^*, \beta^*)$ of (4.1). in the next section we will show that under the same conditions and in the presence of various delays , the steady state $(\alpha^*, \beta^*)$remains (locally) asymptotically stable.

To examine the asymptotic stability of the nontrivial steady state $(\alpha^*, \beta^*)$ in (4.1) we let

$$y_1(t) = \alpha^* + U_1(t), \qquad y_2(t) = \beta^* + U_2(t)$$

In (4.1) and derive the liner variational system in the perturbations $U_1, U_2$ as follows:

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &= \frac{\partial f_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_1}U_1(t) - \frac{\partial g_1}{\partial y_2}U_2(t), \\ \frac{dU_2(t)}{dt} &= \frac{\partial f_2}{\partial y_1}U_1(t) - \frac{\partial f_2}{\partial y_2}U_2(t) - \frac{\partial g_2}{\partial y_2}U_2(t), \end{aligned}\right\} \quad (4.7)$$

Where all the partial derivatives in (4.7) are evaluated at $(\alpha^*, \beta^*)$. The steady state $(\alpha^*, \beta^*)$ of (4.1) is (locally) asymptotically stable if and only if the trivial solution of (4.7) is asymptotically stable which will be the case if and only if the roots of the characteristic equation

$$\text{Det} \begin{pmatrix} \lambda - (f_{11} - g_{11}) & g_{12} \\ -f_{21} & \lambda - (f_{22} - g_{22}) \end{pmatrix} = 0,$$

Where $\left.\begin{aligned} f_{ij} &= \frac{\partial f_1}{\partial y_l}, \\ g_{ij} &= \frac{\partial g_1}{\partial y_j}, \end{aligned}\right\}$ at $(\alpha^*, \beta^*)$ (4.8)

$(i,j = 1, 2)$

Have negative real parts. If $\lambda_1, \lambda_2$ are the riits of (4.8) then

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= (f_{11} - g_{11}) + (f_{22} - g_{22}), \\ \lambda_1\lambda_2 &= (f_{11} - g_{11})(f_{22} - g_{22}) + f_{21}g_{12}. \end{aligned}\right\} \quad (4.9)$$

The proof of the following result is an immediate consequence of (4.7)-(4.9)

THEOREM 4.1 *In the CREDIT-DEBIT system (4.1)-(4.6) assumes that the partial derivatives satisfy the following additional conditions:*

$$\left.\begin{aligned} \frac{\partial g_1}{\partial y_1} &> \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_1}, \\ \frac{\partial g_2}{\partial y_2} &> \frac{\partial f_2}{\partial y_2} + \frac{\partial g_1}{\partial y_2}, \end{aligned}\right\} \text{ Evaluated at } (\alpha^*, \beta^*), \quad (4.10)$$

*Then the steady state $(\alpha^*, \beta^*)$, is (locally) asymptotically stable for the prey- predator system.*

## 16. Delays in reproduction, predation and gestation

A number of authors have proposed and analysed several models of prey- predator systems with time delays. Most of the models have been derived from the so called models of the Kolmogorov-type f (1.2) with time delays incorporated in the average growth rates of the prey and predator species. Volterra [32] proposed the following model for a prey-predator system. The equations are used for the formulation of Credit-debit system of atoms, accounts, economic variable, cosmological variable, astrophysical; variables, and qubit systems for which the model is being applied.

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \delta_1 N_1(t) - \int_{\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 + \delta_2 N_1(t) - \int_{\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.1)$$

Where $r_1, r_2, \delta_1, \delta_2$ *are positive constant and* $F_1, F_2$ is nonnegative continuous delay kernel defined and integrable on $[0, \infty]$. (5.1) has no predator self-regulating negative feedback. Brelot [6] considered a modified format of (5.1) in the form

$$\frac{dN_1(t)}{dt} = N_1(t)[r_1 - \lambda_1 N_1(t) - \delta_1 N_2(t) - \int_{-\infty}^{t} F_1(t-s)N_2(s)ds],$$

$$\frac{dN_2(t)}{dt} = N_2(t)[-r_2 - \lambda_2 N_2(t) - \delta_2 N_1(t) - \int_{-\infty}^{t} F_2(t-s)N_1(s)ds], \quad (5.2)$$

Which has term representing intraspecific competition among both predator and prey species. Under suitable condition on the various parameters in (5.2) one can show that (5.2) has a (locally) asymptotically stable steady state. A number of intergrodifferential equation models of the types (5.1) and (5.2) have been investigated by Cushing [12]

Starting from Hutchinson's (1948)[19] delay logistic equation, May[23] has proposed the following system

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t-\tau)}{K}\right] - \alpha N_1(t)N_2(t),$$

$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_2(t)N_1(t), \quad (5.3)$$

Where r, $\tau$, K, $\alpha, \beta, b$ are positive constants; (5.3) contains a single discrete delay; one can modify (5.3) and incorporate a continuously distributed delay in it so that (5.3) can be written as (May [23])

$$\frac{dN_1(t)}{dt} = rN_1(t)[1 - \int_{-\infty}^{t} F(t-s)N_1(s)ds] - \alpha N_1(t)N_2(t),$$
$$\frac{dN_2(t)}{dt} = -bN_2(t) + \beta N_1(t)N_2(t). \quad (5.4)$$

The first model of a prey-predator type which departs from the Kolmogorov-type formulation is due to Wangersky and Cunningham [33] who considered the model system

$$\frac{dN_1(t)}{dt} = a_1N_1(t) - b_1N_1^2(t) - c_1N_1(t) - N_2(t),$$
$$\frac{dN_2(t)}{dt} = a_2N_2(t) + c_2N_1(t-\tau)N_2(t-\tau), \quad (5.5)$$

Where $a_1, a_2, b_1, c_1, c_2, \tau$ are positive constants; (5.5) means that a duration of $\tau$ time units elapses when an individual prey is killed and the moment when the corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) Recently Cushing [13] has considered a model of the form

$$\frac{dN_1(t)}{dt} = rN_1(t)\left[1 - \frac{N_1(t)}{k} - \alpha N_2(t)\right],$$
$$\frac{dN_2(t)}{dt} = -d_2^*(t) + bN_1(t)\int_0^{\infty} \delta(a)N_2(t-a)e^{-d*a}\,da,$$
(5.6)

Where $r, k, \alpha, d^*, b$ are positive constants and $\delta$ is related to the age dependent fecundity of the predator species.

With the models (5.1)-(5.6) in the background let us now consider a time delayed model similar to (4.1) in the following form:

$$\frac{dN_1(t)}{dt} = f_1\big(N_1(t-\tau_{11})\big) - g_1\big(N_1(t), N_2(t-\tau_{12})\big),$$
$$\frac{dN_2(t)}{dt} = f_2\big(N_1(t-\tau_{21})\big), N_2(t-\tau_{12}) - g_1(N_2(t)),$$
(5.7)

Where $\tau_{ij}$ $(i,j=1,2)$ are nonnegative constants with $\tau=\max\tau_{ij}, i,j=1,2$; the birth and death rate functions $f_1,f_2,g_1,g_2$ satisfy the same conditions as in (4.1)-(4.6). Along with (5.7) we specify the initial conditions as follows:

$$N_1(s)=\phi_1(s)>0,\quad s\in[-\tau,0].$$

$$N_1(s)=\phi_1(s)>0,$$

We3 will assume that in addition to (4.2) we have for (5.7) that

$$\frac{\partial f_2}{\partial N_1}>0,\frac{\partial f_2}{\partial N_2}>0, f_2(N_2,N_2)>0 \text{ whenever } N_1>0,\ N_2>0. \quad (5.9)$$

It is not difficult to show by the methods previously used for the competition system that whenever $N_1(s)>0$ and $N_2(s)>0$ for $s\in[-\tau,0]$, solution of (5.7) exist for all $t\geq 0$ and remain nonnegative for all $t\geq 0$. also $(\alpha^*,\beta^*)$ of (4.5) is a steady state of (5.7).

We will discuss the form and meaning of the delays in (5.7) in the next section. The effects of the various delays in (5.7) on the (local) asymptotic stability of $(\alpha^*,\beta^*)$ can be derived from the following theorem whose proof is quite similar to that of Theorem 3.1 and hence the details of the proof are omitted.

THEOREM 5.1 *In the time delayed prey-predator system(* corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) *) (5.7) let the conditions (4.2)-(4.6), (5.8) and (5.9) hold for the birth and death rates; furthermore suppose the condition (4.10) hold. Then for all nonnegative delays in (5.7), its nontrivial steady state $(\alpha^*,\beta^*)$ is (locally) asymptotically stable.*

## 6. Estimation of decay rates

We will first derive a convention sufficient condition for the asymptotic stability of the trivial solution of a linear system of the form

$$\frac{dY_1(t)}{dt}=-m_{11}Y_1(t)+b_{11}Y_1(t-\tau_{11})-m_{12}Y_2(t-\tau_{12}),$$

$$\frac{dY_2(t)}{dt}=-m_{22}Y_1(t)+m_{21}Y_1(t-\tau_{21})-b_{22}Y_2(t-\tau_{22}),\quad t>0, \quad (6.1)$$

Where$m_{1J}, b_{i1}$ $(i,j=1,2)$ are positive constants while $\tau_{1J}(i,j=1,2)$ are non negative with $\tau=\max(\tau_{1J})(i,j=1,2.$as is shown in the appendix, the following system dominates (6.1*)*

$$\frac{dZ_1(t)}{dt}=-m_{11}Z_1(t)+b_{11}Z_1(t-\tau_{11})-m_{12}Z_2(t-\tau_{12}),$$

$t > 0,$ (6.2)

$$\frac{dZ_2(t)}{dt} = -m_{22}\, Z_1(t) + m_{21} Z_1(t-\tau_{21}) - b_{22} Z_2(t-\tau_{22}),$$

$Z_i(s) = |Y_i(s)|, \qquad i = 1,2 \ \ for\ s \le 0 \ \ for\ which\ Y_i(s)\ is\ defined.$

Thus the asymptotic stability of the trivial solution of (6.2) implies that of (6.1). it is also shown in the appendix that the solutions of (6.2) are nonnegative for all t ≥ 0. we will use the following vector-matrix notation:

$$Z = (Z_1, Z_2)^{\tau}, \qquad ||Z|| = |Z_1(t)| + |Z_2(t)| = Z_1(t) + Z_2(t)$$

$$B = \begin{bmatrix} b_{11} & m_{12} \\ m_{21} & b_{12} \end{bmatrix}, \quad ||B|| = max \begin{matrix} b_{11} + & m_{21} \\ b_{21} + & m_{12} \end{matrix}, \qquad (6.3)$$

$$A = \begin{bmatrix} -m_{11} & 0 \\ 0 & -m_{22} \end{bmatrix}.$$

The logarithmic norm or the measure of any square matrix B, denoted by $\mu(B)$is defined as follows:

$$\mu(B) = \lim_{\theta \to 0+} \frac{/\!/I + \theta B/\!/ - 1}{\theta}$$

($I$ being an identity matrix of the same size as $B$).The matrix measure compatible with the norms in (6.3) is given by

$\mu(B) = \max_j [b_{jj} + \sum_{\substack{i=1 \\ i \neq J}}^{2} | b_{jj} | ]$ $B=(b_{1j})$.

We refer to Coppel [11]or Vidyasager [31]for a number of properties of matrix measure We have shown in the appendix that the derivative of $H(Z_1,Z_2)(t)$(seeA.10)satisfies the relation

$\frac{d}{dt} H(Z_1,Z_2)(t) \le [-m_{11}+b_{11}+m_{21}]Z_1(t)+[-m_{22}+b_{22}+m_{12}]Z_2(t)$, and hence we have

$\frac{d}{dt} H(Z_1,Z_2)(t) \le \{\mu(A) + /\!/B/\!/\}\{ Z_1(t)+ Z_2(t)\}$

If we assume for the matrix norm ‖B‖ as in (6.3)that

$$\mu(A) + /\!/B/\!/ < 0 \ or \ -\mu(A) > /\!/B/\!/$$

Then from the results of the appendix (Lemmas 5 and 6) it will follow that $Z_1(t)+Z_2(t) \to 0$ as $t \to \infty$ and thus we have the following:

Lemma 6.1 Assume that the Coefficients of the linear system (6.1)satisfy the condition (6.7). Then the trivial solution of(6.1)is asymptotically stable.

When (6.7)holds an estimate for the decay rate of solutions of (6.1)can be obtained as follows A necessary and sufficient condition for the asymptotic stability of the trivial solution is that all the roots of the characteristic equation

Det $G(\lambda,T_{iJ})=0$

Have negative real parts where

$$G(\lambda,T_{1J})=\begin{bmatrix}\lambda+m_{11} & 0\\ 0 & \lambda+22\end{bmatrix}-b_{11}^{e-\lambda T}{}_{11}\begin{pmatrix}-m_{12}^{e-\lambda T}{}_{12}\\ -m_{21}^{e-\lambda T}{}_{21} \quad b_{22}^{e-\lambda T}{}_{22}\end{pmatrix}$$

$=\lambda I-A-B(\lambda,T_{ij})$(say),

A positive number $\alpha$(if it exists)is said to be a decay rate of solution of (6.1) if and only if every root of (6.8)satisfies the condition

$Re(\lambda)+\boldsymbol{\alpha}<0.$

When (6.11)holds the solution of (6.1)will be of the order of $e^{-at}$ and hence the supremum of all such $\boldsymbol{\alpha}>0$ $a$ satisfying (6.11)will be a desirable parameter to estimate However such a supremum of all such (6.11)may not be possible to determine since it has to be done directly from (6.8)-(6,10)(see also Section 3).We will obtain an estimate of $\alpha$ (not necessarily the supremum)I (6.11)by the following procedure.

Replace $\lambda$ byz-$\boldsymbol{\alpha}$ in (6.8)and define

$G(z-\boldsymbol{\alpha},T_{IJ})=M(z)=(z-a)I-A-B(z-\boldsymbol{\alpha},T_{IJ})$.

By Lemma 6.1 a Sufficient condition for all roots of M(z)=0 to have negative real parts equivalently to satisfy

$Re(\lambda)<-a$

is the following :

$-\mu(aI+A)>\|B_1(a,T_{ij})\|,$

Where

$$B_1(a,T_{ij})=\begin{pmatrix}b_{11}^{e-\lambda T}{}_{11} & -m_{12}^{e-\lambda T}{}_{12}\\ -m_{21}^{e-\lambda T}{}_{21} & b_{22}^{e-\lambda T}{}_{22}\end{pmatrix}$$

Since $\mu(\boldsymbol{\alpha}\, I+A)=\boldsymbol{\alpha}+\boldsymbol{\mu}(\boldsymbol{A})$ we have from (6.13) that

$\boldsymbol{\alpha}+\mu(A)+\|B_I(\boldsymbol{\alpha},T_{IJ})\|<0$

Is a sufficient condition for Re $(\lambda) < -a$If we denote the left side of (6.15)by $\mathfrak{a}f(\mathfrak{a},T_{Ij})$ we note that

$f(0, T_{Ij}) = \|B\| + \mu(A) < 0 \;\; if \;\; (6.7) holds;$

Al so $\partial f / \partial \mathfrak{a} > 0, f$ is continuous and monotonic in $(\mathfrak{a}); \to \infty as \; \mathfrak{a} \to \infty$; thus by the continuity of $f$ on $\mathfrak{a} \geq 0$, there exists an interval of the from $[0, ^{*})$such that

$$f(0, T_{Ij}) < 0 \;\; for \; \mathfrak{a} \in [0, \mathfrak{a}^{*})$$

And hence

$$f(\mathfrak{a}, T_{Ij}) = 0$$

Thus an upper bound $\mathfrak{a}^{*}$ of $\mathfrak{a}$ for which we have $f(\mathfrak{a}, T_{Ij}) < 0 \;\; \mathfrak{a} \in [0, \mathfrak{a}^{*})$ can be obtained as the unique (by the monotonicity of $f$ on $\mathfrak{a}$) positive root ofmax $b_{11}^{e-\lambda T}{}_{11} - m_{12}^{e-\lambda T}{}_{21} + \mathfrak{a}$-min $. m_{11} =0$

$b_{22}^{e-\lambda T}{}_{22} - m_{21}^{e-\lambda T}{}_{12} \qquad m_{22}$

Where the maximum in (6.18) is taken over $b_{11}, b_{22}, m_{12}, m_{21}$ and $T_{ij} (i, = j \; 1,2)$. A numerical procedure can be devised to solve equations of the form (6.18) in concrete cases. If one is interested in a smaller bound on the decay rate it can be obtained as the unique root of

$$\|B\| e^{ar} + a + \mu(A) = 0, \tau = \max_{i,J} T_{ij}. \qquad (6.19)$$

It will follow from (6.19) than an increase in T

Will reduce the value of the root of (6.19). We remark that since our estimate $a^{*}$ of dacay rate is based on a sufficient condition of the form (6.7), we can conclude that the solotions of (6.1) will converge at least as fast as $e^{-a*t}$ where $a^{*}$ is the unique root of either of (6.18) or (6.19).

## Implications

Authors have formulated a class of two systems model systems staring from the elementary principal that the rate of change of a population confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) equals the creation rate minus the destruction rate. This is a departure from the conventional model building in population ecology where one usually starts from the Malthusian formulation, now known as the Kolmogorov formulation in the nonlinear context. In our analysis of the models proposed we have obtained sufficient conditions for the global existence of nonnegative solutions of the model systems (this is trivial in the Kolmogorov formulation) and the other desirable properties such as the existence of suitable steady states. The two classes of models proposed for competition and debt contain intraspecific self – regulation biologists that each systems is under the action of a self-regulating negative feedback force, opinions and arguments differ regarding the mechanism by which such an effect is realized; we refer for more details on this aspect to a collection of articles compiled by Tamarin [30].

Gopalasamy's main result is the following : if a self –regulating negative feedback mechanism (intraspecific oppugn ant) can act without time delay and if such an effect can keep an ecosystem ( a community of one or more

species of populations) asymptotically stable ( at least locally about a non trivial steady state, then time delays of any magnitude in any other interaction involving positive or negative feedbacks cannot destabilize such a stable ecosystem ( in a mathematical sense); that is such delays cannot cause a hopf-type bifurcation leading to persistent and undamped fluctuations. The delay-induced bifurcations to oscillations in models have been investigated by many authors. (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Qualitatively result is not new to the literature the effects of delays on otherwise stable and unstable systems have been considered by Cushing[12]where (pages 35-39)he has shown that an otherwise unstable system can be made less unstable by delays, Such an effect of delays on unstable systems appear to have been first proposed by Beddinton and May [2]who have also shows that delays can make stable systems less stable leading to a conclusion "time delays are not necessarily destabilizing" It is also shown by Cushing([12], pages 75-103)that while time delays cannot stabilize(in the mathematical sense)an otherwise unstable two species system in competition or mutualism time delays in there species systems can stabilize otherwise unstable systems (even in a mathematical sense)Cushing and Saleem [14]have formulated a model where delays in the positive feedback can have a stabilizing effect: Hale[18]discusses briefly the stabilizing nature of delays and given an example to show that an increase in delays can make an unstable system stable Thus for a better understanding of the effects of delays on otherwise stable systems one has to estimate the rate of decay of perturbations (or the characteristic return time )in terms of the parameters of the system especially when the delays do not cause a mathematical instability This aspect has been investigated by a procedure slightly different from that of the characteristic equation we have proposed (section6) method for the estimation of decay rate of perturbations when there are multiple delays Since many species systems with multiple delays cannot be analysed (due to complexity)by the characteristic equation our methods using comparison lead to easily verifiable sufficient conditions The sufficient conditions obtained are in the form of a diagonal dominance type; these conditions can be weakened to quasi-diagonal dominance type as in Ladde [20], Siljak[20]Anderson [1], Lewis and Anderson [21,22](Ladde's result [20] is delay dependent due to the fact that delay appears in self –regulation also).Verification of quasi –diagonal dominance conditions in arbitrary matrices (except for Metzler matrices )is an impossible task and quasi –diagonal dominance is hard to interpret.

If follows from our discussion that those confederation systems which display delay induced instability leading to permanent and undamped oscillations should have one or more of the following aspects:

(1)time delays are in intraspecific interaction terms (and possibly in other terms also): (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

(2) Intraspecific self-regulation terms are absent in density dependent from in one or more populations of the system;

(3) Intraspecific negative feedback effect are not strong enough to maintain the stability of the relevant steady state of the nondelayed system.

We can now infer how the delay induced stability leads to fluctuations; a primary mechanism of such fluctuations can be described at least theoretically as follows : in its attraction of a system to a steady state, intraspecific self-regulation fails to be effective and the system overhoots the steady state and at such higher densities intraspecific competition becomes operationally effective leading the system to understood the steady state and this process is repeated again and again resulting in undamped oscillations about the steady state. It follows

from our results that if intraspecific competition is sufficiently strong, the system cannot become oscillatory even when there are time delays, so long as they do not interfere with the strength of the intraspecific self – regulation.

We are thus led to the question of existence and importance of intraspecific competition in population systems. In the introduction to an anthology of articles by several authors, Tamarin [ 30] begins with the statement "all animal populations have the reproductive capacity to increase without limit, yet none di" and in his concluding remarks writes " this leaves us with only one great concept in the area of confederation do main regulation: that all credit-debit are regulated" More specifically this is applicable to all the confederation of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

In his discussion on the effect of competition on the densities of populations, Nicholson ([ 25], $page$ [ 147] ) has reached the conclution "Intraspecific competition automatically regulates the severity of its action to the requirements of each denizen , provided the inherent resistance of the environment is sufficiently low to permit the species to exist." In order to look for a mechanism by which intraspecific competition plays its role of decreasing growth rate as confederation density increases, Calhoun [ 7] set up an experiment with Norway rates (Rattus norvegicus) and found that increased density leads to increased hostility which disturbs the psychology and physiology causing decreased reproductive output. Many other mechanisms by which intraspecific compition self-regulate populations have been put forward by a number of authors (see Tamarin [30]). This is true of (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

If we accept the view that intraspecific competition is really predominant in population regulation, then we can understand how and why time delays destabilize stable system leading to oscillations; time delays render intraspecific competitive regulation territorially (in case of particles we take the field in which it operates) weak so that self –regulatory forces become temporarily insufficient. If we now agree that all population models should incorporate self-regulating intraspecific competition terms in surfeit strength along with other tropic intraspecific competition terms in sufficient strength along with other atrophic interactions then it is not fair to say " time delays destabilize" and we note from our analysis that it is usually a failure in self- regulation which can cause instability rather than delays by themselves.

We will give one example to illustrate our point. Consider a credit-debit system in which the debits is self – regulating by a logistic load as follows: (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

$$\frac{dx}{dt} = r\left(x - \frac{x^2}{K}\right) - axy = b_{I}(x) - m_{I}(x,y) \quad (say),$$

$$\frac{dy}{dt} = \beta xy - \delta y = b_{2}(x,y) - m_{2}(y) - \quad (say),$$

Where $K, a, \beta, \delta$ are positive constants such that the steady state $x^*\ y^*$ exists where

$$x^* = \frac{\delta}{\beta}; \quad y^* = \frac{r}{a}\left(-\frac{\delta}{K\beta}\right) > 0.$$

In this example the predator self –regulation $(\partial m_2/\partial y)$ is not sufficiently strong in the sense

$$x^* = \left(\frac{\partial m_2}{\partial y}\right)_{(x^* y^*)} \not> \left(\frac{\partial b_2}{\partial y} + \frac{\partial m_1}{\partial y}\right)_{(x^* y^*)}$$

If in such a model there is a time delay in the credit creation rate, such a delay can lead to instability of $(x^*\ y^*)$ of setting up undamped oscillations. If however we modify the growth rate by the inclusion of a density dependent self-regulation so that credit

$$\frac{dy}{dt} = \beta\, xy - \delta y - \eta y^2$$

For some positive constant η then it is possible, depending on η, for such a modified system to remain stable without oscillations even if there is a time delay in the credit creations rate. We have another completely different justification to have $\eta y^2$ in the credit dynamics: for instance if the debit is removed from this cannot happen. While this example serves to illustrate an aspect of our result, our advocacy of the significance is not based on this example.

Gopalasamy has thus derived conclusion that "time delays need not destabilize" in models with several discrete delays. However we have provided in the appendix the necessary tools (Lemmas 3,4,5) to prove all our results when the relevant delays are all (or some) continuously distributed. Some authors (Caswell [9], May [23], Cushing [12] appear to regard continuously distributed dalays as financially/cosmologically credit wise obtained when he subjected algae to a variable nitrate environment and obtained a better fit for his data with a continuous delay. The questions of whether a discrete or continuous delay is more realistic may well depend on what process is being modeled. For instance an increased supply of some resource at some point in time may trigger reproduction at a particular moment in time later irrespective of the conditions of the resource during the intervening period. If the duration of increased resource supply necessary to elicit such a response is short compared to the intervening period before the reproduction occurs, then a discrete time delay may be appropriate. In a large number of cases of animal. Note fungus algae are prey. Here it shall be an account which loses the energy, amount, or matter for that matter. Same result holds good for (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.)

Populations of particles, atoms, molecules, propositions, systems types chosen the past dietary and nutritional history of an animal over a long period plays an influential role in determining the current bahaviour of the dietary and nutritional history of an animal over a long period plays an influential role in determining the current behavior of the animal and in such cases continuous time delays will be appropriate.

In conclusion the following comments are appropriate. The time delays since the current creation rate is usually due to a contribution of those mature animals living a few units of time earlier depending on the type of animal or organism; the time delays appearing in interspecific interactions can be due to the fact that in order to successfully interact and compete interferentially a developmental maturity may be necessary. In the case of

predation, delays in the birth rate of prey can be due to delayed maturation while that in the birth rate of a predator is due to gestation delay with delayed fecundity; the delays in the predation rate are again are again as before due to the maturity needed for a predator to successfully organize and succeed in an attack.

In our analysis we have assumed throughout a temporally uniform environment neglecting dispersal of populations and their age structure. We expect our results to hold under suitable conditions both in temporally periodic environments with the steady state of a constant environment being replaced by a periodic solution (nonstationary steady state) and when the effects of dispersion and age structures are considered. It is worthwhile to modify our purely ecological, cosmological, financial, propositional, problematic systems models so as to provide them with a genetic foundation and examine the effects of time delays since forces of selection act with some time delay. Such a generalization will definitely lead to a better understanding of "how confederations regulate themselves".

## Appendix

First we derive a comparison result.

Lemma 1. Consider the delay differential system

$$\left.\begin{aligned}\frac{dY_1(t)}{dt} &= -m_{11}\ Y_1(t) + b_{11}Y_1(t-\tau_{11}) - m_{12}\ Y_2(t-\tau_{12}),\\ \frac{dY_2(t)}{dt} &= -m_{22}Y_2(t) - m_{21}Y_1(t-\tau_{21}) + b_{22}\ Y_2(t-\tau_{22}),\end{aligned}\right\} (A.1),$$

Where the $m_{ij}$, $b_{ij}$ $(i.j = 1,2)$ are positive while $\tau_{ij}$ $(i.j = 1,2)$ nonnegative with $\tau = \max \tau_{ij}$, $i,j = 1,2$. Then

$$\left.\begin{aligned}\frac{d|Y_1(t)|}{dt} &\le -m_{11}\,|Y_1(t)| + b_{11}|Y_1(t-\tau_{11})| + m_{12}\ Y_2(t-\tau_{12})|,\\ \frac{d|Y_1(t)|}{dt} &\le -m_{22}\,|Y_2(t)| + m_{21}|Y_1(t-\tau_{21})| + b_{22}\ |Y_2(t-\tau_{21})|,\end{aligned}\right\} (A.2)$$

Where

$$\frac{d}{dt}\,|\,Y_1(t)\,| = \left\{\begin{aligned}&\frac{dY_1(t)}{dt}\ if\ \ Y_1(t) \ge 0,\\ &\frac{dY_1(t)}{dt}\ if\ \ Y_1(t) < 0,\end{aligned}\right\}\ i = 1,2. \qquad (A.3)$$

The proof is by direct verification.

LEMMA 2. Consider now (A.2) and the following system

$$\left.\begin{aligned}\frac{dZ_1(t)}{dt} &= -m_{11}\ Z_1(t) + b_{11}Z_1(t-\tau_{11}) + m_{12}\ Z_2(t-\tau_{12}),\\ \frac{dZ_2(t)}{dt} &= -m_{22}Z_2(t) + m_{21}Z_1(t-\tau_{21}) + b_{22}\ Z_2(t-\tau_{22}),\end{aligned}\right\} \qquad (A.4)$$

Together with the initial conditions

$Z_1(s) > |Y_1(s)|, \quad s \in [-\tau, 0], i = 1,2.$ (A.5)

Then

$|Y_1(t)| < Z_1(t)$ for $t \geq 0$ and $i = 1,2.$ (A.6)

PROOF. Let $U_1(t) = Z_i(t) - |Y_1(s)|, \quad i = 1,2.$ $t \geq -\tau.$ Then we have from (A.2) and (A.4),

$$\left.\begin{aligned} \frac{dU_1(t)}{dt} &\geq -m_{11}\ U_1(t) + b_{11}U_1(t-\tau_{11}) + m_{12}\ U_2(t-\tau_{12}), \\ \frac{dU_2(t)}{dt} &\geq -m_{22}U_2(t) + m_{21}U_1(t-\tau_{21}) + b_{22}\ U_2(t-\tau_{22}), \end{aligned}\right\} t > 0, \quad \text{(A.7)}$$

With

$U_1(s) = Z_i(s) - |Y_i(s)| > 0, \quad i = 1,2, s \in [-\tau, 0].$ (A.8)

Consider (A.7)-(A.8) first on the interval $[-\tau, 0]$ : It will follow from (A.7) that $U_1(t) \geq U_1(0) \exp[-m_{11}t]$

$+\int_0^t [b_{11}U_1(s-\tau_{11}) + m_{12}\ U_2(s-\tau_{12})] \exp[-m_{11}(t-s)ds > 0,]$ (A.9)

For all $t \in [0, \tau^*]$ since $U_i(s) > 0$ on $[-\tau^* 0]$ where $\tau^*$ = positive minimum of $(\tau_{11,}\tau_{12,}\tau_{21,}\tau_{22,})$. Similarly $U_2(t) > 0$ for $t \in [0, \tau^*]$. Now by a similar argument it will follow that $U_1(t) > 0, U_2(t) > 0$ for $t \in [\tau^*, 2\tau^*]$. Repeating the above procedure we obtain the result (A.6).

It follows from Lemmas 1 and 2 that in order to prove theorem 3.1 it is enough to prove that $Z_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$ . which we will do as follows.

PROOF OF THEOREM 3.1 Define a Lyapunov-like function

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t)$

$+\int_{1-T}^t b_{11}\, Z_1(s)ds + \int_{1-T}^t m_{12}\, Z_2(s)ds$

$+\int_{1-T}^t m_{21}\, Z_1(s)ds + \int_{1-T}^t b_{22}\, Z_{22}(s)ds$

Since each term of H in (A.10)is nonnegative for t>0, if we show that $H \to 0$ as $t \to \infty$ it will follow from Lemma 2 that

$0 \leq |Y_{1,}(t)| \leq Z_1(t) \to 0$ as $t \to \infty$,

Implying that the perturbations $Y_{1,}$, $Y_2$, governed by (A.1)approach zero as $t \to \infty$, and this will prove our result Calculate the rate of change of H along the solutions of (A.4)so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z_1}\frac{dZ_1}{dt} + \frac{\partial H}{\partial Z_2}\frac{dZ_2}{dt}$$

$$= -[Z_1(t)\{m_{11} - (b_{11} + m_{12})\} + Z_2(t)\{m_{22} - (b_{22}m_{12})\}] < 0$$

On using (2.10)since $Z_1(t)$,$Z_2(t)$ are positive for all t>0. Thus $H(Z_1, Z_2)(t)\rightarrow 0$ as $t \rightarrow \infty$ since H is bounded below by zero: it follows from Lemma 6 below that $Z_1(t)\rightarrow 0$,$Z_2(t)\rightarrow 0$ as $t\rightarrow \infty$ the proof is complete.

The following set of results are intended to be a guide to prove result similar to the above in the case of continuously distributed delays An analogue of Lemma I is the following.

LEMMA 3 Let $K_{ij}(s)$,$s\in[0,\infty)$, $i,j = 1,2$,denote nonnegative continuous functions integrable on $[0,\infty)$ normalised such that

$\int_0^\infty sk_{ij}(s)ds< \infty$; $\int_0^\infty k_{ij}(s)ds=1$, I,j=1,2.

Consider the intergrodifferential system

$$\frac{dY_1(t)}{dt} = -m_{11}Y_1(t) + b_{11}\int_0^\infty k_{ij}(s)\, Y_1(t-s)ds$$

$$-m_{12}\int_0^\infty k_{12}(s)\, Y_2(t-s)ds$$

$$\frac{dY_2(t)}{dt} = -m_{22}Y_2(t) + m_{21}\int_0^\infty k_{21}(s)\, Y_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)\, Y_2(t-s)ds$$

*Where* $m_{ij}, b_{ij}$ $(i,j = 1,2)$ *are positive constants Then*

$$\frac{d|Y_1(t)|}{dt} \leq -m_{11}|Y_1(t)| + b_{11}\int_0^\infty k_{11}(s)|Y_1(t-s)|ds,$$

$$+m_{12}\int_0^\infty k_{12}(s)|Y_2(t-s)|ds,$$

$$\frac{d|Y_2(t)|}{dt} \leq -m_{22}|Y_2(t)| + m_{21}\int_0^\infty k_{21}(s)|Y_1(t-s)|ds,$$

$$+b_{22}\int_0^\infty k_{22}(s)|Y_2(t-s)|ds,$$

*Where* $d|Y_1(t)||/dt$ *is defined by* (A.3).

Proof is again by direct verification.

The proof of the following lemma is similar to that of Lemma 2 above.

Lemma 4. *Consider (A.14) and the following intergrodifferential system:*

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+m_{12}\int_0^\infty k_{12}(s)Z_2(t-s)ds,$$

$$\frac{dZ_1(t)}{dt} = -m_{11}Z_1(t) + b_{11}\int_0^\infty k_{11}(s)Z_1(t-s)ds$$

$$+b_{22}\int_0^\infty k_{22}(s)Z_2(t-s)ds,$$

Together with the initial conditions

$$\infty > Z_i(s) > |Y_i(s)| for\ s \in (-\infty, 0).$$

(A.16)

Then $|Y_i(t)| < Z_i(t) for\ t \geq 0, i = 1,2.$

The following lemma will be useful to prove a result analogous to that of Theorem 3.1 corresponding to a class of continuously distributed time delays.

LEMMA 5. Let $Z_1(t), Z_2(t)$ be the solutions of (A.15)- (A.16) where $(Y_1\ Y_2)$ corresponds to an arbitrary but fixed solutions of (A.13). Then as a consequence of Lemma & it will follow that

$$0 \leq |Y_i(t)| \ Z_i(t) \to \infty \leq as\ t \to \infty, \qquad i = 1,2,$$

(A.17)

Whenever

$$\left\{ \begin{array}{l} M_{11} > b_{11} + m_{21}, \\ M_{22} > b_{22} + m_{12} \end{array} \right\}$$

PROOF. Define a Lyapunov –like function H where

$H(Z_1,Z_2)(t) = Z_1(t) + Z_2(t) + b_{11}\int_0^\infty k_{11}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+ m_{12}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

$+ m_{21}\int_0^\infty k_{12}(u)\left(\int_{t-u}^t Z_1(s)ds\right)du$

$+ b_{22}\int_0^\infty k_{22}(u)\left(\int_{t-u}^t Z_2(s)ds\right)du$

Where $(Z_1,Z_2)$is any solution of (A.15)-(A.16) for arbitrary $(Y_1,Y_2)$defined on $(-\infty, 0]$ Computing the rate of change of H in (A.19)along the solution of (A.15)we derive that (on using (A.12))

$$\left(\frac{dH(Z_1Z_2(t)}{dt} = [Z_1(t)\{m_{11} - (b_{11} + m_{21})\} + Z_2(t)\{m_{22} - (b_{22} + m_{12})\}]\right.$$

$< 0$(using (A.18)) for $Z_1(t) > 0\ and\ Z_2(t) > 0.$

By Lemma 6 below the result will follow

Lemma 6. Let $H(t)=H(Z_1, Z_2)(t)$ be as defined in (A.10) Suppose

$$m_{11} > b_{11} + m_{21} \text{ and } m_{22} > b_{22} | m_{12}$$

Then

$$Z_1(t) + Z_2(t) \to 0 \text{ast} \to \infty.$$

PROOF. Suppose (A.21) dose not hold; then there exists at least one sequence

$\{t_n\} \to \infty$ as n $\to \infty$ such that

$$[m_{11} - (b_{11} + m_{21})]\, Z_1(t_n) + [m_{22} - (b_{22} + m_{12})]Z_2(t_n) > \varepsilon \qquad (A.22)$$

For some sufficiently small positive number $\varepsilon$ and hence

$$\frac{dH(t_n)}{dt} < -\varepsilon, n = 1,2,3, \ldots.$$

By the continuity and differentiability of H with respect to t,

$$\mathrm{H}(t_n - \varepsilon) - \mathrm{H}(t_n) = \frac{dH(t^*)}{dt}(-\varepsilon), \quad t_n - \varepsilon < t^* < t_n$$

If $\varepsilon$ is small we will have

$$= \frac{dH(t^*)}{dt}\varepsilon/2$$

So that

$$\mathrm{H}(t_n) \leq \mathrm{H}(t_n - \varepsilon) - \varepsilon^2/2$$

$$\leq H(t_{n-1}) - \varepsilon^2/2$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2)$$

$$\leq H(t_{n-2}) - 2(\varepsilon^2/2):$$

$$\leq H(t_0) - \mathrm{n}(\varepsilon^2/2): \mathrm{t}_0 = 0.$$

(A.12) and Lemma 5 will imply that $H(t_0) < \infty$ for any $t_0 > 0$. If n is sufficiently large it will follow that $H(t_n) < 0$ since $H(t_0) < \infty$,which contradicts the nonnegativity of H for all $t \geq 0$. Hence $Z_1(t) + Z_2(t) \to 0$ as $t \to \infty$ implying the asymptotic stability of the trivial solution of (A.1)(since $Z_1$ dominates $y_1$)

We note the result of Lemma 5 and 6 will remain valid if (A.18) is replaced by the condition $\mu(A) + /\!/B/\!/ < 0$

Where A and B are defined as in (6.3).

We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical

propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems

**Gopalsamy**

# Model For Innocuously Benign Delay In A Periodic System

**Abstract**

Sufficient conditions are obtained for the existence of a unique linearly denomination stable positive periodic solution of a system model of two competitions in a periodic environment with time lags in interspecific interactions. It is shown that if the self- regulating intraspecific interaction effects are strong enough and act without time delays then time delays of any length in the interspecific interactions cannot destabilize an otherwise stable system in a periodic domination. An example in finance will make things clear. Corporate finance for one big plant generates lot of accounts of the workers whose salary shall be disbursed to the debit of cash credit account. Here Cash credit account is a resource as also the one to which debits are raised, a la predator. Some of the accounts from which the credits come are demand bills purchase, usance bills discount, vostro and nostro accounts in respect of foreign bills and Letters of credit. In economics inventories are used to produce output, output is sold to produce income which might be saved in part and recycled in investment. Cosmologically take an example of a Black hole which gormandizes matter and radiates energy. Notwithstanding information paradox, one thing is clear that matter is being created in to radiation. We show how the model could be applied to axiomatic predications, postulation alcovishness, phenomenological methodologies, propositional statement, and other problems in cosmology, philosophy, and even in Green House effects. We study the sine qua non properties of such a system. In fact these models can be applied to such unsolved problems like Yang mills Theory and mass gap, and other problems. As we apply the model, we shall go on dilating the position status and the applicability of the model. We shall not touch the electrical engineering problems which might rise questions like what is charge of a Bank. We may say when a customer enters a Bank; we can say the Bank is charged. These factors are still relativistic and we shall make an in-depth study as we progress. Gopalsamy's research was published in Australian Journal of Mathematical Society vide: B 25349=365 It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 34. Introduction

At the outset we assume that the system satisfies Haimovici is caution. The purpose of this article is to establish a set of sufficient conditions for the existence of a unique linearly asymptotically stable positive periodic solution for a nonautonomous system of the form

$$\frac{dx(t)}{dt} = x(t)f_1\big(t, x(t), y(t-\tau_1)\big),$$

**(1.1)**

$$\frac{dy(t)}{dt} = y(t)f_2\,\big(t, x(t-\tau_2), y(t)\big).$$

In the context of confederation dynamics a model system of the form (1.1) means that the interspecific interactions occur with time delays $\tau_1$ 0, $\tau_2$ 0 in a periodic environment, if $f_1$ *and* $f_2$ are assumed to be periodic in $t$ with a common period $\omega$ 0. Note the interactions are inter specific.

Although model systems in periodic domains have been investigated by a large number of authors (for some references see Gopalsamy (4], [5]), periodically varying environments in time delayed model systems of two or more interacting species have not been investigated. The need for Multitype system models in periodic environments and time delayed interactions are evident from the extensive literature of that type relating to single species. While it is known that periodic ecosystems can support more species than uniform environments (Gopalsamy [4]), time delays might destabilize the system leading to a possible loss of one or more species. Furthermore, periodic environments in population dynamics have some relevance to natural selection and evolution; we will consider this aspect briefly at the end.

The principal finding of the following analysis of (1.1) is that, under suitable hypotheses on $f_1$ *and* $f_2$, the system (1.1) has a linearly asymptotically stable unique positive periodic solution for all nonnegative delays $\tau_1, \tau_2$ (while our numerical experiments suggest that the periodic solution of (1.1) is globally asymptotically stable and is independent of the delay parameters $\tau_1, \tau_2$ it is not known how one can prove such a strong result). Since time delays are usually associated with possible destabilization of otherwise stable systems, we can describe the time delays appearing on (1.1) as "harmless".

It has been recently shown by the author (Gopalsamy [6]) that if the self –regulating negative feedback in each denomination is strong enough and acts without time delays, then time delays of any length in the interspecific

interactions cannot destabilize an otherwise stable system in a temporally uniform domain. The result of this paper is thus an extension to a periodic environment of the "harmless" nature of interspecific interaction delays in a temporally constant environment mentioned above.

## 35. Existence and uniqueness of periodic solutions

We will now formulate the problem of existence of periodic solutions of the periodic delay differential system

$$\frac{dx_1(t)}{dt} = x_1(t) f_1\big(t, x_1(t), x_2(t-\tau_{12})\big),$$

$$t > 0, \qquad \textbf{(2.1)}$$

$$\frac{dx_2(t)}{dt} = x_2(t) f_2\big(t, x_1(t-\tau_{21}), x_2(t)\big),$$

(Where $\tau_{12}, \tau_{21}$ are nonnegative constants) in a form suitable for the application of fixed point theory of positive operators. We make the following assumptions regarding $f_1$ and $f_2$ in (2.1).

($H_1$) The partial (Frechet) derivatives of $f_1, f_2$ satisfy the following:

$$\frac{\partial f_i}{\partial x_j} < 0, \qquad i, j = 1, 2 \text{ For all } t > -\tau \text{ where } \tau = \max(\tau_{12}, \tau_{21}). \qquad \textbf{(2.2)}$$

($H_2$) $f_1(t, x_1, x_2) \equiv f_1(t + \omega, x_1, x_2)$ for all $t > -\tau, x_1 \geq 0, x_2 \geq 0$. **(2.3)**

($H_3$) There exist positive constants $b_{il}$, $b_i^u$, $a_{il}$, $a_i^u$ ($l = 1, 2$) such that

$$b_{il} - a_{il} x_i < f_i(t, x_1, x_2) < b_i^u - a_i^u x_i,$$

$$b_{il}/a_{il} < b_i^u / a_i^u, \qquad \textbf{(2.4)}$$

For $i$= 1, 2 and $(x_1, x_2) \in R_+ \times R_+$. If $\tau = 0$ then one can show that (2.1)-(2.4) will imply that the nondelayed periodic system (2.1) will have an asymptotically stable strictly positive periodic solution (for details see Gopalsamy [5]); this result will become a special case of what we obtain in the following.

We will use the following notation:

$$B_\omega^\tau = \{h: [-\tau, \infty) \to R, h \text{ is continuous}, h(t + \omega) = h(t), t \geq -\tau\},$$

$$B = B_\omega^\tau \times B_\omega^\tau,$$

$$K_1 = \{h \in B_\omega^\tau | h(t) \geq 0, t \in [-\tau, \infty)\},$$

$$K = K_1 \times K_1,$$

$$||h|| = \left\{\sup_{t \in [0,\omega],} | h(t)|, h \in B_\omega^\tau \right\}$$

$$||x_1, x_2|| = \max\{||x_1||, ||x_2||, (x_1, x_2) \in B\},$$

$$G_1 = \{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b_l = \min(b_{1l} / a_{1l}, b_{2l}/a_{2l})\}$$

$$G_2 = \left\{(x_1, x_2) \in K, ||(x_1, x_2)|| \; b^u = \max\left(\frac{b_1^u}{a_1^u}, \frac{b_2^u}{a_2^u}\right)\right\},$$

$\bar{G}$ = closure of $G_2$ *in* $B$ and

$\frac{\bar{G}_2}{G_1}$ = complement of $G_1 in\ \bar{G}_2$; $G_1 \subset G_2$ by **(2.4)**

The cone $K = K_1 \times K_1$ induces a partial order in $B$ and we will write $(x_1, x_2) \; (y_1, y_2)$ if and only if $x_1 - y_1 \in K_1$. we define an operator $N$ on $B$ as follows

$$N(x_1, x_2)(t) = \begin{matrix} N_1(t, x_1, x_2) \\ N_2(t, x_1, x_2) \end{matrix} \quad \text{for t> } 0, \qquad \textbf{(2.5)}$$

Where

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}(m, \omega, t-s) g_1\left(m, s, x_1(x) x_2(s - \tau_{12})\right) ds$$

$$+ \int_0^1 G_{12}(m, 0, t-s) g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) ds,$$

$$N_2(t, x_1, x_2) = \int_0^\omega G_{22}(m, \omega, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds \qquad \textbf{(2.6)}$$

$$+ \int_0^1 G_{21}(m, 0, t-s) g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) ds,$$

For $t > 0$, with

$$g_1\left(m, s, x_1(s), x_2(s - \tau_{12})\right) = mx_1(s) + x_1(s) f_1\left(s, x_1(s), x_2(s - \tau_{12})\right),$$

**(2.7)**

$$g_2\left(m, s, x_1(s - \tau_{21}), x_2(s)\right) = mx_2(s) + x_2(s) f_2\left(s, x_1(s - \tau_{21}) x_2, (s)\right),$$

$$G_{ll}(m,\omega,t-s) = e^{-m\omega}(1-e^{-m\omega})^{-1}e^{-m(t-s)}, \qquad i = 1,2,$$

**(2.8)**

$$G_{ij}(m,0,t-s) = e^{-m(t-s)}, i \neq j, i,j = 1,2,$$

$$N(x_1,x_2)(t) = \begin{cases} N_1(n\omega + t, x_1, x_2) \\ N_2(n\omega + t, x_1, x_2) \end{cases} for\ t \in [-\tau, 0], \qquad \textbf{(2.9)}$$

*m* being a suitably selected positive constant (more will be said below) while *n* is a positive constant such that $n\omega > \tau$.

We note immediately the following from the definition of $N$: $if\ (x_1,x_2) \in \bar{G}_2/G_1$ then $N(x_1,x_2)(t)$ [illegible] and if $N(x_1,x_2)$ is periodic $\omega$ for $t$ [illegible] then $N(x_1,x_2)$ is periodic in $t$ with period $\omega$ for $t$ [illegible] $-\tau$. The existence of at least one positive periodic solution of (2.1)-(2.4) is established by means of a few lemmas and a result of Schmitt [9].

Lemma 1. *A necessary and sufficient condition for* $(x_1(t), x_2(t))$, $t$ [illegible] $-\tau$ *to be a periodic solution of (2.1)-(2.4) of period* $\omega$ *is that* $(x_1,x_2)$ *be a fixed point of N such that*

$$N(x_1,x_2)(t) = (x_1(t), x_2(t)) \quad and\ (x_1,x_2) \in K \cap \left(\frac{\bar{G}_2}{G_1}\right).$$

PROOF. The details of proof are similar to those of Lemma 2 of Busenberg and Cooke [1]. We first verify the continuity of $N(x_1,x_2)$ assuming that $N(x_1,x_2)(t)$ is periodic in $t$ of period $\omega$. The continuity of $N(x_1,x_2)$ for $t > 0$ follows from the definition in (2.5)-(2.8) WHICH ALSO PROVIDES $\lim_{t\to 0+} N(x_1,x_2)(t)$. It will follow from (2.9) and the assumed periodicity of $N(x_1,x_2)$ $that\ N(x_1,x_2)$ is continuous

For $t$ [illegible] $-\tau$. Now we will verify the periodicity of $N(x_1,x_2)$ in $t$. A simple computation leads to

$$N_1(t+w,x_1,x_2) - N_1(t,x_1,x_2) = e^{-mt}F_1(t),$$

**(2.10)**

Where

$$F_1(t) = \int_0^{\omega} G_{11}(m,\omega,t-s)\, g_1\big(m,s,x_1(s),x_2(s-\tau_{12})\big)ds$$

$$+\int_0^{\omega} G_{12}(m,0,t-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_{\omega}^{\omega+t} G_{12}(m,0,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$-\int_0^{\omega} G_{11}(m,\omega-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

$$+\int_0^{t} G_{12}(m,0,-s)g_1\left(m,s,x_1(s),x_2(s-\tau_{12})\right)ds$$

**(2.11)**

Which on simplification using the periodicity of $f_1$ shows that $F_1(t) \equiv 0$ whenever $x_1, x_2, \in B,$ thus implying the periodicity of $N_1$ in $t$ of period$\omega$. A similar argument for $N_2$ proves that $(x_1, x_2) \in B \Rightarrow N(x_1, x_2) \in B$.

Now suppose $(x_1, x_2)$ is a fixed point of *N and let* $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$. Then it is easily seen that $(x_1, x_2)$ is a solution of (2.1) and since $(x_1, x_2) \in K \cap (\bar{G}_2/G_1)$ it will follow that $(x_1, x_2)$ is a periodic solution of (2.1).

Conversely, let $(x_1, x_2)$ be a periodic solution of period $\omega$ of (2.1). Then $(x_1, x_2)$ is necessarily a periodic solution of

$$\frac{dx_1(t)}{dt} + mx_1(t) = g_1\left(m,t,x_1(t),x_2(t-\tau_{12})\right)$$

**(2.12)**

$$\frac{dx_2(t)}{dt} + mx_2(t) = g_2\left(m,t,x_1(t-\tau_{21}),x_2(t)\right),$$

For any arbitrary real constant $m$ and hence for any fixed constant $m > 0$. It follows from (2.12) that any periodic solution of (2.1) and hence of (2.12) is a fixed point of $N(x_1, x_2)$. We claim that such a fixed point is in $K \cap (\bar{G}_2/G_1)$. Suppose $(x_1, x_2) \notin K \cap (\bar{G}_2/G_1)$. the following possibilities can arise:

| | |
|---|---|
| (xxxiv) | Either $x_1, x_2$ or both $x_1$ $and$ $x_2$ may become negative for some value of *t* and it will follow from (2.1) that subsequent value of $x_1, x_2$ or $x_1$ $and$ $x_2$ will be negative and hence from (2.4) and (2.1) their derivatives will be negative for all subsequent *t* , implying that $x_1, x_2$ or $x_1$ $and$ $x_2$ cannot be periodic. |
| (xxxv) | While $(x_1, x_2)$ $\in K \subset B$ there may be a value of *t*, say $t^*$, such that $x_1(t^*) > b_1^u/a_1^u$ or $x_2(t^*) > b_1^u/a_1^u$ and in such a case $x_1(t) or\ x_2(t)$ cannot be periodic since respectively $dx_1(t^*)/dt < 0$ $or$ $dx_2(t^*)/dt < 0,$ which will imply that the values of $x_1(t^*)$ $or$ $x_2(t^*)$ cannot be repeated from below. |
| (xxxvi) | Similarly if there is a $t^*$ such that $0 < x_1(t^*) < b_{1l}/a_{1l}$ $or$ $0 < x_2(t^*) < b_{2l}/a_{2l}$ then such solutions cannot be periodic since $dx_1(t^*)dt > 0$ or $dx_2(t^*)dt > 0$, which will mean that the values of $x_1(t^*)$ or $x_2(t^*)$ cannot be repeted from above. |

Thus every periodic solution of (2.1)-(2.4) is a fixed point of *N* in $K \cap \left(\frac{G_2}{G_1}\right) \subset B$.

**LEMMA 2.** *Let N be as in (2.5)-(2.6). Suppose* $\lambda \in (0,1) and\ \lambda N(x_1, x_2) = (x_1, x_2)$ *for some* $((x_1, x_2) \in K - \{0\}.$ *then there exist* $\alpha_1^* > 0, \alpha_2^* > 0$ *such that*

$$0 < x_1(t) \leq \alpha_1^* < b_1^u / a_1^u,$$

$$0 < x_2(t) \leq \alpha_2^* < b_2^u / a_2^u,$$

*for* $t \in [-\tau, \infty)$.

PROOF. If $\lambda N(x_1, x_2) = (x_1, x_2)$ then from the definition of N we get for all $t > 0$ and $(x_1, x_2) \in K - \{0\}$,

$$\frac{dx_1(t)}{dt} = \lambda x_1(t) f_2(t, x_1(t), x_2(t - \tau_{12})) + x_1(t) m(\lambda - 1)$$

**(2.13)**

$$\frac{dx_2(t)}{dt} = \lambda\, x_2\,(t) f_2(t, x_1(t-\tau_{21}), x_2\,(t)) + x_2\,(t) m(\lambda - 1)$$

For all $t$ such that $x_1(t) = b_1^u/a_1^u$ or $x_2(t) = b_2^u/a_2^u$ we have $\frac{dx_1}{dt} < 0$ or $\frac{dx_2}{dt} < 0$; but these are impossible since $x_1, x_2$ are periodic in t. Thus by the continuity and periodicity of $x_1, x_2$ we have $x_1(t) \geq x_1^* < b_1^u/a_1^u$, $x_2(t) \geq x_2^* < b_2^u/a_2^u$ for all $t \geq 0$; by the definition and periodicity the same will follow for all $t \geq -\tau$. A similar argument holds when $x_1(t) < 0$ or $x_2(t) < 0$, noting that in such a case $dx_1/dt > 0$ or $dx_2/dt > 0$, again contradicting the periodicity of $(x_1, x_2)$ in *t*.

LEMMA 3. *The operator N maps* $K \cap (G_2/G_1)$ *in to K and is completely continuous*

PROOF. From the definition of N and (2.4) we derive that

$$N_1(t, x_1, x_2) = \int_0^\omega G_{11}\,(m, \omega, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds$$

$$+ \int_0^\omega G_{12}\,(m, 0, t-s) x_1(s)[m + b_{1l} - a_{1l} x_1(s)] ds. \quad \textbf{(2.14)}$$

For $(x_1, x_2) \in K \cap (G_2/G_1)$, we have $0 < x_1 < b_1^u/a_1^u$ and hence we get $N_1(t, x_1, x_2) > 0$ for all $t \geq 0$ by a proper choice of the positive constant say $m_1$.

We make a similar choice say $m_2$ for *m* for the case $N_1(t, x_1, x_2) \geq 0$ and then choose *m* = max $(m_1, m_2)$. Thus we have

$$N: K \cap (G_2/G_1) \to K \qquad for\ t \geq 0,$$

And by the periodicity of N we have the same for $t \geq -\tau$.

To prove the complete continuity of N we proceed as follows. We know that if $(x_1, x_2) \in B$ then $N(x_1, x_2) \in B$ and so we can restrict $t$ to the set $[0, n\,\omega], n\omega > r$. We write $N = L \circ Q$ where $L$ is the linear operator, $L : B \rightarrow B$, $L(u_1, u_2)(t)$

$$= \int_0^{\omega} G_{11}(m, \omega, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds, + \int_0^{t} G_{12}(m, 0, t-s) \begin{bmatrix} u_1 & (s) \\ u_2 & (s) \end{bmatrix} ds$$

**(2.15)**

And Q is the nonlinear operator

$$Q(u_1, u_2)(t) = \begin{matrix} g_1(m, t, u_1(t), u(t-\tau_{12})) \\ g_2(m, t, u_1(t-\tau_{21}), u_2(t)) \end{matrix} \; for \;\; t > 0,$$

**(2.16)**

$$Q(u_1, u_2)(t) = Q\,(u_1, u_2)(n\omega + t) \quad for\, t \in [-\tau, 0].$$

It can be found that

$$||L(u_1, u_2)(t)|| \leq max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||,$$

**(2.17)**

$$\left\| \frac{d}{dt} L(u_1, u_2) \right\| \leq 2\, max \left\{ \frac{m+b_1^u}{m}, \frac{m+b_2^u}{m} \right\} ||u_1, u_2||.$$

The complete continuity of the linear map $L$ follows from (2.17). Now since Q is continuous and maps bounded sets into bounded sets it will follow that $L \circ Q$ is completely continuous.

To prove the existence of periodic solution of (2.1)-(2.4) we will use the following result of Schmitt [9].

**THEOREM (Schmitt).** *Let $H_1$ and $H_2$ be nonempty bounded open neighborhoods of the origin in a Banach space B with a positive cone K such that $H_1 \subset H_2$ and let N: $K \cap (H_2 / H_1) \rightarrow K$ be completely continuous and which satisfies the following conditions:*

***(w)*** *There exists* $(z_1, z_2) \in K \subset B, \|(z_1, z_2)\| = 1$ *and a* $\mu$ *satisfying*

$\mu > sup\{\|(x_1, x_2) - N(y_1, y_2)\|, (x_1, x_2) \in K \cap H_1, (y_1, y_2) \in K \cap \partial H_2\}$ *such that all solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2) with\ 0 < \lambda < \mu\ satisfy\ (x_1, x_2) \notin \partial H_1.$

*(x) All solution* $(x_1, x_2) \in K \cap (H_2/H_1)$ *of* $(x_1, x_2) = \lambda N(x_1, x_2), 0 < \lambda < 1\ satisfy\ (x_1, x_2) \notin \partial H_2.$

*Then N has at least one fixed point in* $K \cap (H_2/H_1)$.

We can now present our existence result.

**THEOREM 1.** *Assume that* $f_1, f_2$, *of* (2.1) *satisfy* (2.2)-(2.4). *then there exists at least one positive periodic solution of* (2.1)-(2.4).

PROOF. By Lemma 1, it is enough to show that the operator $N: K \cap (G_2/G_1) \to K \subset B$ has at least one fixed point in $G_2/G_1$ and we will do this by applying the result of Schmitt [9]. First we let

$$H_2 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \max\{b_1^u/a_1^u, b_2^u/a_2^u)\} \equiv G_2,$$

$$H_1 = \{(x_1, x_2) \in B | \ ||x_1, x_2|| < \min\{b_{1l}/a_{1l}, b_{1l}/a_{1l})\} \equiv G_1.$$

By Lemma 3, $N: K \cap (H_2/H_1) \to K$ is completely continuous. Choose $\mu = \infty$ and $(z_1, z_2) \equiv (1.1)$. Then all solution $(x_1, x_2) \in K \cap (H_2/H_1)$ of $(x_1, x_2) = N(x_1, x_2) + \lambda(z_1, z_2), 0 < \lambda < \infty$, satisfy (2.1) and hence by their periodicity $(x_1, x_2) \notin \partial H_1$. The condition (a) of Schmitt's theorem is satisfied since $\mu = \infty$ by choice and $(x_1, x_2)$ and $(z_1, z_2)$ are periodic. By Lemma 2 the condition (b) of Schmitt's theorem is satisfied. Thus there exists at least one fixed point of N in $K \cap (H_2/H_1)$ and hence there exists at least one positive periodic solution of (2.1)-(2.4) and the proof is complete.

We remark that so far no restriction on the magnitudes of the delays $\tau_{12}, \tau_{21}$ has been imposed except that they be nonnegative.

## 36. Uniqueness of the periodic solution

Applications of fixed point theorems to prove the existence of solutions of nonlinear problems usually suffer from the drawback that the uniqueness of solutions has to be proved often by other methods. We will use the following result of Smith [11] which is a slight variation of his Theorem 3.5 (see also Busenberg and Cooke [1]).

THEOREM (Smith). *Let N be an operator on a cone K in the Banach space B and let* $K^* = \{(x_1, x_2) \in K \cap B, 0 \leq$ *x1,x2≤C∗ where C∗some positive constant is. Assume the following:*

(hh) If $(x_1, x_2) \in K^* - \{(0,0)\}$ and $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

(ii) N is monotonic nondecreasing on $K^*$.

(jj) If $(x_1, x_2)$ is in the interior of $K^*$ and if $0 < \lambda < 1$ then there exists an $\eta > 0$ depending on $(x_1, x_2)$ and $\lambda$ such that $N(\lambda x_1, \lambda x_2) \geq \lambda(1 + \eta) N(x_1, x_2)$. Then N has at most one fixed point different from $(0,0)$ in $K^* \subset B$.

The cone K= $K_1 \cap K_1$ where $K_1 = \{h \in B_\omega^\tau : h(t) \geq 0, t \in [-\tau, \infty)\}$ is slightly too wide in that solutions of the form $\{x_1(t), 0\}$ and $\{0, x_2(t)\}$ for (2.1) are also possible. We choose a narrower cone $K^*$ in $B$ as follows: define

$$S = \{(x_1, x_2) \in B \mid b_{1l}/a_{1l} \geq \|x_1\| \geq b_1^u/a_1^u,\ b_{2l}/a_{2l} \geq \|x_2\| \geq b_2^u/a_2^u,\}, \quad \textbf{(3.1)}$$

And let

$$K^* = \{\lambda(x_1, x_2) \in B \mid (x_1, x_2) \in S, \lambda \geq 0\}. \quad \textbf{(3.2)}$$

Since $\hat{G}_2 / G_1 \subset K^*$, it is not difficult to see that N: $\hat{G}_2 / G_1 \to B$ has at least one fixed point which is a periodic solution of (2.1)-(2.4). We will prove that this solution is unique by applying the result of Smith cited above. The following lemmas are needed first.

LEMMA 4. If $(x_1, x_2) \in K^* - \{0.0\}$ and if $N(x_1, x_2) = (x_1, x_2)$ then $(x_1, x_2) \in$ interior of $K^*$.

PROFF. If $(x_1, x_2) = N(x_1, x_2)$ and $(x_1, x_2) \in K^* - \{0,0\}$, then $(x_1, x_2)$ is a periodic solution of (2.1) –(2.4) and hence we have

$$(b_{1l} - a_{1l}x_1)x_1 < \frac{dx_1}{dt} < (b_1^u - a_1^u x_1)x_1,$$
$$(b_{2l} - a_{2l}x_2)x_2 < \frac{dx_2}{dt} < (b_2^u - a_2^u x_2)x_2, \quad (\mathbf{3.3})$$

And hence any nonzero periodic solution $(x_1, x_2)$ of (2.1)-(2.4) has to be such that

$$b_{1l}/a_{1l} < x_1(t) < b_1^u - a_1^u,$$

$$b_{2l}/a_{2l} < x_2(t) < b_2^u - a_2^u,$$

For $t > -\tau$, which proves the lemma.

LEMMA 5. The mapping $N$ defined by (2.5)-(2.9) is monotone nondecreasing on $K^*$.

PROOF. Let $(x_1, x_2) \in K^*$, $(y_1, y_2) \in K^*$ such that $(x_1 - y_1, x_2 - y_2) \in K^*$ and $(x_1, x_2) \neq (y_1, y_2)$. Then $x_1 \geq y_1$, $x_2 \geq y_2$. From the definition of $N$ we derive that

$N_1(t, x_1, x_2) - N_1(t, y_1, y_2)$

$$= \int_0^\omega G_{11}(m, \omega, t-s)\{m[x_1(s)] + x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))\}ds$$

$$+ \int_0^t G_{12}(m, 0, t-s)$$

$$\times \{m[x_1(S) - y_1(s)] + [x_1(s)f_1(s, x_1(s), x_2(s-\tau_{12})) - y_1(s)f_1(s, y_1(s), y_2(s-\tau_{12}))]\}ds \quad \mathbf{(3.4)}$$

With a similar relation for $N_2(t, x_1, x_2) - N_2(t, y_1, y_2)$. Since $f_1$ is continuous, $f$ is bounded on $K^*$; also it will follow from $(x_1, x_2) \neq (y_1, y_2)$ that

$$0 \not\equiv x_1(s) - y_1(s) \geq 0 \text{ and } 0 \not\equiv x_2(s) - y_2(s) \geq 0;$$

A sufficiently large positive constant $m$ will make the right side of (3.4) positive.A similar argument will lead to $N_2(t, x_1, x_2) - N_2(t, y_1, y_2) > 0$ perhaps with a larger constant $m$. Finally we can choose the bigger of these two constants $m$ to yield

$N(x_1,\ x_2)\ -N\ (y_1,y_2)$ for $(x_1-y_1,\ x_2-y_2) \in K^*-\{(0,0)\}$.

The monotonicity of $N$ on $K^*$ follows.

LEMMA 6. Let $(x_1,\ x_2) \in$ interior of $K^*$ and let $\lambda$ be $a$ constant such that $0<\lambda<1$. Suppose $f_1, f_2$ satisfy (2.2) - (2.4). Then there exists $a$ Positive constant $\eta$ such that

$$N(\lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta)\, N(x_1,\ x_2). \qquad \textbf{(3.5)}$$

PROOF. A direct computation leads to

$$N_1(t,\ \lambda x_1,\ \lambda x_2) - \lambda N_1(t, x_1,\ x_2)$$

$$= \lambda \int_0^{\omega} G_{11}\ (m,\omega,t-s) g_1\big(s, x_1(s),\ x_2(s-\tau_{12})\big) H_1(s,\lambda,\ x_1,\ x_2) ds$$

$$+ \lambda \int_0^{t} G_{11}\ (m,0,t-s) g_1\big(s, x_1(s),\ x_2(s-\tau_{12})\big) H_1(s,\lambda,\ x_1,\ x_2) ds$$

**(3.6)**

Where

$$H_1(s,\lambda,\ x_1,\ x_2) = \frac{m\ x_1(s)+x_1(s) f_1\big(s,\lambda\ x_1(s),\lambda\ x_2(s-\tau_{12})\big)}{m\ x_1(s)+x_1(s) f_1\big(s, x_1(s), x_2(s-\tau_{12})\big)} - 1. \qquad (\mathbf{3.7})$$

Since $f_1$ is decreasing (see (2.2)) with respect to $x_1$ and $x_2$ there exists a positive constant say $\eta_1$ such that

$$H_1(s,\lambda,\ x_1,\ x_2) \geq \eta_1 \text{ for } 0<\lambda<1, s>0, (x_1,\ x_2) \in K^*, \qquad \textbf{(3.8)}$$

And hence

$$N_1(t,\lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta_1) N_1(t, x_1,\ x_2).$$

A similar argument will show that there exists an $\eta_2 >0$ such that

$$N_2(t,\lambda x_1,\ \lambda x_2) \geq \lambda(1+\eta_2) N_2(t, x_1,\ x_2),$$

And the result follows if we choose $\eta = \min(\eta_1, \eta_2)$.

Now the following uniqueness result is a direct consequence of the result of Smith [11] and the Lemmas 4, 5, 6.

THEOREM 2. *Suppose $f_1, f_2$ of (2.1) satisfy (2.2) – (2.4). Then there exists a unique ω - periodic strictly positive periodic solution of (2.1).*

## 4. Linear stability of the periodic solution

Let $\{x_1^*(t), x_2^*(t), t \text{ -}\tau\}$ be the strictly positive unique periodic solution of (2.1). Let $\{x_1(t), x_2(t), t \text{ -}\tau\}$ be any other solution of (2.1). To study the linear stability of $(x_1^*, x_2^*)$ we let

$$x_1(t) \equiv x_1^*(t)\{1 + X_1(t)\}, \quad x_2(t) \equiv x_2^*(t)\{1 + X_2(t)\}, \quad t \text{ -}\tau, \tag{4.1}$$

And derive that the perturbations $(x_1, x_2)$ are governed by the linear variational system (after neglecting the non linear terms in $x_1, x_2$)

$$\frac{dX_1(t)}{dt} = -a_{11}(t)\, x_1^*(t)X_1(t) - a_{12}(t)x_2^*(\mathrm{t} - \tau_{12})\, X_2(\mathrm{t} - \tau_{12})$$
$$\frac{dX_2(t)}{dt} = -a_{21}(t)x_1^*(t - \tau_{21})x_1(t - \tau_{21}) - a_{22}(t)x_2^*(t)x_2(t), \tag{4.2}$$

For $t > 0$, where

$$\left.\begin{aligned} -a_{11}(t) &= \frac{\partial f_1}{\partial x_1} \\ -a_{12}(t) &= \frac{\partial f_1}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t), x_2^*(\mathrm{t} - \tau_{12})),$$

**(4.3)**

$$\left.\begin{aligned} -a_{22}(t) &= \frac{\partial f_2}{\partial x_1} \\ -a_{22}(t) &= \frac{\partial f_2}{\partial x_2} \end{aligned}\right\} \text{Evaluated at } (t, x_1^*(t - \tau_{21}), x_2^*(\mathrm{t})).$$

To establish the linear asymptotic stability of $(x_1^*(t), x_2^*(t))$ it is enough to prove that of the trivial solution of (4.2) - (4.3) We will do this by means of a combination of a comparison differential system and a Lyapunov function. First we define

$$\frac{d|X_1(t)|}{dt} = \begin{cases} \frac{dX_1(t)}{dt} & if\ X_i(t) \geq 0, \\ -\frac{d\backslash X_1(t)}{dt} & if\ X_i(t) < 0, \end{cases} \quad i = 1,2, \tag{4.4}$$

And derive from (4.2)- (4.3) the following comparison system of differential inequalities:

$$\begin{aligned} \frac{d|X_1(t)|}{dt} &\leq -b_{11}, x_1^*(t)|X_1(t)| + b_{12}x_2^*(t-\tau_{12})|X_2(t-\tau_{12})|, \\ \frac{d|X_2(t)|}{dt} &\leq -b_{21}, x_1^*(t-\tau_{21})|X_1(t-\tau_{12})| - b_{22}x_2^*(t)|X_2(t)|, \end{aligned} \tag{4.5}$$

For $> 0$, where

$$-b_{11} = \max_{t \geq -\tau}\{-a_{11}(t)\},$$

$$\begin{aligned} b_{12} &= |\min_{t \geq -\tau}\{-a_{12}(t)\}| \\ b_{21} &= |\min_{t \geq -\tau}\{-a_{21}(t)\}| \end{aligned} \tag{4.6}$$

$$-b_{22} = \max_{t \geq -\tau}\{-a_{22}(t)\}$$

We can now formulate the result as follows. By the periodicity of $a_{ij}$, the constants $b_{ij}$ are independent of $\tau$ (see also (2.2)).

THEOREM 3. *The unique strictly positive periodic solution* $\{x_1^*(t), x_2^*(t), t \geq -\tau\}$ of $(2.1) - (2.4)$ *is linearly asymptotically stable provided*

$$|-b_{11}| > b_{21} \text{ and } |-b_{22}| > b_{12}, \tag{4.7}$$

*Where the constants* $b_{ij}$ $(i, j = 1,2)$ *are as defined in* (4.6).

PROOF. Consider the scalar valued Lyapunov-like function $v(t) = V(t, X_{1,}X_{2,})$where

$$v(t) = V(t, X_{1,}X_{2,}) = |X_{1,}(t)| + |X_{2,}(t)| \int_{t-\tau_{12}}^{t} b_{12} x_2^*(u)|X_{2,}(u)|du$$

$$+\int_{t-\tau_{21}}^{t} b_{21} x_1^*(u)|X_{1,}(u)|\mathrm{du}. \quad \textbf{(4.8)}$$

Since $x_1^*, x_2^*$ are strictly positive, it is found from (4.7)-(4.) that

$$\frac{dv}{dt} \leq -[(|b_{11}| - b_{21})\, x_1^*(t)|X_{1,}(t)| + (|b_{22}| - b_{22})\, x_2^*(t)|X_2(t)|] \quad \textbf{(4.9)}$$

For $t > 0$. It follows from (4.7)-(4.9) that $V(t, X_{1,}X_{2,}) \geq |X_{1,}(t)| + X_{2,}(t)|$ and $dv/dt \leq 0$,which will imply the stability of the trivial solution of (4.2)-(4.3). We claim that (4.9) also implies the asymptotic stability of the trivial solution of (4.2)-(4.3)in the sense that

$$\lim_{t-\infty} [\,|X_{1,}(t)| + |X_{2,}(t)|\,] = 0,$$

**(4.10)**

For all bounded initial conditions $X_{1,}(s), X_{2,}(s), s \in \{\mathcal{T}, 0\}. Suppose (4.10) is\ not$ true Then we can find an increasing sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$(\,|b_{11}| - b_{21})\, x_1^*(t_m)|X_{1,}(t_m)| + (|b_{22}| - b_{12} x_2^*(t_m)|X_2(t_m)| > \varepsilon,$$

M=1, 2, 3…

**(4.11)**

Using the right derivative *dv/dt* of $v$ we derive that

$$v(t_m - \varepsilon) - v(t_m) \leq -\int_{t_m-\varepsilon}^{t_m} \frac{dv(T)}{dT} d\tau.$$

**(4.12)**

If $\varepsilon$ is small enough we have from (4.9)(4.11),(4.12)that $dv(\tau)/dr < (-\varepsilon/2)$ and

$v(t_m - \varepsilon) - v(t_m)$ $\varepsilon^2/2$ and hence

$v(t_m)$ $v(t_m - \varepsilon) - (\varepsilon^2/2)$

$v(t_m - 1)$-$(\varepsilon^2/2)$

$v(t_m - 2)$-$(\varepsilon^2/2)$

$v(t_0)$-m $(\varepsilon^2/2)$

Thus for sufficiently large $m, v(t_m)$becomes negative which contradicts the Nonnegative nature of $v\ for\ t$ 0. this contradiction shows that $|X_{1,}(t)| + |X_2(t)| \to 0$ as $t \to \infty$ and the proof is complete.

As an example let us consider the case of two species Lotka- Volterra type competition in a periodic environment for for which we have

$$f_1(t, x_1(t), x_2(t - T_{12}))1 \quad \frac{x_1(t) + ax_2(t - T_{12})}{K_1(t)}$$

$$f_2(t, x_1(t - {}_{21})x_2(t)) = 1 - \frac{\beta x_1(t - T_{21}) + x_2(t)}{K_2(t)}$$

Where $\alpha, \beta$ are positive constants and $k_{1,}k_{2,}$ are strictly positive continuous periodic functions with a common period Equation (4.6)and (4.7) for this special case lead to

$$\left|\text{axm}_{t>T}\{-a_{11}(t)\}\right| = \max_{t>T}(1/K_1(t)) > \min_{t>T}\left(-a_{21}(t)\right)\Big| = \min_{t}, \beta/K_2(t)$$

$$\max_{t T}\{-a_{22}(t)\}\Big| = \max_{t T}(1/K_2(t)) > \min_{t T}\left(-a_{12}(t)\right)\Big| = \min_{t} \alpha/K_1(t)$$

The inequalities reduce in the absence of temporal variations in $k_{1,}k_{2,}$ to the inequality $0 < \alpha, \beta < 1$, which is well known for the global asymptotic stability of the positive steady state corresponding to the above example in a

constant systemic domain the above example shows also that delays in interspecific interactions have no influence on the local asymptotic stability of the unique periodic solution of the periodic ecosystem A number of cases of this type in a constant environment have been established recently by the author (Gopalsamy [6]).A numerical example is illustrated at the end.

**Consequences**

It can be found that all the conclusions on the existence uniqueness and stability of a periodic solution of (2.1)-(2.4) are valid when $OT_{12}=0$ or $T_{12}=0$ or both $T_{12}=0$ and $T_{21}=0$: thus it will follow that delays in interspecific competition do not destabilize an otherwise stable coexistence (oscillatory or nonoscillatory) of two competing types of interacting particles atoms molecules transactions interaction. In forthcoming work we will extend the analysis of the system (2.1)-(2.4) to a system of n-competing species with a hypothesis similar to that of (2.4) It is however an open problem to investigate the effects of delays in intraspecific competition in a periodic milieu

Pianka ([8],pages 139-140) remarks that "In real population rate of increase competitive abilities and carrying capacities do very from individual to individual with population density in space and time Indeed temporal variation in the environment may often allow coexistence by continually altering the competitive abilities of population inhabiting it Moreover time lags are doubt lessly of some importance in real population Finally a heterogeneous environment may allow real competitors to evolve divergent resource utilization patterns and to reduce interspecific competitive inhibition

To understand the mechanism of selection pressure against competitive inhibition we first write the Malthusian parameters $\mu_1$, $\mu_2$ of fitness for the two species as follows (see MacArthur[7],page 226-230):

$$\mu_1=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_1\left(\mathrm{t},x_1(\mathrm{t}),x_2(\mathrm{t}-T_{12})\right)$$

$$\mu_2=\frac{1}{x_1(\mathrm{t})}\frac{dx_1(\mathrm{t})}{dt}=f_2(\mathrm{t},x_1(\mathrm{t}-T_{21}),x_2(\mathrm{t}))$$

Since $\left(x_1(\mathrm{t}),x_2(\mathrm{t})\right)\rightarrow\left(\mathrm{x}_1^*(\mathrm{t}),\mathrm{x}_2^*(\mathrm{t})\right)$ for large t we can write the parameters of finess in the from

$$\mu_1=f_1\left(t,x_1^*(t),x_2^*(t-T_{12})\right)$$

$$\mu_2=f_2\left(t,x_1^*(t-T_{12}),x_2^*(t)\right)$$

In these relations we can use

$$x_2^*(t-T_{12}),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}$$

$$x_1^*(t-T_{12}),x_1^*(t)-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt}$$

$\theta_1,\theta_2,\in(0,1)$ and rewrite $\mu_1,\mu_2$ in the form

$$\mu_1=f_1\left(t,x_1^*(T),x_2^*(t)-T_{12}\frac{dx_2^*(t-T_{12}+\theta_1T_{12}}{dt}\right.$$

$$\mu_2=f_2\left(t,x_1^*(T),-T_{21}\frac{dx_1^*(t-T_{21}+\theta_2T_{21}}{dt},x_2^*(t)\right)$$

$\theta_1,\theta_2,\in(0,1)$

It is seen that the presence of the delays $T_{12}$ and $T_{21}$ provides additional sources of variability in the finess parameters and thereby in the selection pressure against interspecific competition Since a necessary and sufficient condition for natural selection to act is differential (variable) rates of confederation growth we find that time delays coupled with periodicity of an domain may allow real to evolve divergent resource utilization patterns and to reduce

Interspecific competitive inhibition for some remarks relating to the significance of a periodic environment to sustain the coexistence of a large number of confederation types in the absence of delays we refer to a recent article (Gopalsamy)The following example has been numerically solved and the solutions are displayed graphically for two different values of the delay parameter $\mathcal{T}$:

$$\frac{dx_1(t)}{dt}x_1(t)[1-x_1(t)+0.4x_2(t-\mathcal{T})/k_1(t)]$$

| |
|---|
| $\frac{dx_2(t)}{dt} x_2(t)[1 - \{0.8\, x_1(t - \mathcal{T}) + x_2(t)\}/k_2(t)]$ |
| Where $k_1(t) = 20 + \sin\ (\pi t)$ and $k_2(t)$=2.5+sin (πt) |
| @@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@@ |
| There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. “All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ....” (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To “deconstruct” is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between “speech” and “writing.” This binary logic functions in an illicit way to establish speech as the means of giving “presence” to the world, while writing is deemed derivative and inferior In Derrida’s sense of “grammatology,” however, all production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of |

"differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. **THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html** Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction,

projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification.

## GOPALSAMY'S MODEL APPIED TO COSMOLOGICAL PROBLEMS

## GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST PERIODIC LOTKA- VOLTERRA SYSTEM

### Abstract

**Gopalsamy studied** Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of a Lotka- Volterra system with almost periodic coefficients. Research was published Australian. Math Soc. B.27 (1986) 346-360. Gopalaswamy in his Delayed responses and stability in two species systems (Journal of Australian mathematical SocietyB25 473-500) has shown that if intraspecific self- regulating negative feedback effects are strong enough such that a nontrivial steady state of a two species system is locally asymptotically stable, then time delays in the positive feedback as well as in other interspecific interactions cannot destabilize the system and hence delay induced instability leading to persistent oscillations is impossible whatever the magnitude of the time delays. A method is also proposed for an estimate of decay rate of perturbations. Sole intention, main objective and primary concern of the study is to show how this model has wide ranging amplitudinal ramifications such as in finance, econometrics, cosmology, propositions, phenomenology, philosophy and other fields. In finance cash credit account is an account that is given by a bank to the customer. This forms the resource which is assiduously and widely competed by such factorial representationalities such as bills received in the name of the company, foreign bills received by the company. On the other hand the bills purchased and discounted also form a creditional value for the cash credit account. Thus there are various debits and credits to the Cash credit account. Thus Cash credit account forms a sort of both predator and prey in consideration to the fact they receive both credits and debits. In econometrics, investment leads total output thus depleting part of the investment, not wholly or in full measure and output leads to savings, consumption, and part of the income so obtained could be ploughed back to the investment portfolio. In this connection reference is invited to the authors DLitt thesis on the Policy studies wherein details studies of the application of the predator and prey models with nearly 57 parameters and concomitant 167 equations have been suggested corresponding to an optimization problem and duality of motion. It is the fundamental theorem that energy is always conserved. And energy is matter. In many of the cosmological interactions this holds good. In many of propositional and problem approaches as in the case of philosophers' like Derrida, Kant, Deleuze, and others. One extant example is the case of Horava Lifshitz gravity in which interactions are of the type to which the model could be applied. All the result hold good for confederation of atoms, molecules, creation rate, destruction

rate, financial account, cosmological interactions, where energy is transferred from one to another, interaction in gravitational field. Quintessentially population is the field and the creation destruction rates or debit credit rates are counted in the same vein as that of predator prey. Note that the model is applicable to each and every module. Quintessentially each parameter should have been given different name such as superscript 1,2, etc., as that has been done in the foregoing and a formal concatenation done at the end. Due to spatial restrictions we have not been able to do the line of thought and manner of analysis as has been done hereinbefore. This is the most important aspect of the model. It is to be stated in unmistakable terms and unequivocal enucleation that the nine module concatenated model becomes applicable for each and every sentence of the model that has been incorporated, including the equation. For example, if there is an equation A+B=C , then A can be differentiated and written in the governing equation form and the corresponding model equations as is stated in the foregoing modular structure gets changed which could simply be represented by a different subscript and superscript for expression felicity. It is in this manner each term has to be solved and finally the present incorporated structure has to be reproduced for ultimate consolidation and corporation by concatenation. This would take lot space to the extent of thousands of pages and long time for expatiation and exposition itself. It is of seminal, cardinal and paramount importance that this has to be borne in mind before the concatenated models are read with each and every module, repeat each and every module. In fact nine module 54 storey structures had to be inculpated in a concatenated form for each and every paper. This again has been refrained from for the spatial restrictions. Once this fundamental truth is stated and understood explicit, the further reading becomes a cake walk.

## 23. Introduction

At the outset we assume that the system. This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (component wise) almost- periodic solution of the Lotka- Volterra system

$$\frac{dx_1(t)}{dt} = x_i(t)\left\{b_1(t) - \sum_{j=1}^{n} a_{ij}\,(t)x_j(t)\right\};$$

t>$t_0$;

$t_0 \in (-\infty, \infty); i = 1, 2, \dots, n,$ (1.1)

where $b_1, a_{ij}$ $(i, j = 1, 2, \dots., n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$.

In question satisfies Haimovici's equation based on age wise stratification given in the foregoing. Note that Haimovici's equations can be applied to Lotka-Volterra system term by term.

Let R and $R_n$ respectively denote the set of all real numbers and the $n$ - dimensional real Euclidean space; $R_n^+$ will denote the nonnegative cone of $R_n$. Since almost- periodic functions defined on R are bounded, we define the constants $b_i^l, b_l^u, a_{ij}^l, a_{ij}^u, (i, j = 1,2, \dots, n)$ by the following:

$$\inf_{t\in R} b_l(t) = b_i^l; \qquad \sup_{t\in R} b_l\,(t) = b_l^u, \qquad i, j = 1, 2, \dots, n.$$

$$\inf_{t\in R} a_{lj}\,(t) = a_{lj}^l; \qquad \sup_{t\in R} a_{lj}\,(t) = a_{ij}^u.$$

We shall study the almost- periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_l^l > 0; \qquad a_{ll}^l > 0, \tag{1.2}$$

$$b_l^l > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{lj}^u \left(b_j^u \,/\, a_{lj}^l\right). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_1}{dt} \leq x_l\left\{b_l^u - a_{ll}^l x_i\right\}; \qquad t > t_0, i = 1, 2, \dots, n, \tag{1.4}$$

As a consequence of which we will have

$$0 < x_l(t_0) \leq b_l^u / a_{ll}^l = x_i^u \Rightarrow x_i(t) \leq b_l^u / a_{ll}^l, \qquad i = 1, 2, \dots, n; t \geq t_0. \tag{1.5}$$

(1.12) And (1.5) together lead to

$$\frac{dx_1}{dt} \leq x_l \left\{ b_l^u \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) - a_{ll}^l x_i \right\}, \qquad i = 1, 2, \dots, n; \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0 \leq \left( b_i^l - \sum_{\substack{j=1 \\ J \neq i}}^{n} a_{ij}^u \left(b_j^u / a_{jj}^l\right) \right) (1/a_{ll}^u) = x_i^l$$

$$\Rightarrow x_l(t) \leq x_i^l \ for\ t \geq t_0,\ i = 1, 2, \dots, n. \tag{1.7}$$

24. It is our aim to show that the mode together with other models stated in the foregoing together with haimovicis model is applicable to whole lot of consummative debit – credit systems in cosmology, astrophysics, finance, economics, sociology, anatomy computation. We will not however divide the system studied here into 9 components/ modules due to restrictions on space….

From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* $b_i, a_{ij} (i, j = 1, 2, \dots, n)$ *are scalar nonnegative almost periodic functions defined on* ***R*** *such that (1.2) and (1.3) hold. Then the set S defined by*

$$S = \{x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R}_n | x_* \leq x_i \leq x^*; i = 1, 2, \dots, n\} \quad (1.8)$$

*Where*

$$x_* = \min_{1 \leq i \leq n} x_i^l ; \qquad x_* = \min_{1 \leq i \leq n} x_i^u \quad (1.9)$$

*is invariant with respect to (1.1)*

**Appellation, cognomen, compellation, designation, denomination of the conducive, infelicitous and unfortutious** responses to environment change may be influenced by changes in available domain and milieu under consideration. An example is the following study: Authors describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. Authors introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero. **Schrodinger equation of the Schwarzschild black hole Jarmo Mäkelä (Cambridge U E-Print: gr-qc/9602008 | PDF** Particles, axions, atoms, molecules, quarks, Majorana fermions, bosons et al in its confederation capacity and aggregation processes, type interactions and interactions between landscape dynamics. Following bears ample testimony, infallible observatory to the fact there might be changes in the quantum behaviour due to merger of black holes. Following example bears enough testimony, infallible observatory and impeccable demonstration and apodictic evidence as how some of the models could be applied to cosmological problems. This is attributable and ascribable to the fact that the classification criterion is based on the age.

**How gravity kills (e) Schrödinger's cat Theorists argue that warped space-time prevents (e) quantum superpositions of large-scale objects. Elizabeth Gibney 17 June 2015 Propositions are written in model able form and can easy be scheduled.**

(243) If the cat in Erwin Schrödinger's famous thought-experiment behaved according to quantum theory, it would be able to exist in multiple states at once: both dead and alive.

(244) Physicists' common explanation for why we don't (e) see such quantum superpositions — in cats or any other aspect of the everyday world — is interference from (e) the environment.

(245) As soon as a quantum object interacts with (e&eb) a stray particle or a passing field, it picks (e) just one state, collapsing (eb) into our classical, everyday view.

(246) But even if physicists could completely isolate (eb) a large object in (eb) a quantum superposition, according to researchers at the University of Vienna, it would still collapse into (e&eb) one state — on Earth's surface, at least.

(247) "Somewhere in interstellar space it could be that the cat has (e) a chance to preserve quantum coherence, but on Earth, or near any planet, there's (=) little hope of that," says Igor Pikovski. The reason, he asserts, is gravity.

(248) Pikovski and his colleagues' idea, laid out in a paper published in Nature Physics on 15 June1, is at present only a mathematical argument. But experimenters hope to test whether gravity really does collapse (eb) quantum superpositions, says Hendrik Ulbricht, an experimental physicist at the University of

Southampton, UK. “This is a cool, new idea, and I’m up for trying to see it in experiments,” he says. Assembling the technology to do so, however, may take as long as a decade, he says.

(249) Cinema-goers who saw the film Interstellar are already familiar with the basic principle behind the Vienna team’s work. Einstein’s theory of general relativity states that an extremely massive object causes clocks near it to (e) run more slowly because (e) its strong gravitational field stretches (e&eb) the fabric of space-time (which is why a character in the film aged only an hour near a black hole, while (e) seven years passed on Earth). On a subtler scale, a molecule placed nearer the Earth’s surface experiences (e&eb) a slightly slower clock than one placed slightly further away.

(250) Because of gravity’s effect on (e&eb) space-time, Pikovski’s team realised that variance in a molecule’s position will also influence (e&eb) its internal energy — the vibrations of particles within (eb) the molecule, which evolve (eb) over time.

(251) If a molecule were put in a quantum superposition of two places, the correlation between position and (e&eb) internal energy would soon cause (eb) the duality to 'decohere' to (e) the molecule taking just one path, they suggest.

(252) “In most situations decoherence is due to (e) something external; here it’s as though the internal jiggling is interacting with (e&eb) the motion of the molecule itself,” adds Pikovski.

**A practical limit**

(253) No one has yet seen this effect because other sources of decoherence — such as magnetic fields, thermal radiation and vibrations — are (=) typically much stronger and cause (eb) quantum systems to collapse long before gravity becomes an issue. But experimenters are keen to try.

(254) Markus Arndt, an experimental physicist also at the University of Vienna, already tests whether quantum superpositions can be observed for (e) large, though not cat-sized, objects (see Fattening up Schrödinger's cat).

(255) He sends large molecules through (e&eb) a matter-wave interferometer, a system that gives (eb) each molecule a choice of (e) two different paths.

(256) In the classical view, a molecule travels down just one path; a quantum molecule effectively passes through (e&eb) both routes at once and interferes with (e&eb) itself to create (eb) a characteristic wave-like pattern (see image, below).

(257) A quantum interference pattern built up from complex molecules known as phthalocyanines.(deleted due to spatial constraints)

(258) A similar set up could be used to (e) test for gravity's ability to destroy (e) quantum behaviour: by comparing a vertical interferometer, in which the superposition should soon decohere due to (e) the stretching of time in one path versus (e&eb) the other, with a horizontal set up, where (e) the superposition could remain.

(259) Arndt, who has tested the effect for (e) molecules of up to 810 atoms2, points out those large molecules, would be (=) good for testing the gravitational effect because (e) they contain (e) lots of particles contributing to (e) internal energy.

(260) **But not only would the researchers have to further suppress the external environment to reduce (e) other decoherence effects, they would also need to (e) increase the separation of the two paths from micrometres to metres, or else use molecules perhaps 1 million times more massive. "It's certainly very challenging," says Arndt**.

(261) If gravity’s effect does limit (e) quantum behaviour on Earth, tests of quantum reality for (e) large objects may eventually have to (e) move into space, says Angelo Bassi, a physicist at the University of Trieste in Italy.

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|---|
| (262) "But from a deep, fundamental point of view, this is nothing new," he says. A gravitational field is merely another environment to interact with, so invoking it does not (e) explain whether quantum behaviour might lead to (eb) classical reality if gravity's influence were mitigated — for example, by doing the experiment in (eb) gravity-free space. |
| (263) Nor does the effect described by Pikovski and colleagues say anything about quantum gravity: a theory that would unite gravity and (e&eb) quantum mechanics in (eb) a single description, which many researchers are working towards. |
| (264) "It's an interesting effect, but it's still quantum physics applied to (e&eb) classical general relativity. In that way it doesn't change our picture of the world," adds Bassi. Nature doi:10.1038/nature.2015.17773 |

Integrated factors these influences because they deal with ambience, mise en scene change or confederation dynamics, but rarely both. Time series of (e) space time suitability models with spatially explicit stochastic models to explore factors that influence the viability of type viability and confederation feasibility under stable and changing scenarios. Methodology could be applied to interrogation propositions and problems which might subsume one with the other notwithstanding that there is a space of singularities and there is space for completion of present from the past and future. An example is Time image movement from Deleuze. We clearly face a stiff challenge in our attempt to proffer Deleuzian treatments of photography as our critical project brushes up against the fact that Deleuze does not invest photography with (e&eb)the creative and re-creative potential that he associates with (e&eb) both cinema and modern painting. In short, while he highly values other visual artistic forms, he seemingly presents photographic texts as stagnate documents or tools that produce certainty, organize bodies and desires, and iterate hackneyed ideas. He even uses photography as something of a foil to demonstrate the innovation of the cinema and the originality of modern painters. In Francis Bacon: The Logic of Sensation (1981), he notes how "photography has taken over the illustrative and documentary role, so that modern painting no longer needs to fulfill this function, which still burdened earlier painters" (10). He later explains that "photographs are ways of seeing, and as such, they are illustrative and narrative reproductions or representations ... . They are what are seen, until finally one sees nothing else." He treats the photo as an instrument for reproducing representations of reality—a device that iterates images until they are ossified as established stories, icons, or even stagnant perceptions. Indeed, like Bergson before him, Deleuze compares photography to what he identifies as a reductionist model of perception: "what we see, what we perceive, are photographs" (74).[4] The photo is, in effect, an always-already passé sensational experience that numbs our sensitivity to the ongoing vitality of life and provides us with an efficient understanding of the dynamism of our world. But it is compelling that Deleuze theorizes both the quotidian nature and the liberatory potential of photography; he treats it as a common, everyday visual experience that, despite its aesthetic and technological limitations, emancipates the modern painter to explore new artistic opportunities. This special issue of Rhizomes repeatedly provides artistic examples and critical strategies that invite us to see photography as replete with the power to unburden and emancipate that Deleuze associates with both film and painting. Indeed, one way to think of this project of reading Deleuze against Deleuze might be to ask: if photography can free painting and painters to pursue novel creative opportunities, can it likewise free itself, photographic practitioners, and even viewers to create and re-create anew? **Rhizomes » Issue 23 (2012) » Michael Kramp Unburdening Life, or the Deleuzian Potential of Photography Michael Kramp Lehigh University** dmk209@lehigh.edu

**Existence of an almost- periodic solution**

DEFINITION 2.1. A function $f(t,x)$,where $f$ is an $m$- vector, $t$ is a real scalar and $x$ is an $n$- vector, is said to be almost periodic in $t$ uniformly with respect to $x \in X \subset R_n$, $if\ f(t,x)$ is continuous in $t \in R\ and\ x \in X$, and if for

any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\tau$ such that the inequality

$$||f(t+\tau,x) - f(t,x)|| = \sum_{l=1}^{m} |f_l(t+\tau,x) = f_l(t,x)|$$

$$< \varepsilon \qquad (2.1)$$

is satisfied for all $t \in (-\infty,\infty), x \in X$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t,x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T\to\infty} T^{-1}\int_0^T f(t,x)\exp(-i\lambda_j t)dt \not\equiv 0 \quad for\ x \in X; i = \sqrt{-1}. \qquad (2.2)$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable. The set $\{\sum_1^N n_j \lambda_j\}$ for all integers N and integers $n_j$ is called the module of $f(t,x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

**MODULE CONTAINMENT THEOREM** (Yoshizawa [7], page 18). *Let $f(t,x)$ and $g(t,x)$ be almost periodic in t uniformly for $x \in D \subset \boldsymbol{R}_n$. If for any compact set $S \subset D$ and for any sequence of real numbers $\{\tau_k\}$ having a limit as $k \to \infty$ (the limit being finite or $\pm\infty$) for which $\{f(t+\tau_k,x)\}$ is uniformly convergent on $\boldsymbol{R} \times S, \{g(t+,x)\}$ is also uniformly convergent on $\boldsymbol{R} \times S$ then the module of $g(t+,x)$ is contained in the module of $f(t+,x)$.*

We can now formulate our existence theorem.

**THEORM 2.1** *Assume that the almost-periodic parameters $b_1, a_{1J}$ $(i,j = 1,2,\dots,n)$ of (1.1) satisfy (1.2)-(1.3). Furthermore assume that there exists a positive number $\mu$ such that*

$$\inf_{1 \in \boldsymbol{R}} a_{11}(t) > \sum_{\substack{J=1 \\ J\neq 1}}^{n} (\sup_{t \in \boldsymbol{R}} a_{Jl}(t)) + \mu, \quad i$$

$$= 1,2,\dots,n. \qquad (2.3)$$

*Then the system (1.1) has a strictly positive (component wise) almost-periodic solution, say $w(t) = w_1(t),\dots,w_n(t), t \in \boldsymbol{R}$, whose module is contained in that of*

$$f(t,x) = \{f_1(t,x_1,\dots,x_n), f_2(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n)\}$$

*Where*

$$f_l(t,x_1,\dots,x_n) = x_l\left\{b_l(t) - \sum_{j=1}^{n} a_{lj}(t)x_j\right\},$$

$$i = 1,2,\dots n; (x_1,x_2,\dots,x_n) \in S.$$

(2.4)

PROOF. The set $S$ defined by (1.8)-(1.9) is compact in $\boldsymbol{R}_n$. Let $\{\tau_h\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{l=1}^{n} |f_i(t+\tau_k, x_1, \dots, x_n) - f_l(t, x_1, \dots, x_n)| \to 0 \quad \text{as } k \to \infty \qquad (2.5)$$

Uniformly in $(x_1,\dots,x_n) \in S$ for $t \in (-\infty,\infty)$. for any real number $\beta$let $k_0(\beta)$ be the smallest value of $k$ such that $\tau k_0 + \beta \geq t_0$. Since the invariance of $S$ implies that any solution of (1.1)

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), x_2(t+\tau_k, t_0, x_2(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\} \in S$$

For all $t \geq \beta$ and $k \geq k_0(\beta)$ whenever $\{x_1(t_0),\dots,n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Converges to a continuous bounded function say $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined for $t \in [\beta,\infty)$ such that $w_1(t) \in [x_*, x^*], t \in [\beta,\infty), i = 1,2,\dots,n$ and the convergence is uniform on all compact subsets of $[\beta,\infty)$. Since $x_*$ and $x^*$ are independence of $\beta$, it is enough to show that

$$\{x_1(t+\tau_k, t_0, x_1(t_0)), \dots, x_n(t+\tau_k, t_0, x_n(t_0))\}, \qquad k \geq k_0(\beta),$$

Is a Cauchy sequence on compact subsets of $[\beta,\infty)$.

Let $U$ be any compact subset of $\beta,\infty)$., and let be any arbitrary positive number. Choose an integer $n_0(\varepsilon,\beta) \geq k_0(\beta)$ so large that for $m \geq k \geq n_0(\varepsilon,\beta)$

and all $t \in (-\infty,\infty)$

$$\sum_{i=1}^{n} |b_1(t+\tau_m) - b_1(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \quad (2.6)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{lj}(t+\tau_m) - a_{lj}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*) \quad (2.7)$$

$$(2\text{nx}^*/x_*)\exp[-\mu x_*(\beta+\tau_k-t_0)] < (\varepsilon/2x^*) \quad (2.8)$$

Denote $x_i(t, t_0, x_1, (t_0))$ $by$ $x_1(t)$ brevity, and consider a function $v(s)$ defined by

$$v(s) = v\left(x(s), x(s+\tau_m+\tau_k)\right)$$

$$= \sum_{i=1}^{n} |\log x_1,(s) - \log x_1(s+\tau_m+\tau_k)|, \qquad x \in t_0, \quad (2.9)$$

Where $x(t) = \{x_1(t), \dots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1,(t_0), \dots, x_n(t)\} \in S$. the invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$(1/x^*)\sum_{l=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)| \in v(s)$$

$$\in (1/x^*)\sum_{i=1}^{n} |x_l(s) - x_1(s+\tau_m+\tau_k)|. \quad (2.10)$$

Calculating the right derivative $D^+v$ of $v$ along the solutions of (1.1) and simplifying

$$D^+v(s) = \sum_{i=1}^{n} \{sign[x_i(s) - x_i(s+\tau_m-\tau_k)]\}$$

$$\left\{\frac{d[logx_i(s)]}{ds} - \frac{d[logx_i(s+\tau_m-\tau_k)]}{ds}\right\}$$

$$\sum_{l=1}^{n} |b_l(s) - b_l(s + \tau_m - \tau_k)|$$

$$+\sum_{l=1}^{n}(-a_{il}(s)|x_i(s) - x_i(s + \tau_m - \tau_k)| + \sum_{\substack{j=1 \\ j\neq l}}^{n} a_{ij}(s)|x_j(s) - x_j(s + \tau_m - \tau_k)|)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(s)|a_{ij}(s + \tau_m - \tau_k)|\, x_j(s + \tau_m - \tau_k) \qquad (2.11)$$

$$-\mu x_* v\big(x(s), x(s + \tau_m - \tau_k)\big) + \sum_{l=1}^{n} |b_l(s) - b_i(s + \tau_m - \tau_k)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{iJ}(s) - a_{iJ}(s + \tau_m - \tau_k)\,|. \qquad (2.12)$$

Let us set $s - \tau_k = \sigma$; then we have from (2.6)-(2.7) that

$$\sum_{i=1}^{n} |b_l(s) - b_1(s + \tau_m - \tau_k)| + x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{iJ}(s) - a_{iJ}(s + \tau_m - \tau_k)\,|.$$

$$= \sum_{l=1}^{n} |b_l(\sigma + \tau_k) - b_i(\sigma + \tau_m)|$$

$$+x^* \sum_{i=1}^{n}\sum_{J=1}^{n} |\, a_{iJ}(\sigma + \tau_k) - a_{iJ}(\sigma + \tau_m)|$$

$$< (\mu\varepsilon x^*)/(2x^*). \qquad (2.13)$$

Now an integration of (2.12) over $(t_0, t - \tau_k)$ together with (2.13) leads to

$$v\big(x(t + \tau_k), x(t + \tau_m)\big) \le v\big(x(t_0), x(t_0 + \tau_m - \tau_k)\big) \exp[-\mu x_*(t + t_k - t_0)]$$

$$+[(\mu\varepsilon x_*)/(2x^*)]/\mu x_*)$$

$[(2x_n^*)/x_*]\exp[-\mu x_*(t+t_k-t_0)] + (\varepsilon/2x^*)$

$< (\varepsilon/x^*)$ On using (2.8), (2.10) (2.14)

For $m \geq n_0(\varepsilon,\beta)$ and for all $t \in \cup \subset [\beta,\infty]$, Thus we have from the definition of $\upsilon$ and (2.14) that

$$\sum_{l=1}^{n} |log x_i\,(t+\tau_k) - log x_l(t+\tau_m)| < (\varepsilon/x^*) \qquad (2.15)$$

And hence using $x_* \leq x_i(t) \leq x^*, i = 1,2,\dots,n; t \geq t_0$,

$$\sum_{i=1}^{n} |x_i\,(t+\tau_k) - x_i(t+\tau_m)| < \varepsilon \text{ for all } t \in \cup \subset [\beta,\infty], m \geq n_0(\varepsilon,\beta). \qquad (2.16)$$

The existence of a function $w(t) = \{w_1(t),\dots,w_n(t)\}$ defined on $[\beta,\infty]$, such that $w_i(t) \in [x_*, x^*], i = 1,2,\dots,n; t \in [\beta,\infty]$ follows from (2.16) and the fact that $x_i(t) \in [x_*, x^*], i = 1,2,\dots,n;$ for $t \geq t_0$. Since $\beta$ is arbitrary, $w$ is in fact

Defined on $(-\infty,\infty)$ and hence we have

$x_i(t+\tau_k) \to w_l(t)$ as $k \to \infty$ uniformly on compact subsets of $(-\infty,\infty)$, $i = 1,2,\dots,n.$

(2.17)

Our next task is to show that $w: R \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t),\dots,x_n(t)\}$ satisfies (1.1) we have $\sum_{i=1}^{n} |\frac{d}{dt}\, x_i(t+\tau_k) - \frac{d}{dt}\, x_i(t+\tau_m)|$

$$\sum_{i=1}^{n} |f_l\left(t+\tau_k, x_1(t+\tau_k),\dots, x_n\,(t+\tau_k)\right)$$

$$- f_l\left(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k)\right)| \qquad (2.18)$$

$$+ \sum_{l=1}^{n} |f_l\,(t+\tau_m, x_1(t+\tau_k),\dots,x_n(t+\tau_k))$$

$$- f_l\big(t + \tau_m, x_1(t + \tau_m), \dots, x_n(t + \tau_m)\big)|$$

Where $f_l\ (i = 1, 2, \dots, n)$are as in (2.4). Since $\mathrm{x}(\mathrm{t} + \tau_k) = x_1(t + \tau_k), \dots, x_n(t + \tau_k) \in \mathrm{S}$ for large $\tau_k$ for $t$ in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m$ $k$ $n_1(\varepsilon)$, then for any $\varepsilon > 0$,

$$\sum_{i=1}^{n} |f_l\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\big)| < (\varepsilon/2). \qquad (2.19)$$

Similarly since $x_l(t + \tau_k) \in S$ and $x_l(t + \tau_m) \in S$ for $i = 1, 2, \dots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough $m, k$, such that $m$ $k$ $n_2(\varepsilon)$, we have

$$\sum_{l=1}^{n} |f_l\left(t + \tau_m, x_1(t + \tau_k), \dots, x_n(t + \tau_k)\right)$$

$$-f_1\big(t + \tau_m, x(t + \tau_m), \dots, x_n(t + \tau_m)\big)| < (\varepsilon/2) \qquad (2.20)$$

This equation will be very much useful in chemistry if used with the modules mentioned and written in concatenation with each other. It should be borne in mind that the applicability is modulewise and in no case concatenation should be marginalized. In all cases parameters are different which could be represented with different superscripts and subscripts like 1,23,…….etc., **Anal Bioanal Chem. 2004 Sep; 380(1):36-45. Epub 2004 Jul 29. Electronic tongue for pharmaceutical analytics: quantification of tastes and masking effects Legin A et al**

The organoleptic aspects of pharmaceutical formulations affect their acceptability to the patient and hence can have an important effect on concordance with treatment. Objective evaluation of these aspects, particularly the taste of the formulation and the drug substance it contains, is difficult. Whilst volunteer taste panels can be used to good effect their utility is limited, particularly during very early stage development when the toxicological profile of the active pharmaceutical ingredient (API) is yet to be established in detail. A potentiometric "electronic tongue" has

been applied to analyse a variety of 41 individual substances and mixtures of particular interest for pharmaceutical research and development. The electronic tongue (ET) was capable of discriminating between substances with different taste modalities and could also distinguish different substances eliciting the same basic taste; the ET is promising in terms of quantifying the content of each substance and has an ability to detect nuances of the basic taste (e.g. lingering or short-lived). After calibration the electronic tongue was successfully applied to predicting bitterness strength of binary mixtures with a sweetener in terms of "apparent" or "perceived" quinine content. In order to render a formulation palatable it is often necessary to mask the (usually bitter) taste of the API by the addition of masking agents such as sweeteners and flavours. The ET proved capable of distinguishing between formulations with different levels of sweetener and/or flavour in a manner that was consistent with their masking efficiency as perceived by a small human taste panel. A suitably calibrated ET could have the benefit of providing the pharmaceutical formulator with reliable data concerning the taste of the product quickly and with a reduced need to ask volunteers to taste active pharmaceutical samples. Early development activities could be facilitated when human tasting is usually not possible in the absence of the required toxicological data.

Thus if $m \geq n_0(\varepsilon) = max\{n_1, n_2\}$, we have

$$\sum_{l=1}^{n} \left| \frac{d}{dt} x_1(t + \tau_k) - \frac{d}{dt} x_1 ( (t + \tau_m) \right| < \varepsilon \qquad (2.21)$$

Which shows that

$$\lim_{k\to\infty} \frac{d}{dt} x_1(t + \tau_k) \quad exists\ for\ i = 1, 2, \dots, n$$

Uniformly on all compact subsets of $(-\infty, \infty)$. Thus we have

$$\lim_{k\to\infty} \lim_{k\to\infty} \frac{x_i(t + \tau_k + h) - x_i(t + \tau_k)}{h}$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{x_l(t + \tau_k + h) - x_l(t + \tau_k)}{h} \qquad (2.22)$$

$$= \lim_{h\to 0} \frac{w_1(t + h) - w_1(t)}{h}, \quad i = 1, 2, \dots, n,$$

Showing that (since the left side of (2.22) exists) $w_i(i = 1, 2, \dots, n)$ is differentiable. Furthermore we have from (2.22) that

$$\frac{dw_l(t)}{dt} = \lim_{k\to\infty}[\, f_i\left(t + \tau_k, x_1(t + \tau_k), \dots, x_n\,(t + \tau_k)\right)$$

$$-f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)$$

$$+ - f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big)]$$

$$= -f_l\big(t + \tau_k, w_1(t), \dots, w_n(t)\big); \qquad i = 1, 2, \dots, n, \qquad (2.23)$$

By the uniform almost periodicity of $f_1, f_2, \dots, f_n$ on $R \times S$ and hence $w(t) = \big(w_1(t), \dots, w_n(t)\big)$is a solution of (1.1).

In order to complete the proof, we have to show that $w(t) = \big(w_1(t), \dots, w_n(t)\big)$ is an almost- periodic function whose module is contained in that of $\{f_1(t, x_1, \dots, x_n), f_2(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n)\}$ for $(x_1, x_2 \dots, x_n) \in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t + \tau_k, x_1, \dots, x_n), f_2(t, + \tau_k, x_1, \dots, x_n), \dots, f_n(t, + \tau_k, x_1, \dots, x_n)\,\}$$

Converges uniformly on R for all $(x_1, \dots, x_n) \in S$, the sequence $\{w_1\,(t, + \tau_k)\ \dots, w_n(t + \tau_k)\}$ converges uniformly on R where $\tau_k$ tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_l\,(t, + \tau_k) \to w_i(t + \tau\,), i = 1, 2, \dots, n$ uniformly on R as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m$ ☒$k$ ☐ *n0(ε)* then

$$\sum_{l=1}^{n} |b_l\,(t + \tau_k) - b_l(t + \tau_m)| < (\varepsilon/4x^*)x_*, \qquad (2.24)$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t+\tau_k) - a_{ij}(t+\tau_m)| < \left(\frac{\varepsilon}{4x^*}\right)\mu x_* \qquad (2.25)$$

$$n(2x^*/x_*)\exp[\mu x_* \tau_k] < (\varepsilon/2x^*). \qquad (2.26)$$

Consider the function $v\big(w(s), w(s+\tau_m-\tau_k)\big)$ for $s \in [t, t+\tau_k]$ defined by

$$v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{l=1}^{n} |\log w_l(s) - \log w_l(s+\tau_m-\tau_k)|. \qquad (2.27)$$

Using the fact $w: \boldsymbol{R} \to S$ and $w$ is a solution of (1.1), calculate the right derivative $D^+v \ of \ v$;

$$D^+v\big(w(s), w(s+\tau_m-\tau_k)\big) = \sum_{i=1}^{n}\{\text{sign}[w_l(s) - w_i(s+\tau_m-\tau_k)]\}.$$

$$\left\{\frac{dw_i(s)}{ds}\cdot\frac{1}{w_l(s)} - \frac{dw_l(s+\tau_m-\tau_k)}{ds}\,\frac{1}{w_l(s+\tau_m-\tau_k)}\right\}$$

$$-\mu\sum_{l=1}^{n}|w_l(s) - w_l(s+\tau_m-\tau_k)| + \sum_{l=1}^{n}|b_i(s) - b_l(s+\tau_m-\tau_k)|$$

$$+x^*\sum_{l=1}^{n}\sum_{J=1}^{n}\left|a_{lJ}(s) - a_{lJ}(s+\tau_m-\tau_k)\right| \qquad (2.28)$$

$$-\mu x_* v\big(w(s), w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{i=1}^{n}|b_l(s) - b_l(s+\tau_m-\tau_k)| \qquad (2.29)$$

$$+x^*\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{lj}(s) - a_{ij}(s+\tau_m-\tau_k)\right|.$$

If we set $s-\tau_k = \sigma$ then we have from (2.29),

$$D^{+}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+\sum_{l=1}^{n}|b_l(\sigma+\tau_k)-b_l(s+\ \tau_m)|$$

$$+x^{*}\sum_{l=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\,(\sigma+\tau_k)-a_{ij}\,(\sigma+\tau_m)\right| \qquad (2.30)$$

$$\leq -\mu x_{*}v\big(w(s),w(s+\tau_m-\tau_k)\big)$$

$$+[(\varepsilon\mu x_{*})/(4x^{*})]\ +[(\varepsilon\mu x_{*})/(4x^{*})]$$

(2.31)

Integrating (2.31) over $(t,t+T_k)$

$$v(w(t+T_k)w(t+T_m))$$

Another MODEL where the model is directly applicable with the scheduled ones are the case of insurance and corporate finance. Insurance: Mathematics and Economics Volume 42, Issue 1, February 2008, Retrieval of Black–Scholes and generalized Erlang models by perturbed observations at a fixed time Dedicated to Professor André Dubey on the occasion of his 60th birthday Daniel Neuenschwander

S-stable laws on the real line (more generally on Hilbert spaces), associated with some non-linear transformations (so-called “shrinking operations”), were introduced in [Jurek, Z.J., 1977. Limit distributions for truncated random variables. In: Proc. 2nd Vilnius Conference on Probability and Statistics, June 28–July 3, 1977. In: Abstracts of Communications, vol. 3, pp. 95–96; Jurek, Z.J., 1979. Properties of s-stable distribution functions. Bull. Acad. Polon. Sci. Sér. Math. XXVII (1), 135–141; Jurek, Z.J., 1981. Limit distributions for sums of shrunken random variables. Dissertationes Math. vol. CLXXXV]. In [Jurek, Z.J., Neuenschwander, D., 1999. S-stable laws in insurance and finance and generalization to nilpotent Lie groups. J. Theoret. Probab. 12 (4), 1089–1107], the authors interpreted s-stable motions on the real line as limits of total amount of claims processes (up to a deterministic premium) of a portfolio of excess-of-loss reinsurance contracts and showed that they led to Erlang’s model or to

Brownian motion. In [Neuenschwander, D., 2000b. On option pricing in models driven by iterated integrals of Brownian motion. In: Mitt. SAV 2000, Heft 1, pp. 35–39], we considered stochastic integrals whose integrand and integrator are both independent Brownian motions, thus modelling a stochastic volatility; as a result we got an analogue of the Black–Scholes formula in this model, confirming a result of Hull and White [Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. J. Finance XLII (2), 281–300]. In the present paper, we will look at a common generalization of these processes, namely s-stable motions on the real line perturbed by a stochastic integral whose integrand and integrator are both (not necessarily independent) s-stable motions. The main result will be that if we can observe the distribution of such so-perturbed s-stable motions (together with the values of the perturbing processes) at time t=1, then we can identify the whole model (including the perturbation) among all models with Lévy processes perturbed by an iterated stochastic integral of two Lévy processes (in the Gaussian case) resp. among all models with a compound Poisson process with drift perturbed by an iterated stochastic integral of two compound Poisson processes (in the completely non-Gaussian case if the perturbing processes have no drift) without knowing anything about the history or about its distribution during 0≤t<1. This applies, e.g., to a situation where several assets obey the same model and one can estimate the distribution at time one by looking at the values of all these assets at time t=1. Interestingly enough, it will be convenient to treat the whole matter in the algebraic framework of the so-called Heisenberg group. This is a concept coming in fact from quantum mechanics and is in a certain sense the simplest non-commutative Lie group.

$ e^{-\mu x^* T_k} v\big(w(t), w(t+T_m-T_k)\big) + (\varepsilon/2x^*)$

$ e^{-\mu x^* T_k} \sum_{i=1}^{n} |\log\ w_i(t) - \log\ w_i(t+T_m-T_k)| + (1/x_*) + (\varepsilon/2x^*)$

$n(2x^*/x_*)\exp[-\mu x_* T_k] + (\varepsilon/2x^*)$

$<(\varepsilon/x^*)$.

Since

$$v\big(w(t+T_k)w(t+T_m)\big) \sum_{i=1}^{n} |\ w_1(t+T_k) - w_1(t+T_m))/x^*)$$

We have from(2.33)that

$$\sum_{i=1}^{n} | w_1(t+T_k) - w_1(t+T_m)| < \varepsilon \; for \; m \; k \; n_0(\varepsilon)$$

Which show that the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}$converges uniformly on **R**. By a similar argument the above conclusion is also valid when $T_k \to -\infty \; as \; k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+T_k),\ldots., w_n(t+T_k)\}\; as\; k \to \infty \; on$ **R** follows and this completes the proof of the theorem.

**COROLLARY.** Suppose $b_i a_{ij}$ (i, j = 1.2, … , n)in (1.1)are periodic with a common period say $T >$ $0. Then\; if$ (1.2), (1.3)$and$ (2.3)$hold\; the\; system$ (1.1)$will\; have\; a\; periodic\; solution\; w(t) =$ $\{w, (t), \ldots\ldots, w_n(t)\} of\; period\; T\; such\; that$

$$w_1(t+T) = w_1(t), \qquad t \in \boldsymbol{R}: i = 1,2, \ldots, n.$$

**PROOF.** Proof follows from that of Theorem 2.1 if we Choose $T_k = KT(K = 0,1,2, \ldots) and\; note\; that$

$$x_1(t+KT) - w_1(t) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+(K+1)t) - w_1(t) \to 0 \; as \; \boldsymbol{K} \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

$$x_i(t+T+KT) - w_1(t+T) \to 0 \; as \; k \to \infty; i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

Implying that

$$w_1(t+T) = w_i(t); \quad i = 1,2, \ldots, n; t \in \boldsymbol{R},$$

## 3. Global asymptotic stability

By definition we say that an almost-periodic solution say $w(t) = \{w_1(t), \ldots, w_n(t)\} of (1.1) is\; globally\; asymptotically$ stable (or attractive) if and only if every other solution x(t)={ $x_1(t), \ldots, x_n(t)$}with $x_1(o) > 0, i = 1.2, \ldots, n$ is defined for all t□0 and satisfies

$$\lim_{t \to \infty} |w_1(t) - x_1(t)| = 0; \quad i = 1,2, \ldots, n.$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1)

**THEOREM 3.1** Suppose the almost- Periodic coefficients $b_1$, $a_{ij}$ (i, j = 1,2, … , n)

Satisfy (1.2)and (1.3)and the following

$$\inf_{t\in R} a_{ij}(t) > \left\{ \sum_{\substack{J=1 \\ J\neq 1}}^{n} \sup a_{ij}(t) \right\} + \mu; \qquad i = 1,2, \dots, n$$

For some positive number $\mu$ $Then$ (1.1)has an almost-periodic solution (with strictly positive components) which is globally asymptotically stable.

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3]

Under condition (3.2) the almost –periodic solution $w(t) = \{w_1(t), \dots . w_n(t)\}$ of (1.1)enjoys a somewhat strong from of stability even if the coefficients of (1.1)are subjected to a class of perturbations We note (see Yoshizawa [7]page17)that if a function b:R→ $R$ is almost periodic then H(b),the "hull of b" is defined as follows:

H(b)= g|g:**R**→ $\boldsymbol{R}$ $and$ $g(t)\lim_{k\to\infty} b(t + T_k) for$ some sequence

$T_k$ of real numbers the limit being uniform in $t \in R$.

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost periodic functions are themselves almost periodic.

A system of the from

$$\frac{dy_1(t)}{dt} = y_1(t)\left\{B_i(t) - \sum_{J-1}^{n} A_{ij}(t) y_1(t)\right\},$$

*i=1,2,…,n; t>*$t_0 \in R$,

Where

$$B_i \in H(b_i); (A_{ij}), \qquad i, j = 1,2, \dots, n,$$

is known as a system from the hull of (1.1) As a consequence of the definition of the hull in (3.3)and (3.5)we shall have

$$\inf_{t\in R} B_i(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} B_i(t) = \mathrm{b}_i^{\,l};$$

$$\inf_{t\in R} A_{ij}(t) = \mathrm{b}_i^{\,l}; \quad \sup_{t\in R} A_{ij}(t) = \mathrm{a}_{ij}^{\,u};$$

and hence the coefficients of(3.4)- (3.5)satisfy the same conditions as (1.2),(1.3)and (3.2)It will then follow from theorems(2.1)and (3.1)that solutions of (3.4)-(3.5)with $y_1(t_0) > 0, i = 1,2,\dots,n$ *will have* the following property:

$$y_1(t) > 0 \ \ for\ t \geq t_0; i = 1,2,\dots,n,$$

$$\sum_{i=1}^{n} y_1(t) remains\ bounded\ for\ t \geq t_0$$

**DEFINITION** A Solution $x(t) = \{x_1(t),\dots,x_n(t)\}, t \in t_0 \in \boldsymbol{R}$,of (1.1)is said to be stable under perturbations from the hull of (1.1)if the following holds :let $y(t) = \{(y_1(t),\dots,y_n(t)\}$ be any solution of (3.4)-(3.5)such that $y_1(t_0) > 0$, i=1,2,…,n,and for some positive constants $\beta_1, \beta_2 \beta_1, \leq y_1(t) \leq \beta_2, t \geq t_0, \ \ i = 1,2,\dots,n$ Then given any $\varepsilon > 0$,there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{n} |\, x_1(t_0) - y_1(t_0)| < \delta$$

$$\sum_{i=1}^{n} \sup_{t\in R} |b_1\,(1) - B_1(t)| < \delta \quad = \sum_{i=1}^{n} |x_1(t\,) - y_i(t\,)| < \varepsilon$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} | \quad \sup_{t\in R} |a_{ij}\ \ (t) - A_{ij}(t)| < \delta$$

For $t \geq t_0$.

We can now formulate the following result which shows that all solution of (1.1)are stable under perturbations from the hull of (1.1)

**THEOREM 3.2**. Assume that the conditions of Theorem (3.1) hold. Then every strictly positive solution of (1.1) ( including its unique almost-periodic solution) is stable under perturbations from the hull of (1.1).

PROOF. Let $x(t) = \{x_1(t), \dots, x_n(t)\}$ and $y(t) = \{y_1(t), \dots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4) – (3.5) respectively for $t \geq t_0 \in R$. Let $\beta_1, \beta_2$ be constants such that

$$\begin{matrix} \beta_1 \leq x_i(t) \leq \beta_2, \\ \beta_1 \leq y_i(t) \leq \beta_2, \end{matrix} \quad i = 1,2, \dots, n; t \geq t_0 \tag{3.7}$$

Consider now a function $V(t) = V(x(t)y(t))$ defined by

$$V\big(x(t)y(t)\big) = \sum_{i=1}^{n} |\log x_1(t) - \log, y_i(t)|, \quad t \geq t_0 \in R. \tag{3.8}$$

It is easy to see that

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V\big(x(t)y(t)\big)$$

$$\frac{1}{\beta_1} \sum_{i=1}^{n} |x_i(t) - y_i(t)|, \; t \geq t_0 \in R. \tag{3.9}$$

Calculating the right derivative $D^+V$ of $V$ and using (1.1) and (3.4) we derive (after simplification) that

$$D^+V\ \big(x(t), y(t)\big) \leq -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+\beta_2 \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0,$$

$$\leq -\mu\beta_1 V\big(x(t), y(t)\big) + \sum_{i=1}^{n} |b_i(t) - B_i(t)|$$

$$+ \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \quad t \geq t_0, \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| \leq V(x(t), y(t)) \leq V(x(t_0), y(t_0))$$

$$+\left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t\in R} |a_{ij}(t) - A_{ij}(t)|\right] \frac{1}{\mu\beta_1},$$

$$t \geq t_0,$$

And hence

$$\sum_{i=1}^{n} |x_1(t) - y_1(t)| \leq \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} |x_1(t_0) - y_1(t_0)|$$

$$+ \frac{\beta_2}{\mu\beta_2} \left[\sum_{i=1}^{n} \sup_{t\in R} |b_i(t) - B_i(t)|\right.$$

The construction reason is because is criticized in a number of usage guides: The reason for the long delays was because the costs greatly exceeded the original estimates. One objection to this construction is based on its redundancy: the word because (literally, by cause) contains within it the meaning of reason; thus saying the reason is because is like saying “The cause is by cause,” which would never be said. A second objection is based on the claim that because can introduce only adverbial clauses and that reason is requires completion by a noun clause. Critics

would substitute that for because in the offending construction: The reason for the long delays in completing the project was that the costs.

$$+\beta_2 \sum_{i=1}^{n} \sum_{i=1}^{n} \sup_{t \in R} \mid a_{ij}(t) - A_{ij}(t) \mid \Big]. \qquad (3.11)$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min \left\{ \frac{\varepsilon}{3} \frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3} \frac{\beta_1}{\beta^2} \mu \right\} \qquad (3.12)$$

Then (3..6) will be satisfied and this completes the proof

## . Jia Li's Presciential Dynamics And Contential Staticity Of Accentuation Attrition Models Applied To Cosmological And Astrophysical And Quantum Mechanical Engineering Problems

**Abstract**: **Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics ET AL ). We shall specifically delineate and disseminate the information and particles and confederation of the financial systems to which the model is applied and in concomitant elucidation of the Haimovici's model we apply this to each and every module. Due to spatial constraints we have not done any separate sets as has been done in the foregoing. It is to be repeated that the model is to be applied for each and every module. In amplification this is true for all the (confederation of corresponding addition is made to the credit –debit system of atoms, molecules, qubits, cosmological interactions, financial transaction, economic exchange confederation as the case may be( Note the author deals for predator – population.) In amplification the model is applicable to various assortment of systems and their perspicuous characterstics, sophisticated seasoning, constituent structure, concept formulation, beat cadence, interiority of perfect causes, incompatibility of conjunctions', resonantial resumptions, hypothetical propositions, logical compatibilities, on causal correspondences, predicational inherence, emphasis related phenomenological methodologies, Partcipational observation, background actuality, addressed aestheticism, polished perception, refined proficiency, existential worldliness, existential strife, predicational anteriorities, apophthegmatic axiom, brocard- gnome, implantation and inculcation, pronouncement and proposition, declaration dogma, character consonance, shift and stratagem, ontological consonance, primordial exactitude, phenomenological correlates, accolytish representation, atrophied asseveration, suppositional surmise, theorization anamensial alienisms, anchorite aperitif, Arcadian Atticism all delineated in the conditionalities and functionalities and orientationalities of the system which incorporate the rules and regulations, axiomatic predications and postulation alcovishness of the foregoing state. Characterstics of such systems are taken in to consideration in the complete consolidation and consummation of the systems. Predator prey model literally form bastion, pillar, post, stylobate, sentinel towards the end of**

**dismasting many of the asymptotic and solutional behaviour in finance, economic, and as we shall see in cosmology, quantum computation and other fields.**

## MODEL FORMULATION

Let $p_1(t, a_1)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t, a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(g_1)Q_1$$

Where $b$ is the natural death rate in the absence of credit types and $g(P)$ is the conversion rate function of debits

into credit creations which is assumed to be differentiable and satisfies

$$g_1(0) = 0, \quad g_1(P_1) > 0, \ P_1 \ 0, \qquad \lim_{P_1 \to \infty} g_1(P_1) = (K_1)_1 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_1 + (\rho_a)_1 + D_1(a_1, P_1, Q_1)\rho_1 + 0, \qquad 0 < a_1 \ \infty,$$

$$\rho_1(t, 0) = \int_0^\infty n_1(a_1, P_1, Q_1)\rho_1(t_1, a_1)da_1$$

Where $D_1$ is the mortality function and $n_1$ is the fertility function.

Put $_1$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_1}{dt}\right]_1 = x_1 \left[\frac{dQ_2}{dt}\right]_1 - x_2 \left[\frac{dQ_1}{dt}\right]_1$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3 \left[\frac{dQ_3}{dt}\right]_1 - x_4 \left[\frac{dQ_2}{dt}\right]_1$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_1 - x_6\left[\frac{dQ_3}{dt}\right]_1$$

Where $x_1, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_1$ and $g_1(P_1)Q_1$

Now when you add the term $(b_1Q_1)$ with $g_1(P_1)(Q_1)$ we are infect see is that $(b_1Q_1)$ is credited incrementally part of $g_1(P_1)(Q_1)$ or instant nearly, when subtracted from $\frac{dQ_1}{dt}$ , it means that $g_1(P_1)(Q_1)$ is slowly eaten by $\frac{dQ_1}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_1 = \mu_1(a_1) + (f_1)_1(P_1)_1 + (f_2)_1(Q)_1 \quad (2)$$

$$n_1 = \beta_1(a_1)h_1(Q_1), \quad (3)$$

Where $\mu_1$the age- dependent debit rate whit is $\mu_1(a_1)$ $\forall a > 0$; $(f_1)_1$ is the density- dependent death rate with $f_1(0) = 0$, $(f_1^1)_1(P_1)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_1)_1(P_1) = (k_1)_1$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_1(0) = 0, (f_2^1)_1(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_1(Q_1) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^1(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_1 + (\rho_a)_1 = -\mu_1(a_1) + (f_1)_1P_1) + \big(f_2)_1(Q_1)\big)(\rho_1)_1 \quad (4)$$

$$\rho_1(t_1, 0) = h_1(Q_1)\int_0^\infty \beta_1(a_1)\rho_1(t, a_1)da, \quad (5)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + g_1(P_1)Q_1 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_1$ and $Q_1$ of (4-6) satisfy

$$0 \le \rho_1(t, a_1) \le \rho_1^*(t, a_1),$$

$$0 \le Q_1(t) \le Q_1^*(t),$$

Where $\rho_1^*(t, a_1)$ and $Q_1^*(t)$, are solutions of

$$(\rho_t^*)_1 + (\rho_a^*)_1 = -(\mu_1(a_1) + (f_1)_1(P_1^*) + (f_2)_1 Q_1^*))\, \rho_1^*, \quad (7)$$

$$\rho_1^*(t, 0) = \int_0^\infty \beta_1(a_1)\, \rho_1^*(t, a_1) da_1 \quad (8)$$

$$\frac{dQ_1^*}{dt} = -b_1 Q_1^* + g_1(P_1^*)Q_1^*, \quad (9)$$

With $P_1^*(t) = \int_0^\infty \rho_1^*(t, a_1) da_1$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_1^*(t)}{dt} = P_1^*(t)\big(r_1 - (f_1)(P_1^*) - (f_2)_1(Q^*)\big), \quad (10)$$

$$\frac{dQ_1^*(t)}{dt} = Q_1^*(t)(-b_1 + g_1(P_1^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_1(a)e^{-\int_0^a \mu_1(\tau)dt}e^{-ra}\,da_1 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_1^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(t)dt}\,da_1 > 1, \qquad (H1)$$

Which guarantees $r_1 > 0$.

*Remark.* If $\beta_1(a) = [\beta_0]_1 a_1 e^{-\alpha_1 a_1}, \mu_1(a_1) \equiv \mu_1$ then the solution of (12) is

$$r_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_1]_1(P_1) = 0, [f_2](Q_1) = \lambda_1 Q_1\ h_1(Q_1) = 1, \qquad and\ \ g_1(P_1) = c_1 P_1$$

$$[\rho_1]_1 + [\rho_a]_1 = -(\mu_1 + \lambda_1 Q_1)\rho_1$$

$$\rho_1(t,0) = \int_0^\infty [\beta_0]_1 a_1 e^{-\alpha_1 a_1}\rho_1(t,a_1)\,da_1$$

$$\frac{dQ_1}{dt} = -b_1 Q_1 + c_1 P_1 Q_1$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_1}{dt} = \omega_1 P_1 - \lambda_1 P_1 Q_1$$

$$\frac{dQ_1}{dt} = b_1 Q_1 - c_1 P_1 Q_1$$

Where $\omega_1 = \sqrt{\beta_1} - \alpha_1 - \mu_1$ THE EQUILIBRIUM $(\rho_0)_1(a_1), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_1(a_1), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_1 + (\rho_a)_1 = -\big(\mu_1(a_1) + (f_1)_1(P_1)\big)\rho,$$

$$\rho_1(t,0) = \int_0^\infty \beta_1(a_1)\rho_1(t,a_1)da_1 \qquad (13)$$

The solution of (13) has the form of

$$\rho_1(t,a_1) = Ce_1^{r(t-a_1)} - \int_0^a (\tau)d\tau - \int_0^1 (f_1)_1\big(P_1(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_1(t) = C_1 e^{r_1 t - \int_0^t (f_1)_1 (P_1(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_1^{-1}(r)$ if $k_1 > r$.

This is equivalent to

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{(k_1)_1 a_1} da_1 < 1.$$

In fact, if we define

$$R_1(s_1) = \int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau)d\tau} e^{-s_1 a_1}\, da_1$$

$R(s)$ is a decreasing function. $R_1(r_1) = 1$ And $R_1(k_1)_1 < 1$ imply $r_1 < (k_1)_1$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_1(a_1), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{k_1 a_1} da_1$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_1 \equiv \int_0^\infty (\rho_0)_{11}(a_1)da_1$ *satisfies*

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau)d\tau} e^{-(f_1)_1(P_0)a_1} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_1(a_1), 0)$

Let $u_1(t, a_1) = \rho(t, a_1) - \rho_0(a_1)$ Then

$$N_1(t) \equiv \int_0^\infty u_1\,(t, a_1)da_1 = \int_0^\infty \rho_1(t, a_1)da_1 - \int_0^\infty (\rho_0)_1(a_1)da = P_1(t) - P_0)_1$$

And

$$(u_1)_1 + (u_a)_1 = (\rho_t)_1 + (\rho_a)_1 - (\rho_0^1)_1(a_1)$$

$$= -\mu_1(a_1) + (f_1)_1(P_1) + (f_2)_1(Q_1))\,(u_1 + (\rho_0)_1) + \mu_1(a_1)(\rho_0)_1 + (f_1)_1(P_0)_1(\rho_0)_1$$

$$= -\mu_1(a_1)u_1 - (f_1)_1(P_1) + (f_2)_1(Q_1))\,u_1 - (\rho_0)_1\,(f_1)_1(P_1) - (f_1)_1(P_0)_1$$

$$+ (f_2)_1(Q_1)), \qquad (15)$$

Where $(\rho_0^1)_1(a_1)$ denotes $d(\rho_0)_1(a_1)/da_1$

Expansion $(f_1)_1(P_1)$ and $(f_2)_1(Q_1)$ at $(P_0)_1$ and 0 yield respectively

$$(f_1)_1(P_1) = (f_1)_1((P_0)_1 + N_1)$$

$$= (f_1)_1(P_0)_1 + (f_1^1)_1(P_0)_1 N_1 + o_1(N_1), \qquad (16)$$

$$(f_2)_1(Q_1)$$

$$= (f_2^1)_1(0)Q + o_1(Q_1) \qquad (17)$$

Write $f_1(P_0) = f_{10}, f_1'(P_0) = f_{10}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_1)_1 + (u_a)_1 \approx -(\mu_1(a_1) + (f_{10})_1)u - (f_{10}^1)_1(\rho_0)_1(N_1) - (f_{20}^1)_1(\rho_0)_1(Q_1)$$

$$u_1(t,0) = \rho_1(t,0)(\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a) + (\rho_0)_1(a_1)da_1 - (\rho_0)_1(0)$$

$$= h_1(Q_1)\int_0^\infty \beta_1(a_1)(u_1(t,a_1)da_1 + (h_1(Q_1) - 1)(\rho_0)_1(0) \qquad (18)$$

Expanding

$$h_1(Q_1) = 1 + (h^1)_1(0)Q_1 + 0(Q_1) \equiv 1 + h_0'Q_1 + 0Q_1$$

And substituting it into (18), we arrive at

$$u_1(t,0) \approx \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + (h_{a_1}')(\rho_0)_1 Q_1$$

The linearization is the following:

$$[u_1] + [u_1] = -(\mu_1(a_1) + [t_{10}]_1 u_1 - [\alpha_1]_1(a_1)N_1 - [\alpha_2]_1(a_1)Q_1$$

$$u_1(t,0) = \int_0^\infty \beta_1(a_1)u_1(t,a_1)da_1 + [\xi_1]Q_1 \qquad (19)$$

$$\frac{dQ_1}{dt} = -b_1 Q_1$$

where

$$[\alpha_1]_1(a_1) = [t_1^1 a]_1[p,o]_1(a_1)d2(a_1) = [f_{20}^1]_1(\rho\, o\,)_1(a_1)\xi = [\mathrm{h\ o}^1]_1[\rho\mathrm{o}]_1(\mathrm{o}) \text{ and } \mathrm{b}_1 = \mathrm{b}_1 - g_1(P_1)$$

If $g_1(P_0) > b$, clearly, $(\rho_0)_1(a_1), 0)$ is unstable and the predator population is persistent. Assume $g_1(P_0)_1 < b$.

Then, $Q_1(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_1 + u_a = -(\mu_1(a_1) + (f_{10})_1)u_1 - (\alpha_1)_1(a_1)N_1$$

$$= -(\mu_1(a_1) + f_{10})u_1 - \alpha_1(a_1)\int_0^\infty u_1(t,a_1)da_1$$

$$u_1(t,0) = \int_0^\infty \beta_1\,(a_1)u_1(t,a_1)da_1$$

Using the method developed by Cushing [2, 3], we suppose

$$u_1(t,a_1) = \tilde{\mathrm{u}}_1(a_1)^{e^{\gamma(t-a_1)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_1}\,da_1$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_1)$ gives,

$$\tilde{\mathrm{u}}_1(a_1) = e^{-\int_0^a \mu_1(\tau_1)d\tau - (f_{10})_1 a}\left(\tilde{\mathrm{u}}_1(0) - w\int_0^a e^{\int_0^\tau \mu_1(s_1)ds + (f_{10})}\tau_1\,(\alpha_1)_1(\tau_1)e^{\gamma_1\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\right)\tilde{u}_1(0)$$

$$+w\int_0^{\infty}\beta_1(a_1)e^{-\gamma_1 a_1-(f_{10})_1 a_1-\int_0^a \mu_1(\tau_1)d\tau}\int_0^a(\alpha_1)_1(\tau_1)e^{-\gamma_1\tau_1}e^{(f_{10})_1\tau_1+\int_0^{\tau}\mu_1(s_1)ds}d\tau_1\,da_1$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}\int_0^a e^{f_{10}\tau+\int_0^{\tau}\mu(s)ds+y\tau}\alpha_1(\tau)d\tau\,da\right)w=0. (23)$$

Define $Y(\gamma,a)=e^{-ya-f_{10}a-\int_0^a\mu(\tau)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta(a)Y(\gamma,a)da\right)u(0)$$

$$+w\int_0^{\infty}\beta(a)Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^1(\gamma,\tau)d\tau\,da=0, (24)$$

$$\left(\int_0^{\infty}Y(\gamma,a)da\right)u(0)$$

$$+\left(-1-\int_0^{\infty}Y(\gamma,a)\int_0^a\alpha_1(\tau)Y^{-1}(\gamma,\tau)d\tau\,da\right)w=0.$$

Equation (20) has a solution of the form of $u(t,a)=u(a)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $u(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I(y,a)=\int_0^a\rho_0(\tau)Y^{-1}(\gamma,\tau)d\tau$. We obtain the characteristic equation

$$f_{10}^{l}\left(\int_0^\infty Y(\gamma,a)da\right)\left(\int_0^\infty \beta(a)Y(\gamma,a)I(\gamma,a)da\right)$$

$$+\left(1-\int_0^\infty \beta(a)Y(\gamma,a)da\right)\left(1+f_{10}^{l}\int_0^\infty Y(\gamma,a)I(\gamma,a)da\right)=0. \quad (25)$$

Since

$$\rho_0(a)=\rho_0(0)e^{-\int_0^a \mu(\tau)d\tau-f_{10}a}$$

$$I(\gamma,a)=\int_0^a \rho_0(\tau)e^{\gamma\tau+f_{10}\tau+\int_0^\tau \mu(s)ds}d\tau$$

$$=\frac{\rho_0(0)}{\gamma}(e^{\gamma a}-1)$$

Let

$$E(a)=e^{-f_{10}a-\int_0^a \mu(\tau)d\tau}$$

Then $[Y_1]_1[\gamma_1]_1,a_1)=E_1(a_1)e^{-\gamma_1 a_1}$. by a simple calculation,

$$\int_0^\infty \beta_1(a_1)Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma}\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}\ da\right);$$

$$\int_0^\infty Y_1(\gamma_1,a_1)I(\gamma_1,a_1)da=\frac{[\rho_0]_1(0)}{\gamma_1}\left(\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}\ da_1\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_1(0)}{\gamma_1}[f_{10}^{1}]\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\left(\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)$$

$$+\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right).$$

$$\left(1+\frac{(\rho_0)_1(0)}{\gamma_1}(f_{10}^{'})_1\left(\int_0^\infty E_1(a_1)da_1-\int_0^\infty E_1(a_1)e^{-\gamma_1 a_1}da_1\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\right)(1+\frac{(f_{10}^{'})_1}{\gamma_1}(\rho_0)_1(0)\int_0^\infty E_1(a_1)da_1)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)da_1=1-\int_0^\infty \beta_1(a_1)E_1(a_1)e^{-\gamma_1 a_1}da_1\neq 0,$$

Since $\int_0^\infty \beta_1(a_1)Y_1(\gamma_1 a_1)\int_0^a \alpha_1(\tau_1)Y^{-1}(\gamma_1,\tau_1)d\tau_1 da_1>0$. Therefore, the characteristic solution is

$$\gamma_1=-(f_{10}^{'})_1(\rho_0)_1\int_0^\infty E_1(a_1)da_1.$$

Since $f_1'(P_0)_1>0$, we obtain the following result

THEOREM 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_1),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_1),0)$ which is related to the extinction of debit summation as has been referred to in the foregoing and to The possible persistence of the prey

population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_1 + (\mu_1(a_1) + [f_1]_1(P_1^*) + [f_2]_1(Q_1^*))P_1^* = 0, \quad (26)$$

$$p_1^*(0) = h_1(Q_1^*)\int_0^\infty \beta_1(a_1)p_1^*(a_J)p_1^*(a_1)da_1 \quad (27)$$

$$g_1(P_1^*) - b_1 = 0, \quad (28)$$

Where $P_1^* = \int_0^\infty p_1^*(a_1)\, da_1$

Equation (28) yields $P^* = g^{-1}(b)$,

From (26),

$$p_1^*(a) = p_1^*(0)e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}$$

Substituting into (27), we need to solve

$$\left(1 - h_1(Q_1^*)\int_0^\infty \beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(Q_1^*)a_1}da_1\right)p_1^*(0) = 0$$

For $Q_1^*$

Fix $p_1^*(0)$ and $P_1^*$ Let

$$H_1(z_1) = 1 - h_1(z_1)\int_0^\infty \beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}da_{,1}$$

And assume that

$$[k_3]_1 \int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(g_1^{-1}(b_1))a_1 - [k_2]_1 a_1}\, da_1 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_1(z_1)] > 0.$$

Clearly,

$$\frac{dH_1}{dz_1} = h_1(z)\left([f_2^{'}]_1(z_1)\right)\int_0^{\infty} a_1\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - f_1(P_1^*)a_1 - f_2(z_1)a_1} da_1$$

$-h^{'}(z_1)\int_0^{\infty}\beta_1(a_1)\, e^{\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1 - [f_2]_1(z_1)a_1}\quad da_1$

By the assumptions on $\left[f_2^1\right]_1 and\ [h^1]^1 dH_1/dz > 0$. Hence, if

$$H_1(0) = 1 - \int_0^{\infty}\beta_1(a_1)e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1(P_1^*)a_1} da_1 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_1(0) < 0$ follows if $g_1^{-}(b_1) < [f_1^{-1}]_1(r_1)$, where $r_1$ is the solution of (12).

Indeed, since

$$R_1(s_1) = \int_0^{\infty}\beta_1(a_1)\, e^{-\int_0^a \mu_1(\tau_1)d\tau_1 e^{-y_1^a}} da_1$$

Is a decreasing function of $s_1$ when $P_1^* = [g^{-1}]_1(b_1) < [f_1^{-1}]_1(r_1)$, that is $[f_1]_1(P_1^*) < r$, $R(f_1(P^*)) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_1(g^{-1}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1)d\tau_1 - [f_1]_1\left(g_1^{-1}(b)\right)a_1} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_1^*(a_1), Q_1^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_1^*(a_1), Q_1^*)$ is more complicated. We again consider perturbations $m_1(t_1 a_1) = p_1(t_1, a_1) - p_1^*(a_1)$ and $W_1(t) = Q_1(t) - Q^*$.Then

$$M_1(t) \equiv \int_0^\infty m_1(t, a_1) da_1 = P_1(t) - P_1^*,$$

$$[m_1]_1 + [m_a]_1 = [p_1]_1 + [p_a]_1 - [p^*]_1(a_1) \approx -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1$$

$$-[f_{1*}^1]_1 p_1^*(a_1)M - [f_{2*}^1]_1 p_1^*(\mathrm{a})W_1,$$

$$m_1(t,0)p_1(t,0) - p_1^*(0) \approx h_* \int_0^\infty \beta_1(a_1) m_1(t, a_1)\, da + \frac{[h_*^1]_1}{[h_*]_1} p_1^*(0) W_1$$

$$\frac{dW_1}{dt} = \frac{dQ_1}{dt} \approx [g_*^1]_1 Q_1^* M_{,1}$$

Where $[f_{1*}]_1$ denoting $[f_1(P^*)]_1$, $[f_{1*}]_1$ denoting $[f_2(P^*)]_1$, $[f_{1*}^{'}]_1$ denoting $[f^{'}{}_1(P^*)]_1 [f_{2*}^{'}]_1$ denoting $[f_2]_1$ $(Q_1^*$, $h_{*1}$ denoting $h_*^{'}$ denoting $h_1^{'}$ $(Q_1^*)$, and $[g_*^{'}]_1$

Denoting $[g^{'}]_1(P_1^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_1]_1 + [m_a]_1 = -(\mu_1(a_1) + [f_{1*}]_1 + [f_{2*}]_1)m_1 - [f'_{1*}]_1[p_1^*](a_1)M_1 - [f'_{2*}]_1 p_1^*(a_1)W_1$$

$$m_1(t,0) = [h_*]_1 \int_0^\infty \beta_1(a_1)m_1(t,a_1)da_1 + \frac{[h_*^1]_1}{[h^*]_1}[p^*]_1(0)\,W_{,1} \qquad (30)$$

$$\frac{dW_1}{dt} = [g_*^1]_1[Q_1^*]M_1$$

Integrating $[m_1]_1(t,a_1)$ with respect to a from 0 to ∞, we have

$$\frac{dM_1(t)}{dt} = \int_0^\infty [mt]_1(t,a_1)da_1$$

$$= -\int_0^\infty [m_a]_1\,(t,a_1)da_1 - \int_0^\infty \mu(a_1)m(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 \int_0^\infty p_1^*(a_1)da_1\right)M_1 - [f_{2*}^1]_1 \int_0^\infty p_1^*(a_1)da_1 W_1$$

$$= m_1(t,0) - \int_0^\infty \mu_1(a_1)\,m_1(t,a_1)da_1$$

$$-\left([f_{1*}]_1 + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 - [f'_{2*}]_1 P_1^* W_{,1}$$

From (30), we arrive at the equations

$$\frac{dW_1(t)}{dt} = \int_0^\infty ([h_*]_1\beta_1(a_1) - \mu_1(a_1))m_1(t,a_1)da_1$$

$$-\left([f_{1*}] + [f_{2*}]_1 + [f'_{1*}]_1 P_1^*\right)M_1 + \left(\frac{[h'_*]_1}{[h_*]^1}\,P_1^*(0) - f'_{2*}P_1^*\right)W_1,$$

(31)

$$\frac{dW_1(t)}{dt}\left[g^{'}{}_{*}\right]_1 Q_1^* M_{.1}$$

By the mean value theorem,

$$\int_0^{\infty}([h_{"*}]_1\beta_1(a_1)-\mu_1(a_1))m_1(t,a_1)da_1=([h_*]_1\beta_1(\theta_1)-\mu_1(\theta_1))M_{,1}$$

Where $0< \theta_1=\theta_1(t)<\infty$.

And if $a_1^2\,[a_1^2]_1<4\,[a_2]_1<4\,a_2 e$,

Re $\lambda_1\left[\lambda_1\right]_1=\text{Re}\lambda_2\left[\lambda_2\right]_1=-\frac{1}{2}a_1[a2\ ]_1<0$.

Let $\Delta=\max_{2<u<\infty}\left|h_*\,[h_*]_1\beta_1(a_1)-\mu_1(a_1)\right|$. Then$|n_1(t)|$ ⊿∀t ⊿. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_1^2<4\,a_2 e$ and $[a_1]>2\Delta$,or $[a_1^2]$ ⊿$[a_2]e$ and $(a_1-\sqrt{(a_1^2)_1-4(a_2)_1 e})>2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{a}{a+1}\right)$ and $\beta_1(a_1)=ae^{-ma}$ Then $\Delta=[\mu_0]+1$ Simple calculation leads to

$$\int_0^{\infty} \beta_1(a_1) e^{-\int_0^a \mu_1(\tau_1) + d\tau_1 da} = \frac{1}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_1 + [\mu_0]$ 0.5. Assume$[f_1](x_1) = K_1 x_1$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_1(x_1) = [0x_1]/(1+x_1)$; the functional responses $[[f_2](x) = [0]\, x_1/(1+x_1)$and $h_1(x_1)$=$(1+\omega_1 x_1)/(1+x_1)$(See Holling [7] and Rashevsky [13])

Equation (14),now has the form of

$$\frac{m_1 + [\mu_0] + [K][P_0])}{[m_1]_1 + (\mu_0)_1 + [K][P_0])_1^3\ [K]_1 [P_0]_1^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_1$ i e $[P_0] < b/(\delta_1 - b_1)$then the equilibrium$(P_0)$ $[P_0]_1(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_1$

$[f_{1*}] = K_1\ P^*\ [f_{2*}] = \frac{\sigma Q_1^*}{1+Q_1^*} \quad [h_*] = \frac{1+\omega_1 Q_1^*}{1+Q_1^*}$

$[f_{1*}] = K_1\ [f_{2*}^1] = \frac{\sigma_1}{(1+Q_1^*)^2} [g_*^1] = \frac{\sigma_1}{(1+P_1^*)^2} \quad [h_*^1] = \frac{\omega_1 - 1}{(1+Q_1^*)^2}$

## AN ANALOGY BETWEEN PREDATOR PREY AND ACCENTUATION ATTRION MODELS

**Best example how the model could be applied is the Schrödinger's equation itself wherein RHS gobbles up LHS, and the terms on RHS do the same with each other. It is like adding milk to water. Whether it happens instantaneously or time lag is there is a different question altogether dealt at the opportune points**

| |
|---|
| **Best example that can be given is the formulation of the quantum wave equation albeit in a nascent form for the recently found gravitational waves. This has been studied in length in various papers of gravitational waves, wherein the attribution of how the model is used is described. Note the calculation holds for only one module. This is in consideration to the fact that the relation is beteen the both debit systems and credit systems which as we have enucleated in the foregoing bears ample testimony for the application of the model** |
| **Gravitational quantum wave equation:** |
| **Equation Time-dependent equation (Wikipedia)** |
| **(1) In quantum mechanics, the Schrödinger equation is (=) a partial differential equation that describes how the quantum state of a quantum system changes with (e&eb) time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.[1]** |
| **(2) In classical mechanics Newton's second law, (F = ma), is used to make (eb) a mathematical prediction as to what path a given system will take following a set of known initial conditions.** |
| **(3) In quantum mechanics, the analogue of Newton's law is (=) Schrödinger's equation for (e) a quantum system (usually atoms, molecules, and subatomic particles whether free, bound, or localised).** |
| **(4) It is not a simple algebraic equation, but in general a linear partial differential equation, describing (eb) the time-evolution of the system's wave function (also called a "state function").[2]:1–2** |
| **(5) The concept of a wavefunction is (=) a fundamental postulate of quantum mechanics.** |
| **(6) Although Schrödinger's equation is often presented as a separate postulate, some authors [3]: Chapter 3 show that some properties resulting from Schrödinger's equation may be deduced just from (e) symmetry principles alone, for example the commutation relations.** |
| **(7) Generally, "derivations" of the Schrödinger equation demonstrate (eb) its mathematical plausibility for describing (eb) wave–particle duality, but to date there are no universally accepted derivations of (e) Schrödinger's equation from (e) appropriate axioms.** |
| **(8) In the Copenhagen interpretation of quantum mechanics, the wave function is (=) the most complete description that can be given of (e) a physical system.** |
| **(9) Solutions to Schrödinger's equation describe (eb) not only molecular, atomic, and subatomic systems, but also macroscopic systems, possibly even the whole universe.[4]:292ff** |

| |
|---|
| **(10) The Schrödinger equation is consistent with (e&eb) classical mechanics but not with (e&eb) special relativity.** |
| **(11) Making quantum mechanics consistent with (e&eb) special relativity is one of the great achievements of (e) Quantum Field Theory and requires (e) the introduction of the creation and annihilation operators so that (e) the number of particles can no longer be considered constant.** |
| **(12) The Schrödinger equation is not the only way to make predictions in (eb) quantum mechanics—other formulations can be used, such as Werner Heisenberg's matrix mechanics, and Richard Feynman's path integral formulation** |
| **(13) The form of the Schrödinger equation depends on (=) the physical situation (see below for special cases).** |
| **(14) The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time: [5]:143** |
| **(15) A wave function that satisfies the non-relativistic Schrödinger equation with V = 0. In other words, this corresponds to (e &eb) a particle traveling freely through (e&eb) empty space. Time-dependent** |
| **Schrödinger equation (general)** |
| **(16) i ℏ ∂ ∂ t Ψ ( r , t ) = H ^ Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)={\hat {H}}\Psi (\mathbf {r} ,t) where i is the imaginary unit, ħ is the Planck constant divided by 2π, the symbol ∂/∂t indicates a partial derivative with respect to time t, Ψ (the Greek letter psi) is the wave function of (e) the quantum system, and Ĥ is the Hamiltonian operator (which characterizes the total energy of any given wave function and takes different forms depending on the situation).** |
| **(17) Each of these three rows is a wave function which satisfies (eb) the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right:** |
| **(18) The probability distribution of finding the particle with (e&eb) this wave function at a given position.** |
| **(19) The top two rows are examples of stationary states, which correspond to (e&eb) standing waves.** |
| **(20) The bottom row is an example of a state which is (=) not a stationary state.** |
| **(21) The right column illustrates (eb) why stationary states are (=) called "stationary".** |

**(22) The most famous example is the non-relativistic Schrödinger equation for (e) a single particle moving in an electric field (but not a magnetic field; see the Pauli equation): [6]**

**Time-dependent Schrödinger equation**

**(Single non-relativistic particle)**

**(23) i ℏ ∂ ∂ t Ψ ( r , t ) = [ − ℏ 2 2 μ ∇ 2 + V ( r , t ) ] Ψ ( r , t ) {\displaystyle i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)} i\hbar {\frac {\partial }{\partial t}}\Psi (\mathbf {r} ,t)=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} ,t)\right]\Psi (\mathbf {r} ,t)**

**(24) Here μ is (=) the particle's "reduced mass", V is its potential energy, ∇2 is the Laplacian (a differential operator), and Ψ is (=) the wave function (more precisely, in this context, it is called the (=) "position-space wave function").**

**(25) In plain language, it means "total energy equals (=) kinetic energy plus potential energy", but the terms take unfamiliar forms for (e) reasons explained below.**

**(26) Given the particular differential operators involved, this is (=) a linear partial differential equation.**

**(27) It is also a diffusion equation, but unlike the heat equation, this one is (=) also a wave equation given (e&eb) the imaginary unit present in the transient term.**

**(28) The term "Schrödinger equation" can refer to (e) both the general equation (first box above), or the specific nonrelativistic version (second box above and variations thereof).**

**(29) The general equation is indeed quite general, used throughout quantum mechanics, for everything from the Dirac equation to quantum field theory, by plugging in various complicated expressions for the Hamiltonian. The specific nonrelativistic version is a simplified approximation to reality, which is quite accurate in many situations, but very inaccurate in others (see relativistic quantum mechanics and relativistic quantum field theory)**

**(30) To apply the Schrödinger equation, the Hamiltonian operator is set up for the system, accounting for the kinetic and potential energy of the particles constituting the system, then inserted into the Schrödinger equation. The resulting partial differential equation is solved for the wave function, which contains information about the system.**

**Time-independent equation**

**(31) The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states (also called "orbitals", as in**

**atomic orbitals or molecular orbitals). These states are important in their own right, and if the stationary states are classified and understood, then it becomes easier to solve the time-dependent Schrödinger equation for any state. Stationary states can also be described by a simpler form of the Schrödinger equation, the time-independent Schrödinger equation. (This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.)**

**Time-independent Schrödinger equation (general)**

**(32) E Ψ = H ^ ⚿ Ψ {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi } {\displaystyle E\Psi =\operatorname {\hat {H}} \Psi }**

**In words, the equation states:**

**(33) When the Hamiltonian operator acts on a certain wave function Ψ, and the result is proportional to the same wave function Ψ, then Ψ is a stationary state, and the proportionality constant, E, is the energy of the state Ψ.**

**(34) The time-independent Schrödinger equation is discussed further below. In linear algebra terminology, this equation is an eigenvalue equation.**

**(35) As before, the most famous manifestation is the non-relativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):**

**Time-independent Schrödinger equation (single non-relativistic particle)**

**(36) E Ψ ( r ) = [ − ℏ 2 2 μ ∇ 2 + V ( r ) ] Ψ ( r ) {\displaystyle E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )} E\Psi (\mathbf {r} )=\left[{\frac {-\hbar ^{2}}{2\mu }}\nabla ^{2}+V(\mathbf {r} )\right]\Psi (\mathbf {r} )**

**Gravitational waves: (Wikipedia)**

**(37) In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×1036 joules of which only 200 Watts (joules per second) is lost through gravitational radiation, leading to a decay in the orbit by about 1×10−15 meters per day or roughly the diameter of a proton. At this rate, it would take the Earth approximately 1×1013 times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the majority of the time the bodies are far apart and only radiating slowly, so the difference is unimportant in this example.**

**More generally, the rate of orbital decay can be approximated by[39]**

**(38) d r d t = − 64 5 G 3 c 5 ( m 1 m 2 ) ( m 1 + m 2 ) r 3 , {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac**

**{(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,} {\displaystyle {\frac {\mathrm {d} r}{\mathrm {d} t}}=-{\frac {64}{5}}\,{\frac {G^{3}}{c^{5}}}\,{\frac {(m_{1}m_{2})(m_{1}+m_{2})}{r^{3}}}\ ,}**

**where r is the separation between the bodies, t time, G Newton's constant, c the speed of light, and m1 and m2 the masses of the bodies. This leads to an expected time to merger of [39]**

**t = 5 256 c 5 G 3 r 4 ( m 1 m 2 ) ( m 1 + m 2 ) . {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.} {\displaystyle t={\frac {5}{256}}\,{\frac {c^{5}}{G^{3}}}\,{\frac {r^{4}}{(m_{1}m_{2})(m_{1}+m_{2})}}.}**

**(39) For example, a pair of solar mass neutron stars in a circular orbit at a separation of 1.89×108 m (189,000 km) has an orbital period of 1,000 seconds, and an expected lifetime of 1.30×1013 seconds or about 414,000 years. Such a system could be observed by LISA if it were not too far away. A far greater number of white dwarf binaries exist with orbital periods in this range. White dwarf binaries have masses in the order of the Sun, and diameters in the order of the Earth. They cannot get much closer together than 10,000 km before they will merge and explode in a supernova which would also end the emission of gravitational waves. Until then, their gravitational radiation would be comparable to that of a neutron star binary.**

**(40) By substituting value of dr/dt in Schrödinger's wave equation we get the quantum wave equation for gravitational waves.**

In order to have (H3) and (H4) hold, $m, \mu_0$, K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_1}{(m+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma)^2}+\frac{2\omega_1}{([m_1]_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1+b_1}+\sigma_1)^3}<1$$

And

$$\frac{1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^2}+\frac{2\omega_1}{(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1})^3}>1$$

When the inequalities (33) and (34) hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_1+[\mu_0]_1+1+\frac{K_1 b_1}{\delta_1-b_1}+(m_1+[\mu_0]_1+\frac{K_1 b_1}{\delta_1-b_1}+\sigma_1)\,Q_1^*)^3$$

$=m_1 + [\mu_0]_1 + 3 + \frac{K_1 b_1}{\delta_1 - b_1}(1+ Q^*)(1 + \omega_1 Q_1^*$

Now

$$\left[a_1^2\right] - 4\left[a_2\right] e = \left(\frac{2K_1 b_1}{\delta_1 - b_1}\right)^2 + \frac{4K_1 b_1 \sigma_1 Q_1^*}{(\delta_1 - b_1)(1 + Q_1^*)} + \frac{\sigma_1^2 Q_1^{*2}}{(1 + Q_1^*)^2}$$

$$\frac{4\, b_1 \sigma_1 (\delta_1 - b) Q^*}{\delta_1 (1 + Q_1^*)^2} - \frac{4\,(\delta_1 - b_1)^2 (1 - \omega_1) Q_1^*}{\delta_1 (1 + \omega_1 Q_1^*)(1 + Q_1^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_1 a e^{mu}{}_1 \; [\mu_0]_1 - \frac{a_1}{a_1+1}$

Hence if $[a_1^2 > 4(a_2)_1 e$ and

$$[a_1] = \frac{2K_1 b_1}{\delta_1 - b_1} + \frac{\sigma_1 Q_1^*}{1 + Q_1^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_1]_1 = [f_{1*}]_1 + [f_{2*}]_1 + [f_{1*}] P_1^*$$

$$[a_2]_1 = \left[f_{2*}^1\right] P_1^* - \frac{h_*^1}{h_*} P^*(0)$$

$$e=[\,g^{1}_{*}]_{\,1}\,Q^{*}_{1}$$

$$\eta_1(t) = \left[h_{*}\right]_{1}\beta_{1}\left(\theta_{1}(t)\right) - \mu_{1}\left(\theta(t)\right),$$

The system(31)can be written as the following system:

$$[M(t)]\left[a_{1}\right]M_1 - [a_2]_{1}W_1 - [n(t)]M_1$$

$$w_1(t) = eM_1$$

Write

$$A=\begin{pmatrix} -[a_1]_{1} - [a_2]_{1} \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{1} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}$$

The solution of (32)is

$$X_1(t) = e^{A,t}[X]_{1}(0) + \int_{0} e^{At}$$

**Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models**

**(Module Three)**

**Abstract: Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-435) 1990. We demonstrate its applicability cosmology/ quantum computation economics et al. As stated in the foregoing we shall apply model to various cosmological problem and show the relevance to the same. . To exemplify model can be applied to the following situational significations: Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 Masaru Shibata and Takashi Nakamura** perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. **Authors** use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice (α=1) it is not possible. As in the axisymmetric case and

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Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(g_3)Q_3$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_3(0) = 0, \quad g_3(P_3) > 0, \ P_3 \ 0, \qquad \lim_{P_3 \to \infty} g_3(P_3) = (K_3)_3 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_3 + (\rho_a)_3 + D_3(a_3, P_3, Q_3)\rho_3 + 0, \qquad 0 < a_3 \ \infty,$$

$$\rho_3(t,0) = \int_0^\infty n_3(a_3, P_3, Q_3)\rho_3(t_3, a_3)da_3$$

Where $D_3$ is the mortality function and $n_3$ is the fertility function.

Put $_3$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_3}{dt}\right]_3 = x_3\left[\frac{dQ_2}{dt}\right]_3 - x_2\left[\frac{dQ_3}{dt}\right]_3$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_3 - x_4\left[\frac{dQ_2}{dt}\right]_3$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_3 - x_6\left[\frac{dQ_3}{dt}\right]_3$$

Where $x_3, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_3$ and $g_3(P_3)Q_3$

Now when you add the term $(b_3Q_3)$ with $g_3(P_3)(Q_3)$ we are infect see is that $(b_3Q_3)$ is credited incrementally part of $g_3(P_3)(Q_3)$ or instant nearly, when subtracted from $\frac{dQ_3}{dt}$ , it means that $g_3(P_3)(Q_3)$ is slowly eaten by $\frac{dQ_3}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_3 = \mu_3(a_3) + (f_3)_3(P_3)_3 + (f_2)_3(Q)_3 \quad (2)$$

$$n_3 = \beta_3(a_3)h_3(Q_3), \quad (3)$$

Where $\mu_3$the age- dependent debit rate whit is $\mu_3(a_3)$ $\forall a > 0$; $(f_3)_3$ is the density- dependent death rate with $f_3(0) = 0$, $(f_3^3)_3(P_3)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_3)_3(P_3) = (k_3)_3$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_3(0) = 0, (f_2^3)_3(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_2)_3(Q_3) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 3, h^3(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_3 + (\rho_a)_3 = -\mu_3(a_3) + (f_3)_3P_3) + \big(f_2)_3(Q_3)\big)(\rho_3)_3 \quad (4)$$

$$\rho_3(t_3, 0) = h_3(Q_3) \int_0^\infty \beta_3(a_3)\rho_3(t, a_3)da, \quad (5)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + g_3(P_3)Q_3 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_3$ and $Q_3$ of (4-6) satisfy

$$0 \leq \rho_3(t, a_3) \leq \rho_3^*(t, a_3),$$

$$0 \leq Q_3(t) \leq Q_3^*(t),$$

Where $\rho_3^*(t, a_3)$ and $Q_3^*(t)$, are solutions of

$$(\rho_t^*)_3 + (\rho_a^*)_3 = -(\mu_3(a_3) + (f_3)_3(P_3^*) + (f_2)_3 Q_3^*))\,\rho_3^*, \quad (7)$$

$$\rho_3^*(t, 0) = \int_0^\infty \beta_3(a_3)\,\rho_3^*(t, a_3)da_3 \quad (8)$$

$$\frac{dQ_3^*}{dt} = -b_3 Q_3^* + g_3(P_3^*)Q_3^*, \quad (9)$$

With $P_3^*(t) = \int_0^\infty \rho_3^*(t, a_3)da_3$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_3^*(t)}{dt} = P_3^*(t)\big(r_3 - (f_3)(P_3^*) - (f_2)_3(Q^*)\big), \quad (10)$$

$$\frac{dQ_3^*(t)}{dt} = Q_3^*(t)(-b_3 + g_3(P_3^*)), \quad (11)$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_3(a) e^{-\int_0^a \mu_3(\tau)dt} e^{-ra}\, da_3 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_3^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(t)dt}\, da_3 > 1, \qquad (H1)$$

Which guarantees $r_3 > 0$.

*Remark.* If $\beta_3(a) = [\beta_0]_3 a_3 e^{-\alpha_3 a_3}, \mu_3(a_3) \equiv \mu_3$ then the solution of (12) is

$$r_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_3]_3(P_3) = 0, [f_2](Q_3) = \lambda_3 Q_3\ h_3(Q_3) = 1, \qquad and\ \ g_3(P_3) = c_3 P_3$$

$$[\rho_3]_3 + [\rho_a]_3 = -(\mu_3 + \lambda_3 Q_3)\rho_3$$

$$\rho_3(t,0) = \int_0^\infty [\beta_0]_3 a_3 e^{-\alpha_3 a_3} \rho_3(t, a_3)\, da_3$$

$$\frac{dQ_3}{dt} = -b_3 Q_3 + c_3 P_3 Q_3$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_3}{dt} = \omega_3 P_3 - \lambda_3 P_3 Q_3$$

$$\frac{dQ_3}{dt} = b_3 Q_3 - c_3 P_3 Q_3$$

Where $\omega_3 = \sqrt{\beta_3} - \alpha_3 - \mu_3$

## THE EQUILIBRIUM $(\rho_0)_3(a_3), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_3(a_3), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_3 + (\rho_a)_3 = -\big(\mu_3(a_3) + (f_3)_3(P_3)\big)\rho,$$

$$\rho_3(t,0) = \int_0^\infty \beta_3(a_3)\rho_3(t,a_3)da_3 \qquad (13)$$

The solution of (13) has the form of

$$\rho_3(t,a_3) = Ce_3^{r(t-a_3)} - \int_0^a (\tau)d\tau - \int_0^3 (f_3)_3\big(P_3(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_3(t) = C_3 e^{r_3 t - \int_0^t (f_3)_3(P_3(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_3^{-3}(r)$ if $k_3 > r$. This is equivalent to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{(k_3)_3 a_3} da_3 < 1.$$

In fact, if we define

$$R_3(s_3) = \int_0^\infty \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau)d\tau} e^{-s_3 a_3}\, da_3$$

$R(s)$ is a decreasing function. $R_3(r_3) = 1$ And $R_3(k_3)_3 < 1$ imply $r_3 < (k_3)_3$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_3(a_3), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{k_3 a_3} da_3 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_3 \equiv \int_0^\infty (\rho_0)_{33}(a_3)da_3$ *satisfies*

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau)d\tau} e^{-(f_3)_3(P_0)a_3} da = 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_3(a_3), 0)$

Let $u_3(t, a_3) = \rho(t, a_3) - \rho_0(a_3)$ Then

$$N_3(t) \equiv \int_0^\infty u_3\,(t, a_3)da_3 = \int_0^\infty \rho_3(t, a_3)da_3 - \int_0^\infty (\rho_0)_3(a_3)da = P_3(t) - P_0)_3$$

And

$$(u_3)_3 + (u_a)_3 = (\rho_t)_3 + (\rho_a)_3 - (\rho_0^3)_3(a_3)$$

$$= -\mu_3(a_3) + (f_3)_3(P_3) + (f_2)_3(Q_3))\,(u_3 + (\rho_0)_3) + \mu_3(a_3)(\rho_0)_3 + (f_3)_3(P_0)_3(\rho_0)_3$$

$$= -\mu_3(a_3)u_3 - (f_3)_3(P_3) + (f_2)_3(Q_3))\,u_3 - (\rho_0)_3\,(f_3)_3(P_3) - (f_3)_3(P_0)_3$$

$$+ (f_2)_3(Q_3)), \qquad (15)$$

Where $(\rho_0^3)_3(a_3)$ denotes $d(\rho_0)_3(a_3)/da_3$

Expansion $(f_3)_3(P_3)$ and $(f_2)_3(Q_3)$ at $(P_0)_3$ and 0 yield respectively

$$(f_3)_3(P_3) = (f_3)_3((P_0)_3 + N_3)$$

$$= (f_3)_3(P_0)_3 + (f_3^3)_3(P_0)_3N_3 + o_3(N_3), \qquad (16)$$

$$(f_2)_3(Q_3)$$

$$= (f_2^3)_3(0)Q + o_3(Q_3) \qquad (17)$$

Write $f_3(P_0) = f_{30}, f_3'(P_0) = f_{30}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_3)_3 + (u_a)_3 \approx -(\mu_3(a_3) + (f_{30})_3)u - (f_{30}^3)_3(\rho_0)_3(N_3) - (f_{20}^3)_3(\rho_0)_3(Q_3)$$

$$u_3(t,0) = \rho_3(t,0)(\rho_0)_3(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a) + (\rho_0)_3(a_3)da_3 - (\rho_0)_3\,(0)$$

$$= h_3(Q_3)\int_0^\infty \beta_3(a_3)(u_3(t,a_3)da_3 + (h_3(Q_3) - 3)(\rho_0)_3\,(0) \qquad (18)$$

Expanding

$$h_3(Q_3) = 1 + (h^3)_3(0)Q_3 + 0(Q_3) \equiv 1 + h_0'Q_3 + 0Q_3$$

And substituting it into (38), we arrive at

$$u_3(t,0) \approx \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3 + (h_{a_3}')(\rho_0)_3Q_3$$

The linearization is the following :

$$[u_3] + [u_3] = -(\mu_3(a_3) + [t_{30}]_3 u_3 - [\alpha_3]_3(a_3)N_3 - [\alpha_2]_3(a_3)Q_3$$

$$u_3(t,0) = \int_0^\infty \beta_3(a_3)u_3(t,a_3)da_3 + [\xi_3]Q_3 \qquad (19)$$

$$\frac{dQ_3}{dt} = -b_3 Q_3$$

where

$$[\alpha_3]_3(a_3) = [t_3^3 a]_3[p,o]_3(a_3)d2(a_3) = [f_{20}^3]_3(\rho\, o\,)_3(a_3)\xi = [\mathrm{h\ o}^3]_3[\rho \mathrm{o}]_3(\mathrm{o}) \text{ and } \mathrm{b}_3 = \mathrm{b}_3 - g_3(P_3)$$

If $g_3(P_0) > b$, clearly, $(\rho_0)_3(a_3), 0)$ is unstable and the predator population is persistent. Assume $g_3(P_0)_3 < b$.

Then, $Q_3(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_3 + u_a = -(\mu_3(a_3) + (f_{30})_3)u_3 - (\alpha_3)_3(a_3)N_3$$

$$= -(\mu_3(a_3) + f_{30})u_3 - \alpha_3(a_3)\int_0^\infty u_3(t,a_3)da_3$$

$$u_3(t,0) = \int_0^\infty \beta_3\,(a_3)u_3(t,a_3)da_3$$

Using the method developed by Cushing [2, 3], we suppose

$$u_3(t,a_3) = \tilde{\mathrm{u}}_3(a_3)^{e^{\gamma(t-a_3)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_3}\, da_3$. Then,

A solution of the first differential equation for $\tilde{u}(a_3)$ gives,

$$\tilde{u}_3(a_3) = e^{-\int_0^a \mu_3(\tau_3)d\tau - (f_{30})_3 a}\left(\tilde{u}_3(0) - w\int_0^a e^{\int_0^\tau \mu_3(s_3)ds + (f_{30})}\tau_3\,(\alpha_3)_3(\tau_3)e^{\gamma_3\tau}d\tau\right).$$

Substituting the expression (13) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\right)\tilde{u}_3(0)$$

$$+w\int_0^\infty \beta_3(a_3)e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a (\alpha_3)_3(\tau_3)e^{-\gamma_3\tau_3}e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds}d\tau_3\,da_3$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}da\right)(\tilde{u}_3(0))_3$$

$$+\left(-3 - \int_0^\infty e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3)d\tau}\int_0^a e^{(f_{30})_3\tau_3 + \int_0^\tau \mu_3(s_3)ds + y\tau}\alpha_3(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_3 a_3 - (f_{30})_3 a_3 - \int_0^a \mu_3(t_3)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_3\,(a_3)Y_3(\gamma_3, a_3)da_3\right)u_3(0)$$

$$+w\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3=0,$$

(24)

$$\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\tilde{u}_3(0)$$

$$+\left(-1-\int_0^{\infty}Y_3(\gamma_3,a_3)\int_0^{a}(\alpha_3)_3(\tau)_3(Y^{-3})_3(\gamma_3,\tau_3)d\tau_3\,da_3\right)w=0.$$

Equqtion (20) has a solution of the form of $u_3(t,a_3)=u_3(a_3)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_3(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)Y^{-3}\,(\gamma_3,a_3)d\tau_3$. We obtain the characteristic equation

$$f_{30_3}^{l}\left(\int_0^{\infty}Y_3(\gamma_3,a_3)da_3\right)\left(\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$+\left(1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3,a_3)da\right)\left(1+(f_{30}^{l})_3\int_0^{\infty}Y_3(\gamma_3,a_3)I_3(\gamma_3,a_3)da_3\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_3(a_3)=(\rho_0)_3(0)e^{-\int_0^{a}\mu_3(\tau_3)d\tau_3-(f_{30})_3a_3}$$

$$I_3(\gamma_3,a_3)=\int_0^{a}(\rho_0)_3(\tau_3)e^{\gamma_3\tau_3+(f_{30})_3\tau+\int_0^{\tau}\mu_3(s_3)ds_3}d\tau$$

$$= \frac{(\rho_0)_3(0)}{\gamma_3}(e^{\gamma_3 a_3} - 1)$$

Let

$$E_3(a_3) = e^{-(f_{30})_3 a_3 - \int_0^a \mu_3(\tau_3) d\tau_3}$$

Then $[Y_3]_3[\gamma_3]_3, a_3) = E_3(a_3)e^{-\gamma_3 a_3}$. by a simple calculation,

$$\int_0^\infty \beta_3(a_3)Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma}\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}\ da\right);$$

$$\int_0^\infty Y_3(\gamma_3, a_3)I(\gamma_3, a_3)da = \frac{[\rho_0]_3(0)}{\gamma_3}\left(\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}\ da_3\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_3(0)}{\gamma_3}[f_{30}^3]\int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\ (\int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3)$$

$$+\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right).$$

$$\left(1 + \frac{(\rho_0)_3(0)}{\gamma_3}(f_{30}^{'})_3\left(\int_0^\infty E_3(a_3)da_3 - \int_0^\infty E_3(a_3)e^{-\gamma_3 a_3}da_3\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3\right)(1 + \frac{(f_{30}^{'})_3}{\gamma_3}(\rho_0)_3(0)\int_0^\infty E_3(a_3)da_3) = 0.$$

We need $\tilde{u}(a) \not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)da_3 = 1-\int_0^{\infty}\beta_3(a_3)E_3(a_3)e^{-\gamma_3 a_3}da_3 \neq 0,$$

Since $\int_0^{\infty}\beta_3(a_3)Y_3(\gamma_3 a_3)\ \int_0^{a}\alpha_3\,(\tau_3)Y^{-3}(\gamma_3,\tau_3)d\tau_3 da_3 > 0$. Therefore, the characteristic solution is

$$\gamma_3 = -(f_{30}^{'})_3(\rho_0)_3\int_0^{\infty}E_3(\,a_3)da_3.$$

Since $f_3^{'}(P_0)_3 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_3),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_3),0)$ which is related to the extinction of debit summation types

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(\,p^*(a),Q^*\,)$. It must satisfy the equations:

$$[p_a^*]_3 + (\mu_3(a_3) + [f_3]_3(\,P_3^*) + [f_2]_3(Q_3^*)\,)P_3^* = 0, \qquad (26)$$

$$p_3^*(0) = h_3(Q_3^*)\int_0^{\infty}\ \beta_3(a_3)p_3^*(a_J)p_3^*(a_3)da_3 \qquad (27)$$

$$g_3(P_3^*) - b_3 = 0, \qquad (28)$$

Where $P_3^*=\int_0^\infty p_3^*(a_3)\,da_3$

Equation (28) yields $P^* = g^{-3}(b)$,

From (26),

$$p_3^*(a) = p_3^*(0)e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}$$

Substituting into (27), we need to solve

$$\left(1 - h_3(Q_3^*)\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(Q_3^*)a_3}da_3\right)p_3^*(0) = 0$$

For $Q_3^*$

Fix $p_3^*(0)$ and $P_3^*$ Let

$$H_3(z_3) = 1 - h_3(z_3)\int_0^\infty \beta_3(a_3)\,e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}da_{,3}$$

And assume that

$$[k_3]_3\int_0^\infty \beta_3(a_3)e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(g_3^{-3}(b_3))a_3 - [k_2]_3a_3}\,da_3 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_3(z_3)] > 0.$$

Clearly,

$$\frac{dH_3}{dz_3} = h_3(z)([f_2^{'}]_3(z_3)) \int_0^{\infty} a_3\beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - f_3(P_3^*)a_3 - f_2(z_3)a_3}\, da_3$$

$-h^{'}(z_3) \int_0^{\infty} \beta_3(a_3)\, e^{\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3 - [f_2]_3(z_3)a_3}\ \ da_3$

By the assumptions on $\left[f_2^3\right]_3 and\ [h^3]^3 dH_3/dz > 0$. Hence, if

$$H_3(0) = 1 - \int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3(P_3^*)a_3} da_3 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_3(0) < 0$ follows if $g_3^{-}(b_3) < [f_3^{-3}]_3(r_3)$, where $r_3$ is the solution of (12).

Indeed, since

$$R_3(s_3) = \int_0^{\infty} \beta_3(a_3)\, e^{-\int_0^a \mu_3(\tau_3)d\tau_3 e^{-y\frac{a}{3}}} da_3$$

Is a decreasing function of $s_3$ when $P_3^* = [g^{-3}]_3(b_3) < [f_3^{-3}]_3(r_3)$, that is $[f_3]_3(P_3^*) < r, R\left(f_3(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_3(g^{-3}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3)d\tau_3 - [f_3]_3\left(g_3^{-3}(b)\right)a_3} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_3^*(a_3), Q_3^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_3^*(a_3), Q_3^*)$ is more complicated. We again consider perturbations $m_3(t_3 a_3) = p_3(t_3, a_3) - p_3^*(a_3)$ and $W_3(t) = Q_3(t) - Q^*$.Then

$$M_3(t) \equiv \int_0^\infty m_3(t, a_3) da_3 = P_3(t) - P_3^*,$$

$$[m_3]_3 + [m_a]_3 = [p_3]_3 + [p_a]_3 - [p^*]_3(a_3) \approx -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3$$

$$-[f_{3*}^3]_3 p_3^*(a_3) M - [f_{2*}^3]_3 p_3^*(\mathrm{a}) W_3,$$

$$m_3(t,0)p_3(t,0) - p_3^*(0) \approx h_* \int_0^\infty \beta_3(a_3) m_3(t, a_3)\, da + \frac{[h_*^3]_3}{[h_*]_3}\, p_3^*(0) W_3$$

$$\frac{dW_3}{dt} = \frac{dQ_3}{dt} \approx [g_*^3]_3 Q_3^* M_{,3}$$

Where $[f_{3*}]_3$ denoting $[f_3(P^*)]_3$, $[f_{3*}]_3$ denoting $[f_2(P^*)]_3$, $[f_{3*}']_3$ denoting $[f'_3(P^*)]_3 [f_{2*}']_3$ denoting $[f_2]_3$ $(Q_3^*$, $h_{*3}$ denoting $h_*'$ denoting $h_3'$ $(Q_3^*)$, and $[g_*']_3$

Denoting $[g']_3(P_3^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_3]_3 + [m_a]_3 = -(\mu_3(a_3) + [f_{3*}]_3 + [f_{2*}]_3)m_3 - [f'_{3*}]_3 [p_3^*](a_3) M_3 - [f_{2*}']_3 p_3^*(a_3) W_3$$

$$m_3(t,0) = [h_*]_3 \int_0^\infty \beta_3(a_3) m_3(t, a_3) da_3 + \frac{[h_*^3]_3}{[h^*]_3} [p^*]_3(0)\, W_{,3} \qquad (30)$$

$$\frac{dW_3}{dt} = [g_*^3]_3 [Q_3^*] M_3$$

Integrating $[m_3]_3(t,a_3)$ with respect to a from 0 to ∞, we have

$$\frac{dM_3(t)}{dt} = \int_0^\infty [mt]_3(t,a_3)da_3$$

$$= -\int_0^\infty [m_a]_3\,(t,a_3)da_3 - \int_0^\infty \mu(a_3)m(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [f'_{3*}]_3 \int_0^\infty p_3^*(a_3)da_3\right)M_3 - [f_{2*}^3]_3 \int_0^\infty p_3^*(a_3)da_3 W_3$$

$$= m_3(t,0) - \int_0^\infty \mu_3(a_3)\, m_3(t,a_3)da_3$$

$$-\left([f_{3*}]_3 + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 - [\dot{f}_{2*}]_3 P_3^* W_{\cdot 3}$$

From (30), we arrive at the equations

$$\frac{dW_3(t)}{dt} = \int_0^\infty ([h_*]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3$$

$$-\left([f_{3*}] + [f_{2*}]_3 + [\dot{f}_{3*}]_3 P_3^*\right)M_3 + \left(\frac{[h'_*]_3}{[h_*]^3}\, P_3^*(0) - f'_{2*}P_3^*\right)W_3,$$

(33)

$$\frac{dW_3(t)}{dt}[g'_*]_3\, Q_3^*\, M_{\cdot 3}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_3\beta_3(a_3) - \mu_3(a_3))m_3(t,a_3)da_3 = ([h_*]_3\beta_3(\theta_3) - \mu_3(\theta_3))M_{\cdot 3}$$

Where $0< \theta_3 = \theta_3(t) < \infty$.

And if $a_3^2 [a_3^2]_3 < 4 [a_2]_3 < 4 a_2 e$,

$\mathrm{Re}\,\lambda_3 [\lambda_3]_3 = \mathrm{Re}\lambda_2 [\lambda_2]_3 = -\frac{3}{2} a_3 [a2\ ]_3 < 0.$

Let $\Delta = \max_{2<u<\infty} \left| h_* [h_*]_3 \beta_3(a_3) - \mu_3(a_3) \right|$. Then $|n_3(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_3^2 < 4 a_2 e$ and $[a_3] > 2\Delta$,or $[a_3^2] \geq 4[a_2]e$ and $(a_3 - \sqrt{(a_3^2)_3 - 4(a_2)_3 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+3}\right)$ and $\beta_3(a_3) = ae^{-ma}$ Then $\Delta = [\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_3(a_3) e^{-\int_0^a \mu_3(\tau_3) + d\tau_3 da} = \frac{3}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_3 + [\mu_0] \geq 0.5$. Assume $[f_3](x_3) = K_3 x_3$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_3(x_3) = [0x_3]/(3 + x_3)$; the functional responses $[[f_2](x) = [0]\, x_3/(1 + x_3)$and $h_3(x_3) = (1+\omega_3 x_3)/(1 + x_3)$(See Holling [7] and Rashevsky [13])

Equation (34),now has the form of

$$\frac{m_3 + [\mu_0] + [\mathrm{K}][P_0])}{[m_3]_{\,3} + (\mu_0)_{\,3} + [\mathrm{K}][P_0])_{\,3}^{\,3}\ [\mathrm{K}]_{\,3}[P_0]_{\,3}^{\,3}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(3 + P_0) < b_3$ i e $[P_0] < b/(\delta_3 - b_3)$then the equilibrium$(P_0)$ $[P_0]_{\,3}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_3$

$[f_{3*}] = K_3\ \mathrm{P}^{\,*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*} \quad [h_*] = \frac{3+\omega_3 \mathrm{Q}_3^*}{3+\mathrm{Q}_3^*}$

$[f_{3*}] = K_3\ [\mathrm{f}_{2*}^{\,3}] = \frac{\sigma_3}{(3+\mathrm{Q}_3^*)^2}[\ \mathrm{g}_*^{\,3}] = \frac{\sigma_3}{(3+\mathrm{P}_3^*)^2} \quad [\mathrm{h}_*^{\,3}] = \frac{\omega_3 - 3}{(3+\mathrm{Q}_3^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_3}{(m + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma)^{\,2}} + \frac{2\omega_3}{([m_3]_{\,3} + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 + b_3} + \sigma_3)^{\,3}} < 1$$

And

$$\frac{1}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,2}} + \frac{2\omega_3}{(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{\,*}$ which can be solved from the

following equation:

$(m_3 + [\mu_0]_{\,3} + 1 + \frac{K_3 b_3}{\delta_3 - b_3} + (m_3 + [\mu_0]_{\,3} + \frac{K_3 b_3}{\delta_3 - b_3} + \sigma_3)\ \mathrm{Q}_3^{\,*})^{\,3}$

$=m_3 + [\mu_0]_3 + 1 + \frac{K_3 b_3}{\delta_3 - b_3}(1+ Q^*)(1 + \omega_3 Q_3^*$

Now

$$\left[a_3^2\right] - 4\left[a_2\right] e = \left(\frac{2K_3 b_3}{\delta_3 - b_3}\right)^2 + \frac{4K_3 b_3 \sigma_3 Q_3^*}{(\delta_3 - b_3)(1 + Q_3^*)} + \frac{\sigma_3^2 Q_3^{*2}}{(1 + Q_3^*)^2}$$

$$\frac{4\, b_3 \sigma_3 (\delta_3 - b) Q^*}{\delta_3 (1 + Q_3^*)^2} - \frac{4\,(\delta_3 - b_3)^2 (1 - \omega_3) Q_3^*}{\delta_3 (1 + \omega_3 Q_3^*)(1 + Q_3^*)} P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_3\, a e_3^{mu}\ [\mu_0]_3 - \frac{a_3}{a_3+3}$

Hence if $[a_3^2 > 4(a_2)_3 e$ and

$$[a_3] = \frac{2K_3 b_3}{\delta_3 - b_3} + \frac{\sigma_3 Q_3^*}{1 + Q_3^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_3]_3 = [f_{3*}]_3 + [f_{2*}]_3 + [f_{3*}] P_3^*$$

$$[a_2]_3 = \left[f_{2*}^3\right] P_3^* - \frac{h_*^3}{h_*} P^*(0)$$

$$e=[\,g_{*}^{3}]_{\,3}\,Q_{3}^{*}$$

$$\eta_3(t) = \left[h_{*}\right]_{\,3}\,\beta_{\,3}\left(\theta_{\,3}(t)\right) - \mu_{\,3}\left(\theta(t)\right),$$

The system(33)can be written as the following system:

$$[M(t)]\left[a_{\,3}\right]M_3 - [a_2]_{\,3}W_3 - [n(t)]M_3$$

$$w_3(t) = eM_3$$

Write

$$A=\begin{pmatrix} -[a_3]_{\,3} & -[a_2]_{\,3} \\ e & 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_{\,3} & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_3 \\ W_3 \end{pmatrix}$$

The solution of (32)is

$$X_3(t) = e^{A,t}\,[X]_{\,3}(0) + \int_0 e^{At}$$

**This equation is useful in following type of problems . For actual model kindly see various papersThere is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's**

**1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin**

**bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses ≳20 M⊙. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201 STOP Advanced LIGO may be the first experiment to (e) detect gravitational waves. Through superradiance of stellar black holes, it may also be (=) the first experiment to discover the QCD axion with (e&eb) decay constant above the GUT scale. When an axion's Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e) the black hole, forming a "gravitational atom." Through the superradiance process, the number of axions occupying (e) the bound levels grows exponentially, extracting (e) energy and angular momentum from the black hole. Axions transitioning between levels of the gravitational atom and (e&eb) axions annihilating to (e) gravitons can produce (eb) observable gravitational wave signals. The signals are long-lasting, monochromatic, and can be distinguished from (e) ordinary astrophysical sources. Authors estimate up to O (1) transition events at aLIGO for (e) an axion between 10^-11 and (e&eb) 10^-10 eV and up to 10^4 annihilation events for (e) an axion between 10^-13 and 10^-11 eV. In the event of a null search, aLIGO can constrain (e) the axion mass for (e) a range of rapidly spinning black hole formation rates. Axion annihilations are also promising for (e) much lighter masses at future lower-frequency gravitational wave observatories; the rates have (e) large uncertainties, dominated by (e) supermassive black hole spin distributions. Projections for aLIGO are (=) robust against perturbations from (e) the black hole environment and account for (e) updated exclusion on (e&eb) the QCD axion of 6*10^-13 eV < ma < 2*10^-11 eV suggested by (e) stellar black hole spin measurements. Subjects: High Energy Physics - Phenomenology (hep-ph); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and**

**Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 91, 084011 (2015) DOI: 10.1103/PhysRevD.91.084011Cite as:arXiv:1411.2263 [hep-ph] (or arXiv:1411.2263v3 [hep-ph] for this version) Discovering the QCD Axion with Black Holes and Gravitational Waves Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang STOP Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for**

**(e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole**

**merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of**

**(e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**

**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain**

**bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating**

**accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared**

**Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments:**

**Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy**

## Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Four

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equailibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-445) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Some of the concepts to which model is applicable is as follows:**

**Historically, the first true GUT which was based on (e) the simple Lie group SU(5), was proposed by (e) Howard Georgi and Sheldon Glashow in 1974.[2] The Georgi–Glashow model was preceded by (e) the semisimple Lie algebra Pati–Salam model by (e) Abdus Salam and Jogesh Pati,[3] who pioneered the idea to unify gauge interactions. The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos, however in (eb) the final version of their paper [4] they opted for (e) the less anatomical GUM (Grand Unification Mass). Nanopoulos later that year was the first to use [5] the acronym in a paper.[6] The fact that the electric charges of (e) electrons and protons seem to cancel (e&eb) each other exactly to (e) extreme precision is essential for (e) the existence of the macroscopic world as we know it, but this important property of (e) elementary particles is not explained in the Standard**

**Model of particle physics. While the description of strong and weak interactions within (eb) the Standard Model is based on (e) gauge symmetries governed by (e&eb) the simple symmetry groups SU(3) and (e&eb) SU(2) which allow (eb) only discrete charges, the remaining component, the weak hypercharge interaction is described by (e) an abelian symmetry U(1) which in principle allows (eb) for arbitrary charge assignments.[note 1] The observed charge quantization, namely the fact that all known elementary particles carry electric charges which appear to be (=) exact multiples of ⅓ of the "elementary" charge, has led to (eb) the idea that hypercharge interactions and possibly the strong and weak interactions might be embedded in (eb) one Grand Unified interaction described by (e) a single, larger simple symmetry group containing (e) the Standard Model. This would automatically predict (eb) the quantized nature and values of (e) all elementary particle charges. Since this also results in a prediction for (e) the relative strengths of the fundamental interactions which we observe, in particular the weak mixing angle, Grand Unification ideally reduces (e) the number of independent input parameters, but is also constrained by (e) observations. Grand Unification is reminiscent of (e) the unification of electric and (e&eb) magnetic forces by (e) Maxwell's theory of electromagnetism in the 19th century, but its physical implications and mathematical structure are qualitatively different. Physics Letters B Volume 315, Issues 3–4, 7 October 1993, Pages 325-330 Supersymmetric unification without proton decay Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. G. Lazarides, C. Panagiotakopoulos Q. Shafi. Some sentences have been deleted attributable and ascribable to restraints on space. Please excuse me STOP On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed (e) a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with (e&eb) a peak gravitational-wave strain of (e) 1.0×10−21. It matches the waveform predicted by general relativity for (e) the inspiral and merger of a pair of black holes and the ringdown of the resulting (eb) single black hole. The signal was observed with a matched-filter signal-to-noise ratio of (e) 24 and a false alarm rate estimated to be (=) less than 1 event per 203 000 years, equivalent to (=) a significance greater than 5.1σ. The source lies at (eb) a luminosity distance of 410+160−180 Mpc corresponding to (e&eb) a redshift z=0.09+0.03−0.04. In the source frame, the initial black hole masses are (=) 36+5−4M⊙ and 29+4−4M⊙, and the final black hole mass is (=) 62+4−4M⊙, with (e&eb) 3.0+0.5−0.5M⊙c2 radiated in (eb) gravitational waves. All uncertainties define (eb) 90% credible intervals. These observations demonstrate (eb) the existence of binary stellar-mass black hole systems. Observation of Gravitational Waves from a Binary Black Hole Merger B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102 – Published 11 February 2016 This is the first direct detection of gravitational waves and (e&eb) the first observation of a binary black hole merger. Received 21 January 2016 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.061102 STOP This is the second in a series of three papers on (e&eb) systematic analysis of rank 1 Coulomb branch geometries of (e) four dimensional N=2 SCFTs. In the first paper authors developed a strategy for (e) classifying physical rank-1 CB geometries of N=2 SCFTs. They show how to carry out this strategy computationally to construct (eb) the Seiberg-Witten curves and (e&eb) one-forms for all the rank-1 SCFTs. Explicit expressions are given for all cases, with the exception of the Nf=4 SU (2) gauge theory and (e&eb) the En SCFTs which were previously constructed. Classification includes all known rank-1 theories plus (+) a new one with (e&eb) an abelian flavor group, plus (+) nine additional theories whose existence is more speculative. Four of those, reported in first paper, depend on (=) the assumption of new frozen rank-1 SCFTs. Here authors also show that the assumption of the existence of certain rank-0 N=2 SCFTs leads to (eb) five additional consistent rank-1 CB geometries. Subjects: High Energy Physics - Theory (hep-th) Cite as:arXiv: 1601.00011 [hep-th] (or arXiv: 1601.00011v2 [hep-th] for this version Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows Philip C. Argyres, Matteo Lotito, Yongchao Lü, Mario Martone (Submitted on 31 Dec 2015 (v1), last revised 6 Jan 2016 (this version, v2)) STOP Authors propose a model of Dark Energy in which the field currently dominating (e) the energy density of the universe is (=) an "axion field" linearly coupled to (e&eb) the Pontryagin density, tr (F∧F), (i.e., the exterior derivative of (e) the Chern-Simons form) of (e) a massive gauge field. Authors assume that the axion has (e) self-interactions corresponding to (e&eb) a non-trivial (exponential) potential. Authors argue that a non-vanishing magnetic helicity of (e) the gauge field triggers (eb) slow-rolling of the axion at field values far below (eb) the Planck scale. Proposal leads to (eb) a "Tracking Dark Energy Scenario" in which the contribution of the axion energy density to (e) the total energy density is (=) constant (and small) during (e&eb) the early radiation phase, until (e) a secular growth term proportional to (e) the Pontryagin density of (e) the gauge field becomes (=) dominant. The initially small contribution of (e) the axion field to (e) the total**

**energy density is related to the observed small baryon-to-entropy ratio. Some speculations concerning the nature of the gauge field are offered.**
**Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Cite as:arXiv: 1601.00057 [hep-th] (or arXiv:1601.00057v2 [hep-th] for this version) Tracking Dark Energy from Axion-Gauge Field Couplings Stephon Alexander, Robert Brandenberger, Juerg Froehlich (Submitted on 1 Jan 2016 (v1), last revised 23 Jan 2016 (this version, v2)) STOP Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version) Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP In this work authors obtain bounds on (e&eb) the topological Abelian string-vortex and on (e&eb) the string-cigar, by using (e) a new measure of configurational complexity, known as (=) configurational entropy. In this way, the information-theoretical measure of (e) six-dimensional braneworld scenarios is capable to (e) probe situations where (e) the parameters responsible for (e) the brane thickness are arbitrary. The so-called configurational entropy (CE) selects (e&eb) the best value of the parameter in (eb) the model. This is accomplished by (e) minimizing the CE, namely, by (e) selecting the most appropriate parameters in the model that correspond to (e&eb) the most organized system, based upon (e) the Shannon information theory. This information-theoretical measure of complexity provides (eb) a complementary perspective to (e) situations where (e) strictly energy-based arguments are (=) inconclusive. Authors show that the higher the energy the higher the CE, what shows (eb) an important correlation between the energy of the localized field configuration and (e&eb) its associated entropic measure. Subjects: High Energy Physics - Theory (hep-th) DOI:10.1016/j.physletb.2016.02.038 Cite as: arXiv: 1601.00076 [hep-th] (or arXiv: 1601.00076v2 [hep-th] for this version) Bounds on topological Abelian string-vortex and string-cigar from information-entropic measure R. A. C. Correa, D. M. Dantas, C. A. S. Almeida, Roldao da Rocha (Submitted on 1 Jan 2016 (v1), last revised 17 Feb 2016 (this version, v2)) STOP Sum of the Holevo quantity (that bounds the capacity of quantum channels to transmit (e&eb) classical information about an observable) and (e&eb) the quantum discord (a measure of the quantumness of (e) correlations of that observable) yields (eb) an observable-independent total given by (e) the quantum mutual information This split naturally delineates ,dilates and disseminates information about (e&eb) quantum systems accessible to (e) observers -- information that is redundantly transmitted by (e&eb) the environment -- while showing that it is (=) maximized for (e) the quasi-classical pointer observable. Other observables are accessible only via (e&eb) correlations with the pointer observable .Further authors prove an anti-symmetry property relating accessible information and (e&eb) discord. It shows that information becomes (=) objective -- accessible to (e) many observers -- only as quantum information is relegated to (e) correlations with (e&eb) the global environment, (eb) and, therefore, locally inaccessible. The resulting complementarity explains (eb) why, in a quantum Universe, we perceive objective classical reality while (e) flagrantly quantum superpositions are out of reach. Subjects: Quantum Physics (quant-ph) Journal reference: Scientific Reports 3, 1729 (2013) Report number: LAUR-13-26386 Cite as:arXiv: 1303.4659 [quant-ph] (or arXiv: 1303.4659v2 [quant-ph] for this version) Complementarity of quantum discord and classically accessible information Michael Zwolak, Wojciech H. Zurek STOP Niels Bohr proposed that the outcome of the measurement becomes (=) objective and real, and, hence, classical, when (e) its results can be communicated by (e) classical means. In this work authors revisit Bohr's postulate using (e) modern tools from (e) the quantum information theory. They find a full confirmation of Bohr's idea: if a measurement device is in a nonclassical state, (eb) the measurement results cannot be communicated perfectly by classical means. In this case some part of**

**information in the measurement apparatus is lost (e) in the process of communication: the amount of this lost information turns out to be (=) the quantum discord. The information loss occurs even when (e) the apparatus is not entangled with the system of interest. The tools presented in this work allow (eb) to generalize Bohr's postulate: authors show that for pure system-apparatus states (eb) quantum communication does not provide any advantage when (e) measurement results are communicated to (e) more than one recipient. They further demonstrate the superiority of quantum communication to (e&eb) two recipients on (e&eb) a mixed system-apparatus state and show that (eb) this effect is fundamentally different from (e) quantum state cloning. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. Lett. 111, 040401 (2013) DOI:10.1103/PhysRevLett.111.040401 Report number: LA-UR-12-25351 Cite as: arXiv: 1304.2914 [quant-ph] (or arXiv: 1304.2914v2 [quant-ph] for this version) Quantum discord cannot be shared Alexander Streltsov, Wojciech H. Zurek STOP Wojciech H. Zurek, Don N. Page investigate thermodynamic equilibrium of (e) a self-gravitating perfect fluid in (eb) a spherically symmetric system containing (e) a black hole of mass M by means of (e) the Tolman-Oppenheimer-Volkoff (TOV) equation. At r >> 2M its solutions describe (eb) a black-body radiation atmosphere with (e&eb) the Hawking temperature T_BH~1/(8 \pi M) that is increasingly blueshifted as r approaches (eb) 2M. However, there is no horizon at (eb) the Schwarzschild radius. Instead, the fluid becomes (=) increasingly hot and dense there, piling up into (e&eb) a "firewall" with the peak temperatures and densities reaching (eb) Planck values somewhat below r = 2M. This firewall surrounds (e&eb) a negative point mass residing at r=0, the only singularity of the solution. The entropy of the firewall is comparable to (e&eb) the Bekenstein-Hawking entropy. Comments: Authors study self-gravitating spherically symmetric fluid with (e&eb) a mass of a black hole surrounded by (e) Hawking radiation. Solutions cross r=2M without (e) encountering coordinate singularity to reach a firewall-like "Planck cocoon" with entropy close to (e) black hole entropy. Authors reproduce paper with an updated title and abstract. For a later study with similar results see G. 't Hooft, gr-qc/9706058 Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Physical Review D, Volume 29, Number 4, pp. 628 - 631 (1984) DOI:10.1103/PhysRevD.29.628 Cite as:arXiv:1511.07051 [gr-qc] (or arXiv:1511.07051v1 [gr-qc] for this version) Firewalls, black-hole thermodynamics, and singular solutions of the Tolman-Oppenheimer-Volkoff equation Wojciech H. Zurek, Don N. Page (Submitted on 22 Nov 2015) STOP Frederic Magniez, Ashwin Nayak consider the problem of testing the commutativity of (e) a black-box group specified by its k generators. The complexity (in terms of k) of (e) this problem was first considered by Pak, who gave (eb) a randomized algorithm involving O(k) group operations. They construct a quite optimal quantum algorithm for this problem whose complexity is in O~(k2/3)O~(k2/3). The algorithm uses and highlights the power of the quantization method of Szegedy. For the lower bound of Ω(k2/3)Ω(k2/3), authors give a reduction from (e) a special case of Element Distinctness to our problem. Along the way, they prove the optimality of the algorithm of (e) Pak for the randomized model. Algorithmica July 2007, Volume 48, Issue 3, pp 221-232 First online: 25 June 2007 Quantum Complexity of Testing Group Commutativity Frederic Magniez, Ashwin Nayak STOP Pawel Wocjan, Jon Yard analyze relationships between quantum computation and (e&eb) a family of generalizations of the Jones polynomial. Extending recent work by Aharonov et al., authors give efficient quantum circuits for (e) implementing the unitary Jones-Wenzl representations of the braid group. We use these to provide new quantum algorithms for (e) approximately evaluating a family of specializations of the HOMFLYPT two-variable polynomial of trace closures of braids. They also give (eb) algorithms for approximating the Jones polynomial of a general class of closures of (e) braids at roots of unity. Next authors provide (eb) a self-contained proof of a result of (e) Freedman et al. that any quantum computation can be replaced by (e&eb) an additive approximation of the Jones polynomial, evaluated at (eb) almost any primitive root of unity. Proof encodes two-qubit unitaries into (e&eb) the rectangular representation of (e) the eight-strand braid group. Authors then give QCMA-complete and (e&eb) PSPACE-complete problems which are based on braids. They conclude with direct proofs that evaluating the Jones polynomial of the plat closure at most primitive roots of unity is a #P-hard problem, while learning its most significant bit is PP-hard, circumventing (e) the usual route through the Tutte polynomial and graph coloring. Subjects: Quantum Physics (quant-ph); Geometric Topology (math.GT) Cite as: arXiv:quant-ph/0603069 (or arXiv:quant-ph/0603069v3 for this version) The Jones polynomial: quantum algorithms and applications in quantum complexity theory Pawel Wocjan, Jon Yard STOP Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan address the task of switching off the Hamiltonian of (e) a system by removing (e) all internal and system-environment couplings. Authors propose dynamical decoupling schemes, which use (e) only bounded-strength controls, for quantum many-body systems with local system Hamiltonians and local**

**environmental couplings. To do so, they introduce (eb) the combinatorial concept of (e) balanced-cycle orthogonal arrays (BOAs) and show how to construct them from (e) classical error-correcting codes. The derived decoupling schemes may be useful as (=) a primitive for more complex schemes, e.g., for Hamiltonian simulation. For the case of n qubits and a 2-local Hamiltonian, the length of the resulting decoupling (e&eb) scheme scales as O (nlogn), improving over (e&eb) the previously best-known schemes that scaled quadratically with (e&eb) n. More generally, using balanced-cycle orthogonal arrays constructed from families of BCH codes, we show that bounded-strength decoupling for any ℓ-local Hamiltonian, where ℓ≥2, can be achieved using decoupling schemes of length at most O(nℓ−1logn). Subjects:Quantum Physics (quant-ph); Emerging Technologies (cs.ET) Journal reference: IEEE Trans. Inf. Theory, 62(5): 2881-2894, 2016 DOI: 10.1109/TIT.2016.2535183 Report number: MIT-CTP /4705 Cite as: arXiv: 1509.00408 [quant-ph] (or arXiv: 1509.00408v2 [quant-ph] for this version) Improved bounded-strength decoupling schemes for local Hamiltonians Adam D. Bookatz, Martin Roetteler, and Pawel Wocjan STOP Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on (e) realistic bounded-strength control Hamiltonians. Each simulation protocol consists of (e) periodic repetitions of a basic control block, constructed as (=) a suitable modification of an "Eulerian decoupling cycle," that would otherwise implement (eb) a trivial (zero) target Hamiltonian. For an open quantum system coupled to (e) an uncontrollable environment, approach may be employed to (e) engineer an effective evolution that simulates (e&eb) a target Hamiltonian on the system, while suppressing (e) unwanted decoherence to the leading order. Authors present illustrative applications to (e) both closed- and open-system simulation settings, with (e&eb) emphasis on simulation of non-local (two-body) Hamiltonians using (e) only local (one-body) controls. In particular, authors provide (eb) simulation schemes applicable to (e&eb) Heisenberg-coupled spin chains exposed to (e) general linear decoherence, and show how to simulate (e&eb) Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as (=) potentially relevant to (e&eb) the dynamical generation of a topologically protected (e&eb) quantum memory. Additional implications for quantum information processing are discussed. Subjects: Quantum Physics (quant-ph) Journal reference: New J. Phys. 16, 045021 (2014) DOI: 10.1088/1367-2630/16/4/045021 Cite as:arXiv: 1310.4153 [quant-ph] (or arXiv: 1310.4153v1 [quant-ph] for this version) Hamiltonian quantum simulation with bounded-strength controls Adam D. Bookatz, Pawel Wocjan, and Lorenza Viola STOP Felix Fontein, Pawel Wocjan present a quantum algorithm for computing the period lattice of (e) infrastructures of fixed dimension. The algorithm applies to (e&eb) infrastructures that satisfy certain conditions. The latter are always fulfilled for (e) infrastructures obtained from (e) global fields, i.e., algebraic number fields and function fields with (e&eb) finite constant fields. The first main contributions is an exponentially better method for (e) sampling approximations of vectors of the dual lattice of (e) the period lattice than the methods outlined in the works of (e) Hallgren and Schmidt and Vollmer. This new method improves (eb) the success probability by (e) a factor of at least 2^ {n^2-1} where (e) n is the dimension. The second main contribution is a rigorous and complete proof that the running time of the algorithm is (=) polynomial in the logarithm of (e) the determinant of the period lattice and (e&eb) exponential in n. The third contribution is the determination of (e) an explicit lower bound on (e&eb) the success probability of our algorithm which greatly improves on (e&eb) the bounds given in the above works. The exponential scaling seems inevitable because (e) the best currently known methods for (e) carrying out fundamental arithmetic operations in infrastructures obtained from (e) algebraic number fields take exponential time. In contrast, the problem of computing (e&eb) the period lattice of infrastructures arising from function fields can be solved without (e) the exponential dependence on (e&eb) the dimension n since this problem reduces efficiently to the abelian hidden subgroup problem. This is also true for (e) other important computational problems in (eb) algebraic geometry. The running time of the best classical algorithms for (e) infrastructures arising from (e) global fields increases subexponentially with (e&eb) the determinant of the period lattice. Subjects: Quantum Physics (quant-ph) Cite as: arXiv: 1111.1348 [quant-ph] (or arXiv: 1111.1348v2 [quant-ph] for this version) Quantum Algorithm for Computing the Period Lattice of an Infrastructure Felix Fontein, Pawel Wocjan STOP Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan advocate a new approach of (e) addressing hidden structure problems and (e&eb) finding efficient quantum algorithms. Authors introduce and investigate the Hidden Symmetry Subgroup Problem (HSSP), which is a generalization of the well-studied Hidden Subgroup Problem (HSP). Given a group acting on (e&eb) a set and an oracle whose level sets define (eb) a partition of the set; the task is to recover (e) the subgroup of symmetries of (e) this partition inside the group. The HSSP provides (eb) a unifying framework that, besides the HSP, encompasses (e) a wide range of algebraic oracle**

**problems, including (e) quadratic hidden polynomial problems. While the HSSP can have (e) provably exponential quantum query complexity, authors obtain efficient quantum algorithms for various interesting cases. To achieve this, authors present a general method for (e) reducing the HSSP to the HSP, which works efficiently in several cases related to (e&eb) symmetries of polynomials. The HSSP therefore connects in a rather surprising way certain hidden polynomial problems with (e&eb) the HSP. Using this connection, authors obtain the first efficient quantum algorithm for (e) the hidden polynomial problem for (e) multivariate quadratic polynomials over fields of (e) constant characteristic. Authors also apply the new methods to (e&eb) polynomial function graph problems and (e&eb) present an efficient quantum procedure for (e) constant degree multivariate polynomials over (e&eb) any field. This result improves in several ways the currently known algorithms. Subjects: Quantum Physics (quant-ph) Journal reference: SIAM Journal on Computing 42:5 (2013) DOI: 10.1137/120864416 Cite as: arXiv: 1107.2189 [quant-ph] (or arXiv: 1107.2189v2 [quant-ph] for this version) Hidden Symmetry Subgroup Problems Thomas Decker, Gábor Ivanyos, Miklos Santha, Pawel Wocjan STOP Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan present an efficient general method for realizing a quantum walk operator corresponding to (e&eb) an arbitrary sparse classical random walk. Approach is based on Grover and Rudolph's method for preparing coherent versions of (e) efficiently integrable probability distributions. This method is intended for (e) use in quantum walk algorithms with polynomial speedups, whose complexity is usually measured in terms of (e&eb) how many times we have to apply a step of a quantum walk, compared to (e&eb) the number of necessary classical Markov chain steps. Authors consider a finer notion of complexity including (e) the number of elementary gates it takes to implement each step of the quantum walk with (e&eb) some desired accuracy. The difference in complexity for (e) various implementation approaches is that method scales linearly in the sparsity parameter and (e&eb) poly-logarithmically with (e&eb) the inverse of the desired precision. Best previously known general methods either scale quadratically in (eb) the sparsity parameter, or polynomially in (eb) the inverse precision. Approach is especially relevant for (e) implementing quantum walks corresponding to (e&eb) classical random walks like those used in (eb) the classical algorithms for approximating permanents and sampling from binary contingency tables. In those algorithms, the sparsity parameter grows with (e&eb) the problem size, while (e) maintaining high precision is required. Subjects: Quantum Physics (quant-ph) Journal reference: Quantum Information and Computation, Vol.10, No.5&6, pp0420-0434 (2010) Cite as: arXiv: 0903.3465 [quant-ph] (or arXiv: 0903.3465v3 [quant-ph] for this version) Efficient Circuits for Quantum Walks Chen-Fu Chiang, Daniel Nagaj, Pawel Wocjan STOP Preparing the ground state of a system of interacting classical particles is (=) an NP-hard problem. Thus, there is in general no better algorithm to (e) solve this problem than exhaustively going through all N configurations of (e) the system to determine (eb) the one with lowest energy, requiring (e) a running time proportional to N. A quantum computer, if it could be built, could solve (e) this problem in time sqrt (N). Here, authors present a powerful extension of this result to (e) the case of interacting quantum particles, demonstrating (eb)that a quantum computer can prepare the ground state of (e) a quantum system as efficiently as it does for (e) classical systems. Subjects: Quantum Physics (quant-ph) Journal reference: Phys.Rev.Lett.102:130503, 2009 DOI:10.1103/PhysRevLett.102.130503 Cite as: arXiv: 0809.2705 [quant-ph] (or arXiv: 0809.2705v1 [quant-ph] for this version) Preparing ground states of quantum many-body systems on a quantum computer David Poulin, Pawel Wocjan STOP Daniel Nagaj, Pawel Wocjan construct a simple translationally invariant, nearest-neighbor Hamiltonian on (e&eb) a chain of 10-dimensional qudits that makes it possible to (e) realize universal quantum computing without (e) any external control during (e&eb) the computational process. Authors only require the ability to prepare an initial computational basis state which encodes (e) both the quantum circuit and its input. The computational process is (=) then carried out by (e) the autonomous Hamiltonian time evolution. After a time polynomially long in the size of (e) the quantum circuit has passed, the result of (e) the computation is obtained with (e&eb) high probability by (e) measuring a few qudits in the computational basis. This result also implies (eb) that there cannot (e) exist efficient classical simulation methods for (e) generic translationally invariant nearest-neighbor Hamiltonians on (e&eb) qudit chains, unless (e) quantum computers can be efficiently simulated by (e&eb) classical computers (or, put in complexity theoretic terms, unless BPP=BQP). Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 78, 032311 (2008) DOI: 10.1103/PhysRevA.78.032311 Cite as: arXiv: 0802.0886 [quant-ph] (or arXiv: 0802.0886v4 [quant-ph] for this version) Hamiltonian Quantum Cellular Automata in 1D Daniel Nagaj, Pawel Wocjan**

## 100.MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^{\infty} \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(g_4)Q_4$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_4(0) = 0, \quad g_4(P_4) > 0, \quad P_4 \; 0, \qquad \lim_{P_4 \to \infty} g_4(P_4) = (K_4)_4 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_4 + (\rho_a)_4 + D_4(a_4, P_4, Q_4)\rho_4 + 0, \qquad 0 < a_4 \; \infty,$$

$$\rho_4(t,0) = \int_0^{\infty} n_4(a_4, P_4, Q_4)\rho_4(t_4, a_4)da_4$$

Where $D_4$ is the mortality function and $n_4$ is the fertility function.

Put $_4$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_4}{dt}\right]_4 = x_4 \left[\frac{dQ_2}{dt}\right]_4 - x_2 \left[\frac{dQ_4}{dt}\right]_4$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3 \left[\frac{dQ_3}{dt}\right]_4 - x_4 \left[\frac{dQ_2}{dt}\right]_4$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5 \left[\frac{dQ_2}{dt}\right]_4 - x_6 \left[\frac{dQ_3}{dt}\right]_4$$

Where $x_4, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_4$ and $g_4(P_4)Q_4$

Now when you add the term $(b_4Q_4)$ with $g_4(P_4)(Q_4)$ we are infect see is that $(b_4Q_4)$ is credited incrementally part of $g_4(P_4)(Q_4)$ or instant nearly, when subtracted from $\frac{dQ_4}{dt}$ , it means that $g_4(P_4)(Q_4)$ is slowly eaten by $\frac{dQ_4}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_4 = \mu_4(a_4) + (f_4)_4(P_4)_4 + (f_2)_4(Q)_4 \quad (2)$$

$$n_4 = \beta_4(a_4)h_4(Q_4), \quad (3)$$

Where $\mu_4$the age- dependent debit rate whit is $\mu_4(a_4)$ $\forall a > 0$; $(f_4)_4$ is the density- dependent death rate with $f_4(0) = 0$, $(f_4^4)_4(P_4)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_4)_4(P_4) = (k_4)_4$ ; $f_2$ is the functional response of creditors to prey with $[f_2]_4(0) = 0, (f_2^4)_4(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_4(Q_4) = [k_2]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 4, h^4(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_4 + (\rho_a)_4 = -\mu_4(a_4) + (f_4)_4P_4) + \big(f_2)_4(Q_4)\big)(\rho_4)_4 \quad (4)$$

$$\rho_4(t_4, 0) = h_4(Q_4)\int_0^\infty \beta_4(a_4)\rho_4(t, a_4)da, \quad (5)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4 + g_4(P_4)Q_4 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_4$ and $Q_4$ of (4-6) satisfy

$$0 \le \rho_4(t, a_4) \le \rho_4^*(t, a_4),$$

$$0 \le Q_4(t) \le Q_4^*(t),$$

Where $\rho_4^*(t, a_4)$ and $Q_4^*(t)$, are solutions of

$$(\rho_t^*)_4 + (\rho_a^*)_4 = -(\mu_4(a_4) + (f_4)_4(P_4^*) + (f_2)_4 Q_4^*))\,\rho_4^*, \quad (7)$$

$$\rho_4^*(t, 0) = \int_0^\infty \beta_4(a_4)\,\rho_4^*(t, a_4)da_4 \quad (8)$$

$$\frac{dQ_4^*}{dt} = -b_4 Q_4^* + g_4(P_4^*)Q_4^*, \quad (9)$$

With $P_4^*(t) = \int_0^\infty \rho_4^*(t, a_4)da_4$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_4^*(t)}{dt} = P_4^*(t)\big(r_4 - (f_4)(P_4^*) - (f_2)_4(Q^*)\big), \tag{10}$$

$$\frac{dQ_4^*(t)}{dt} = Q_4^*(t)(-b_4 + g_4(P_4^*)), \tag{11}$$

Where $r$ is the real solution of the equation

$$\int_0^\infty \beta_4(a)e^{-\int_0^a \mu_4(\tau)dt}e^{-ra}\,da_4 = 1. \tag{12}$$

If $r < 0$, the trivial solution $P_4^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(t)dt}\,da_4 > 1, \tag{$H1$}$$

Which guarantees $r_4 > 0$.

*Remark.* If $\beta_4(a) = [\beta_0]_4 a_4 e^{-\alpha_4 a_4}, \mu_4(a_4) \equiv \mu_4$ then the solution of (12) is

$$r_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_4]_4(P_4) = 0, [f_2](Q_4) = \lambda_4 Q_4\ h_4(Q_4) = 1, \qquad and\ \ g_4(P_4) = c_4 P_4$$

$$[\rho_4]_4 + [\rho_a]_4 = -(\mu_4 + \lambda_4 Q_4)\rho_4$$

$$\rho_4(t,0) = \int_0^\infty [\beta_0]_4 a_4 e^{-\alpha_4 a_4}\rho_4(t,a_4)\,da_4$$

$$\frac{dQ_4}{dt} = -b_4Q_4 + c_4P_4Q_4$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_4}{dt} = \omega_4P_4 - \lambda_4P_4Q_4$$

$$\frac{dQ_4}{dt} = b_4Q_4 - c_4P_4Q_4$$

Where $\omega_4 = \sqrt{\beta_4} - \alpha_4 - \mu_4$

## 101.THE EQUILIBRIUM $(\rho_0)_4(a_4), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_4(a_4), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_4 + (\rho_a)_4 = -\big(\mu_4(a_4) + (f_4)_4(P_4)\big)\rho,$$

$$\rho_4(t,0) = \int_0^\infty \beta_4(a_4)\rho_4(t,a_4)da_4 \qquad (13)$$

The solution of (13) has the form of

$$\rho_4(t,a_4) = Ce_4^{r(t-a_4)} - \int_0^a (\tau)d\tau - \int_0^4 (f_4)_4\big(P_4(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_4(t) = C_4 e^{r_4 t - \int_0^t (f_4)_4(P_4(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_4^{-4}(r)$ if $k_4 > r$. This is equivalent to

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{(k_4)_4 a_4} da_4 < 1.$$

In fact, if we define

$$R_4(s_4) = \int_0^\infty \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau)d\tau} e^{-s_4 a_4}\, da_4$$

$R(s)$ is a decreasing function. $R_4(r_4) = 1$ And $R_4(k_4)_4 < 1$ imply $r_4 < (k_4)_4$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_4(a_4), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{k_4 a_4} da_4$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_4 \equiv \int_0^\infty (\rho_0)_{44}(a_4)da_4$ *satisfies*

$$\int_0^\infty \beta_4(a_4) e^{-\int_0^a \mu_4(\tau)d\tau} e^{-(f_4)_4(P_0)a_4} da$$

$$= 1 \qquad (11)$$

Next, we study the stability of $(\rho_0)_4(a_4), 0)$

Let $u_4(t, a_4) = \rho(t, a_4) - \rho_0(a_4)$ Then

$$N_4(t) \equiv \int_0^\infty u_4(t, a_4)da_4 = \int_0^\infty \rho_4(t, a_4)da_4 - \int_0^\infty (\rho_0)_4(a_4)da = P_4(t) - P_0)_4$$

And

$$(u_4)_4 + (u_a)_4 = (\rho_t)_4 + (\rho_a)_4 - (\rho_0^4)_4(a_4)$$

$$= -\mu_4(a_4) + (f_4)_4(P_4) + (f_2)_4(Q_4))\,(u_4 + (\rho_0)_4) + \mu_4(a_4)(\rho_0)_4 + (f_4)_4(P_0)_4(\rho_0)_4$$

$$= -\mu_4(a_4)u_4 - (f_4)_4(P_4) + (f_2)_4(Q_4))\,u_4 - (\rho_0)_4\,(f_4)_4(P_4) - (f_4)_4(P_0)_4$$

$$+ (f_2)_4(Q_4)), \qquad (15)$$

Where $(\rho_0^4)_4(a_4)$ denotes $d(\rho_0)_4(a_4)/da_4$

Expansion $(f_4)_4(P_4)$ and $(f_2)_4(Q_4)$ at $(P_0)_4$ and 0 yield respectively

$$(f_4)_4(P_4) = (f_4)_4((P_0)_4 + N_4)$$

$$= (f_4)_4(P_0)_4 + (f_4^4)_4(P_0)_4N_4 + o_4(N_4), \qquad (16)$$

$$(f_2)_4(Q_4)$$

$$= (f_2^4)_4(0)Q + o_4(Q_4) \qquad (17)$$

Write $f_4(P_0) = f_{40}, f_4'(P_0) = f_{40}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_4)_4 + (u_a)_4 \approx -(\mu_4(a_4) + (f_{40})_4)u - (f_{40}^4)_4(\rho_0)_4(N_4) - (f_{20}^4)_4(\rho_0)_4(Q_4)$$

$$u_4(t,0) = \rho_4(t,0)(\rho_0)_4(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a) + (\rho_0)_4(a_4)da_4 - (\rho_0)_4\,(0)$$

$$= h_4(Q_4)\int_0^\infty \beta_4(a_4)(u_4(t,a_4)da_4 + (h_4(Q_4) - 4)(\rho_0)_4\,(0) \qquad (18)$$

Expanding

$$h_4(Q_4) = 1 + (h^4)_4(0)Q_4 + 0(Q_4) \equiv 1 + h_0'Q_4 + 0Q_4$$

And substituting it into (48), we arrive at

$$u_4(t,0) \approx \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4 + (h'_{a_4})(\rho_0)_4 Q_4$$

The linearization is the following :

$$[u_4] + [u_4] = -(\mu_4(a_4) + [t_{40}]_4 u_4 - [\alpha_4]_4(a_4)N_4 - [\alpha_2]_4(a_4)Q_4$$

$$u_4(t,0) = \int_0^\infty \beta_4(a_4)u_4(t,a_4)da_4 + [\xi_4]Q_4 \qquad (19)$$

$$\frac{dQ_4}{dt} = -b_4 Q_4$$

where

$$[\alpha_4]_4(a_4) = [t_4^4 a]_4[p,o]_4(a_4)d2(a_4) = [f_{20}^4]_4(\rho\, o\,)_4(a_4)\xi = [\mathrm{h\ o}^4]_4[\rho\mathrm{o}]_4(\mathrm{o}) \text{ and } \mathrm{b}_4 = \mathrm{b}_4 - g_4(P_4)$$

If $g_4(P_0) > b$, clearly, $(\rho_0)_4(a_4), 0)$ is unstable and the predator population is persistent. Assume $g_4(P_0)_4 < b$.

Then, $Q_4(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_4 + u_a = -(\mu_4(a_4) + (f_{40})_4)u_4 - (\alpha_4)_4(a_4)N_4$$

$$= -(\mu_4(a_4) + f_{40})u_4 - \alpha_4(a_4)\int_0^\infty u_4(t,a_4)da_4$$

$$u_4(t,0) = \int_0^\infty \beta_4\,(a_4)u_4(t,a_4)da_4$$

Using the method developed by Cushing [2, 3], we suppose

$$u_4(t,a_4) = \tilde{\mathrm{u}}_4(a_4)^{e^{\gamma(t-a_4)}}$$

And define $w = \int_0^\infty \tilde{u}(a) e^{-\gamma a_4}\, da_4$. Then,

A solution of the first differential equation for $\tilde{u}(a_4)$ gives,

$$\tilde{u}_4(a_4) = e^{-\int_0^a \mu_4(\tau_4)d\tau - (f_{40})_4 a}\left(\tilde{u}_4(0) - w\int_0^a e^{\int_0^\tau \mu_4(s_4)ds + (f_{40})\tau_4}\,(\alpha_4)_4(\tau_4)e^{\gamma_4 \tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\right)\tilde{u}_4(0)$$

$$+w\int_0^\infty \beta_4(a_4) e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a (\alpha_4)_4(\tau_4)e^{-\gamma_4\tau_4} e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds} d\tau_4\, da_4$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\, da\right)(\tilde{u}_4(0))_4$$

$$+\left(-4 - \int_0^\infty e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4)d\tau}\int_0^a e^{(f_{40})_4\tau_4 + \int_0^\tau \mu_4(s_4)ds + y\tau}\alpha_4(\tau)d\tau\, da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_4 a_4 - (f_{40})_4 a_4 - \int_0^a \mu_4(t_4)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_4\,(a_4) Y_4(\gamma_4, a_4) da_4\right)u_4(0)$$

$$+w\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4=0,$$

(24)

$$\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\tilde{u}_4(0)$$

$$+\left(-1-\int_0^{\infty}Y_4(\gamma_4,a_4)\int_0^{a}(\alpha_4)_4(\tau)_4(Y^{-4})_4(\gamma_4,\tau_4)d\tau_4\,da_4\right)w=0.$$

Equqtion (20) has a solution of the form of $u_4(t,a_4)=u_4(a_4)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_4(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)Y^{-4}(\gamma_4,a_4)d\tau_4$. We obtain the characteristic equation

$$f_{40}^{l}{}_4\left(\int_0^{\infty}Y_4(\gamma_4,a_4)da_4\right)\left(\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$+\left(1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4,a_4)da\right)\left(1+(f_{40}^{l})_4\int_0^{\infty}Y_4(\gamma_4,a_4)I_4(\gamma_4,a_4)da_4\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_4(a_4)=(\rho_0)_4(0)e^{-\int_0^{a}\mu_4(\tau_4)d\tau_4-(f_{40})_4a_4}$$

$$I_4(\gamma_4,a_4)=\int_0^{a}(\rho_0)_4(\tau_4)e^{\gamma_4\tau_4+(f_{40})_4\tau+\int_0^{\tau}\mu_4(s_4)ds_4}d\tau$$

$$= \frac{(\rho_0)_4(0)}{\gamma_4}(e^{\gamma_4 a_4} - 1)$$

Let

$$E_4(a_4) = e^{-(f_{40})_4 a_4 - \int_0^a \mu_4(\tau_4) d\tau_4}$$

Then $[Y_4]_4[\gamma_4]_4, a_4) = E_4(a_4)e^{-\gamma_4 a_4}$. by a simple calculation,

$$\int_0^\infty \beta_4(a_4)Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma}\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}\ da\right);$$

$$\int_0^\infty Y_4(\gamma_4, a_4)I(\gamma_4, a_4)da = \frac{[\rho_0]_4(0)}{\gamma_4}\left(\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}\ da_4\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_4(0)}{\gamma_4}[f_{40}^4]\int_0^\infty E_4(a_4)e^{-\gamma_4 a_4}da_4\ (\int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4)$$

$$+\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right).$$

$$\left(1 + \frac{(\rho_0)_4(0)}{\gamma_4}(f_{40}^{'})_4\left(\int_0^\infty E_4(\,a_4)da_4 - \int_0^\infty E_4(\,a_4)e^{-\gamma_4 a_4}da_4\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4\right)(1 + \frac{(f_{40}^{'})_4}{\gamma_4}(\rho_0)_4(0)\int_0^\infty E_4(\,a_4)da_4) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)da_4 = 1-\int_0^{\infty}\beta_4(a_4)E_4(a_4)e^{-\gamma_4 a_4}da_4 \neq 0,$$

Since $\int_0^{\infty}\beta_4(a_4)Y_4(\gamma_4 a_4)\ \int_0^{a}\alpha_4\,(\tau_4)Y^{-4}(\gamma_4,\tau_4)d\tau_4 da_4 > 0.$ Therefore, the characteristic solution is

$$\gamma_4 = -(f_{40}^{'})_4(\rho_0)_4\int_0^{\infty}E_4(a_4)da_4.$$

Since $f_4^{'}(P_0)_4 > 0,$ we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_4), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_4), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_4 + (\mu_4(a_4) + [f_4]_4(P_4^*) + [f_2]_4(Q_4^*))P_4^* = 0, \quad (26)$$

$$p_4^*(0) = h_4(Q_4^*)\int_0^{\infty}\beta_4(a_4)p_4^*(a_J)p_4^*(a_4)da_4 \quad (27)$$

$$g_4(P_4^*) - b_4 = 0, \quad (28)$$

Where $P_4^*=\int_0^\infty p_4^*(a_4)\,da_4$

Equation (28) yields $P^* = g^{-4}(b)$,

From (26),

$$p_4^*(a) = p_4^*(0)e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}$$

Substituting into (27), we need to solve

$$\left(1 - h_4(Q_4^*)\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(Q_4^*)a_4}da_4\right)p_4^*(0) = 0$$

For $Q_4^*$

Fix $p_4^*(0)$ and $P_4^*$ Let

$$H_4(z_4) = 1 - h_4(z_4)\int_0^\infty \beta_4(a_4)\,e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4}da_{,4}$$

And assume that

$$[k_3]_4\int_0^\infty \beta_4(a_4)e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(g_4^{-4}(b_4))a_4 - [k_2]_4a_4}\,da_4 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_4(z_4)] > 0.$$

Clearly,

$$\frac{dH_4}{dz_4} = h_4(z)([f_2^{'}]_4(z_4)) \int_0^{\infty} a_4\beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - f_4(P_4^*)a_4 - f_2(z_4)a_4} da_4$$

$-h^{'}(z_4)\int_0^{\infty} \beta_4(a_4)\, e^{\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4 - [f_2]_4(z_4)a_4} \quad da_4$

the assumptions on $\left[f_2^4\right]_4 and\ [h^4]^4 dH_4/dz > 0$. Hence, if

$$H_4(0) = 1 - \int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4(P_4^*)a_4} da_4 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_4(0) < 0$ follows if $g_4^-(b_4) < [f_4^{-4}]_4(r_4)$, where $r_4$ is the solution of (12).

Indeed, since

$$R_4(s_4) = \int_0^{\infty} \beta_4(a_4)\, e^{-\int_0^a \mu_4(\tau_4)d\tau_4 e^{-\gamma_4^a} da_4}$$

Is a decreasing function of $s_4$ when $P_4^* = [g^{-4}]_4(b_4) < [f_4^{-4}]_4(r_4)$, that is $[f_4]_4(P_4^*) < r, R\left(f_4(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_4(g^{-4}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4)d\tau_4 - [f_4]_4\left(g_4^{-4}(b)\right)a_4} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_4^*(a_4), Q_4^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_4^*(a_4), Q_4^*)$ is more complicated. We again consider perturbations $m_4(t_4 a_4) = p_4(t_4, a_4) - p_4^*(a_4)$ and $W_4(t) = Q_4(t) - Q^*$.Then

$M_4(t) \equiv \int_0^\infty m_4(t, a_4) da_4 = P_4(t) - P_4^*$,

$$[m_4]_4 + [m_a]_4 = [p_4]_4 + [p_a]_4 - [p^*]_4(a_4) \approx -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4$$

$$-[f_{4*}^4]_4 p_4^*(a_4)M - [f_{2*}^4]_4 p_4^*(\mathrm{a})W_4,$$

$$m_4(t,0)p_4(t,0) - p_4^*(0) \approx h_* \int_0^\infty \beta_4(a_4) m_4(t, a_4)\, da + \frac{[h_*^4]_4}{[h_*]_4}\, p_4^*(0) W_4$$

$$\frac{dW_4}{dt} = \frac{dQ_4}{dt} \approx [g_*^4]_4 Q_4^* M_{,4}$$

Where $[f_{4*}]_4$ denoting $[f_4(P^*)]_4$, $[f_{4*}]_4$ denoting $[f_2(P^*)]_4$, $[f_{4*}']_4$ denoting $[f'_4(P^*)]_4 [f_{2*}']_4$ denoting $[f_2]_4$ $(Q_4^*$, $h_{*}_4$ denoting $h_*'$ denoting $h_4'$ $(Q_4^*)$, and $[g_*']_4$

Denoting $[g']_4(P_4^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$[m_4]_4 + [m_a]_4 = -(\mu_4(a_4) + [f_{4*}]_4 + [f_{2*}]_4)m_4 - [f'_{4*}]_4 [p_4^*](a_4)M_4 - [f_{2*}']_4 p_4^*(a_4)W_4$

$m_4(t,0) = [h_*]_4 \int_0^\infty \beta_4(a_4) m_4(t, a_4) da_4 + \frac{[h_*^4]_4}{[h^*]_4} [p^*]_4(0)\, W_{,4}$ (30)

$$\frac{dW_4}{dt} = [g_*^4]_4 [Q_4^*] M_4$$

This is equation is important in the following cases: problem of using observed correlations to infer causal relations is relevant to (e) a wide variety of scientific disciplines. Yet given correlations between just two classical variables, (eb) it is impossible to determine whether they arose from (e) a causal influence of one on the other or a common cause influencing both, unless one can implement (eb) a randomized intervention. **Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. Resch** here consider the problem of causal inference for (e) quantum variables. Authors introduce causal tomography, which unifies and generalizes (e&eb) conventional quantum tomography schemes to provide (eb) a complete solution to the causal inference problem using (e) a quantum analogue of a randomized trial. Authors furthermore show that, in contrast to the classical case, observed quantum correlations alone can sometimes provide (eb) a solution. They implement a quantum-optical experiment that allows (eb) to control (e&eb) the causal relation between two optical modes, and two measurement schemes -- one with (e&eb) and one without (e) randomization -- that extract this relation from (e) the observed correlations. Results show that entanglement and (e&eb) coherence, known to be central to (e) quantum information processing, also provide (eb) a quantum advantage for (e) causal inference. Subjects: Quantum Physics (quant-ph); Learning (cs.LG); General Relativity and Quantum Cosmology (gr-qc); Machine Learning (stat.ML) Journal reference: Nat Phys 11, 414-420 (2015) DOI: 10.1038/nphys3266 Cite as: arXiv: 1406.5036 [quant-ph] (or arXiv: 1406.5036v1 [quant-ph] for this version) **Inferring causal structure: a quantum advantage Katja Ried, Megan Agnew, Lydia Vermeyden, Dominik Janzing, Robert W. Spekkens, Kevin J. ReschSTOP D. T. Son, A. O. Starinets** review recent progress in applying the AdS/CFT correspondence to (e&eb) finite-temperature field theory. In particular, they show how the hydrodynamic behavior of field theory is reflected in (e&eb) the low-momentum limit of (e) correlation functions computed through (e&eb) a real-time AdS/CFT prescription, which they formulate. Authors also show how the hydrodynamic modes in field theory correspond to (e&eb) the low-lying quasinormal modes of (e) the AdS black p-brane metric. They provide a proof of the

universality of (e) the viscosity/entropy ratio within (eb) a class of theories with (e&eb) gravity duals and formulate (eb) a viscosity bound conjecture. Possible implications for real systems are mentioned. Subjects: High Energy Physics - Theory (hep-th) Journal reference: Ann.Rev.Nucl.Part.Sci.57:95-118,2007 DOI: 10.1146/annurev.nucl.57.090506.123120Report number: INT PUB 07-02Cite as: arXiv:0704.0240 [hep-th] (or arXiv:0704.0240v2 [hep-th] for this version) An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society **Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP** Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in

(eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version **Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016) STOP Viscosity, Black Holes, and Quantum Field Theory D. T. Son, A. O. Starinets STOP Authors** study information retrieval from evaporating black holes, assuming that the internal dynamics of a black hole is unitary and rapidly mixing, and assuming that the retriever has unlimited control over the emitted Hawking radiation. If the evaporation of the black hole has already proceeded past the ``half-way" point, where (e) half of the initial entropy has been radiated away, then (eb) additional quantum information deposited in (eb) the black hole is revealed in (eb) the Hawking radiation very rapidly. Information deposited prior to the half-way point remains concealed until (e) the half-way point, and then emerges (eb) quickly. These conclusions hold because (e) typical local quantum circuits are efficient encoders for (e) quantum error-correcting codes that nearly achieve (eb) the capacity of the quantum erasure channel. Estimate of a black hole's information retention time, based on (e) speculative dynamical assumptions, is just barely compatible with (e&eb) the black hole complementarity hypothesis. **Black holes as mirrors: quantum information in random subsystems Patrick Hayden1 and John Preskill2 Published 26 September 2007 Journal of High Energy Physics, Volume 2007, JHEP09 (2007) STOP** By analyzing the key properties of black holes from the point of (e) view of quantum information, authors derive a model-independent picture of (e) black hole quantum computing. It has been noticed that this picture exhibits (eb) striking similarities with (e&eb) quantum critical condensates, allowing (eb) the use of a common language to describe (eb) quantum computing in both systems. Authors analyze such quantum computing

by allowing coupling to (e&eb) external modes, under the condition that the external influence must be (=) soft-enough in order not to (e) offset the basic properties of the system. Authors derive model-independent bounds on some crucial time-scales, such as (=) the times of gate operation, decoherence, maximal entanglement and total scrambling. They show that for black hole type quantum computers all these time-scales are of (e) the order of the black hole half-life time. Furthermore, authors construct explicitly a set of Hamiltonians that generates (eb) a universal set of quantum gates for (e) the black hole type computer. They find that the gates work at maximal energy efficiency. Furthermore, authors establish a fundamental bound on the complexity of quantum circuits encoded on (eb&e) these systems, and characterize the unitary operations that are (=) implementable. It becomes apparent that the computational power is very limited due to (e) the fact that the black hole life-time is of the same order of (e) the gate operation time. As a consequence, it is impossible to retrieve its information, within (eb) the life-time of a black hole, by externally coupling to (e&eb) the black hole qubits. However, authors show that, in principle, coupling to (e&eb) some of the internal degrees of freedom allows (eb) acquiring knowledge about the micro-state. Still, due to the trivial complexity of operations that can be performed, (eb) there is no time advantage over the collection of Hawking radiation and subsequent decoding. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Report number: LMU-ASC 03/16, MPP-2016-1 Cite as: arXiv:1605.01407 [hep-th] (or arXiv:1605.01407v1 [hep-th] for this version) **Universality of Black Hole Quantum Computing Gia Dvali, Cesar Gomez, Dieter Lust, Yasser Omar, and Benedikt Richter (Submitted on 4 May 2016) STOP** In this addendum to [arXiv: 1402.5674] two points are discussed. In the first additional evidence is provided (eb) for a dual connection between the geometric length of an Einstein-Rosen bridge and (e&eb) the computational complexity of the quantum state of the dual CFT's. The relation between growth of complexity and (e&eb) Page's “Extreme Cosmic Censorship” principle is also remarked on. The second point involves a gedanken experiment in which Alice measures (eb) a

complete set of commuting observables at her end of (e) an Einstein-Rosen bridge is discussed. An apparent paradox is resolved by appealing to (e) the properties of GHZ tripartite entanglement. **Addendum to computational complexity and black hole horizons Leonard Susskind* Version of Record online: 18 JAN 2016 DOI: 10.1002/prop.201500093 STOP** Analyzing some well established facts, authors give a model-independent parameterization of black hole quantum computing in terms of (e&*eb) a set of macro and micro quantities and their relations. These include the relations between the extraordinarily-small energy gap of (e) black hole qubits and important time-scales of (e) information-processing, such as, scrambling time and Page's time. Authors then show, confirming and extending previous results, that other systems of nature with (e&eb) identical quantum informatics features are (=) attractive Bose-Einstein systems at (eb) the critical point of quantum phase transition. Here authors establish a complete isomorphy between (e&eb) the quantum computational properties of these two systems. In particular, they show that the quantum hair of a critical condensate is (=) strikingly similar to (=) the quantum hair of a black hole. Irrespectively whether one takes the similarity between the two systems as a remarkable coincidence or as (=) a sign of a deeper underlying connection, the following is evident. Black holes are not unique in their way of (e) quantum information processing and they can manufacture black hole based quantum computers in labs by (e) taking advantage of quantum criticality. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Atomic Physics (physics. Atom-ph); Quantum Physics (quant-ph) Cite as: arXiv: 1601.01329 [hep-th] (or arXiv: 1601.01329v1 [hep-th] for this version) **Black Hole Based Quantum Computing in Labs and in the Sky Gia Dvali, Mischa Panchenko (Submitted on 6 Jan 2016) STOP Gia Dvali, Alexander Gußmann** show that the existence of black holes with (e&eb) classical skyrmion hair invalidates (e) standard proofs that global charges, such as (=) the baryon number, cannot be conserved by (e) a black hole. By carefully analyzing the standard arguments based on (e) a Gedankenexperiment in which a black hole is seemingly-unable to (e) return the baryon number that it swallowed,

authors identify inconsistencies in this reasoning, which does not (e) take into the account neither the existence of skyrmion black holes nor the baryon/skyrmion correspondence. Authors then perform a refined Gedankenexperiment by incorporating (e) the new knowledge and show that (eb) no contradiction with conservation of baryon number takes place at (eb) any stage of black hole evolution. Analysis also indicates (eb) no conflict between semi-classical black holes and (e&eb) the existence of baryonic gauge interaction arbitrarily-weaker than gravity. Next, authors study classical cross sections of a minimally-coupled massless probe scalar field scattered by (e) _ a skyrmion black hole. Authors investigate how the skyrmion hair manifests itself by (e) comparing this cross section with (e&eb) the analogous cross section caused by (e) a Schwarzschild black hole which has (e) the same ADM mass as (=) the skyrmion black hole. Here authors find an order-one difference in the positions of (e) the characteristic peaks in (eb) the cross sections. The peaks are shifted to smaller scattering angles when (e) the skyrmion hair is present. This comes from the fact that the skyrmion hair changes (e&eb) the near horizon geometry of the black hole when compared to (e&eb) a Schwarzschild black hole with same ADM mass. Authors keep the study of this second aspect general so that (e) the qualitative results obtained can also be applied to(e&eb) black holes with classical hair of different kind. Subjects: High Energy Physics - Theory (hep-th); High Energy Astrophysical Phenomena (astro-ph.HE); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU-ASC 18/16 Cite as: arXiv:1605.00543 [hep-th] (or arXiv:1605.00543v2 [hep-th] for this version) A preliminary discussion is given of (e) the prospects that gravitational-wave observations of (e) binary inspiral of black holes could reveal or constrain (e&eb) quantum modifications to (e) black hole dynamics, such as are required to (e) preserve postulates of quantum mechanics. Different proposals for such modifications are characterized by (e) different scales, and the size of these scales relative to (e&eb) those probed by observation of (e) inspiral signals is important in determining (eb) the feasibility of finding experimental signatures. Certain scenarios with strong quantum modifications in (eb) a region extending

well outside the horizon are expected to modify (e&eb) classical evolution, and distort (e&eb) the near-peak gravitational wave signal, suggesting (eb) a search for departures from (e) waveforms predicted by general relativity. The near agreement of the GW150914 signals with (e&eb) such waveforms is discussed, and indicates (eb) constraints on some such scenarios. Important strategies for more precise future tests are 1) to develop more precise predictions from (e) scenarios proposing quantum modifications, and 2) searching for observed deviations from (e) numerical relativity predictions via (e&eb) analysis of gravity wave data, particularly focussing on (e&eb) the signal region corresponding to plunge and merger. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1602.03622 [gr-qc] (or arXiv: 1602.03622v2 [gr-qc] for this version) **Gravitational wave tests of quantum modifications to black hole structure -- with post-GW150914 update Steven B. Giddings STOP** Direct LHC bounds on (e&eb) colored SUSY particles now corner naturalness more than the measured value of the Higgs mass does. Bounds on the gluino are of (e) particular importance, since it radiatively "sucks" up (e) the stop and Higgs-up soft masses. As a result, even models that easily accommodate a 125 GeV Higgs are almost as tuned as (=) the simplest version of SUSY, the MSSM: at best at the percent level. In this paper, authors further examine how current LHC results constrain (e) naturalness in three classes of models that may relax LHC bounds on (e&eb) sparticles: split families (e&eb), baryonic RPV (e&eb), and Dirac gauginos. In models of split families and bRPV, the bounds on the gluino are (=) only slightly reduced, resulting in (eb) a few percent tuning. In particular, having a natural spectrum in bRPV models typically implies (eb) that tops, Ws, and Zs are easily produced (eb) in the cascade decays of (e) squarks and gluinos. The resulting leptons and missing energy push (eb) the gluino mass limit above 1 TeV. Even when the gluino has a Dirac mass and does not contribute to the stop mass at one loop, tuning reappears in calculable models because there is no symmetry imposing the supersoft limit. Authors conclude that, even if sparticles are found at LHC-14, naturalness will not emerge triumphant. Subjects: High Energy Physics - Phenomenology (hep-ph)DOI:10.1007/JHEP03(2014)022 Cite

as:rXiv:1309.3568 [hep-ph] (or arXiv:1309.3568v2 [hep-ph] for this version) **The Last Vestiges of Naturalness Asimina Arvanitaki, Masha Baryakhtar, Xinlu Huang, Ken Van Tilburg, Giovanni Villadoro STOP** The lack of evidence for new physics beyond (e) the standard model at the LHC points to (e) a paucity of new particles near the weak scale. This suggests that the weak scale is tuned and that supersymmetry, if present at all, is realized (eb) at higher energies. The measured Higgs mass constrains (e) the scalar sparticles to be below 10^5 TeV, while (e) gauge coupling unification favors (eb) Higgsinos below 100 TeV. Nevertheless, in many models gaugino masses are (=) suppressed and remain within reach of the LHC. Tuning the weak scale and (e&eb) the renormalization group evolution of (e) the scalar masses constrain (e) Split model building. Due to the small gaugino masses, either the squarks or the up-higgs often run tachyonic; in the latter case, successful electroweak breaking requires heavy higgsinos near the scalar sparticles. We discuss the consequences of tuning the weak scale and the phenomenology of several models of Split supersymmetry including anomaly mediation, U(1)_(B-L) mediation, and Split gauge mediation. Subjects: High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Experiment (hep-ex) DOI: 10.1007/JHEP02 (2013)126 Cite as: arXiv: 1210.0555 [hep-ph] (or arXiv: 1210.0555v2 [hep-ph] for this version) **Mini-Split Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Giovanni Villadoro STOP** Light states associated with (e&eb) the hierarchy problem affect (e&eb) the Higgs LHC production and decays. Authors illustrate this within the MSSM and two simple extensions applying (e&eb) the latest bounds from (e) LHC Higgs searches. Large deviations in the Higgs properties are expected in (eb) a natural SUSY spectrum. The discovery of a non-Standard-Model Higgs may signal (eb) the presence of light stops accessible at (eb) the LHC. Conversely, the more the Higgs is Standard-Model-like, (eb) the more tuned the theory becomes. Taking the ratio of different Higgs decay channels at the LHC cancels (er&eb) the leading (eb) QCD uncertainties and potentially improves (eb) the accuracy in Higgs coupling measurements to (e) the percent level. This may lead to (eb) the possibility of doing precision Higgs physics at (eb) the LHC. Finally, authors entertain the

possibility that the ATLAS excess around (e&eb) 125 GeV persists with (e&eb) a Higgs production cross-section that is enhanced compared to (e&eb) the SM. This increase can only be accommodated in extensions of (e) the MSSM and it may suggest (eb) that stops lie below 400 GeV, likely within (eb) reach of next year's LHC run. Subjects: High Energy Physics - Phenomenology (hep-ph) DOI: 10.1007/JHEP02 (2012)144 Cite as: arXiv: 1112.4835 [hep-ph] (or arXiv: 1112.4835v2 [hep-ph] for this version) **A Non Standard Model Higgs at the LHC as a Sign of Naturalness Asimina Arvanitaki, Giovanni Villadoro STOP** It has recently been suggested that the presence of a plenitude of light axions, an Axiverse, is (=) evidence for the extra dimensions of string theory. Authors discuss the observational consequences of (e) these axions on astrophysical black holes through (e&eb) the Penrose superradiance process. When an axion Compton wavelength is comparable to (e&eb) the size of a black hole, the axion binds to (e&eb) the black hole "nucleus" forming (eb) a gravitational atom in the sky. The occupation number of superradiant atomic levels fed by (e) the energy and angular momentum of the black hole grows exponentially. The black hole spins down and an axion Bose-Einstein condensate cloud forms (eb) around it. When the attractive axion self-interactions become (=) stronger than the gravitational binding energy, (eb) the axion cloud collapses, a phenomenon known in condensed matter physics as "Bosenova". The existence of axions is first diagnosed by (e) gaps in the mass vs (e&eb) spin plot of astrophysical black holes. For young black holes the allowed values of (e) spin are quantized, giving rise to (eb) "Regge trajectories" inside the gap region. The axion cloud can also be observed directly either through (e&eb) precision mapping of the near horizon geometry or through (e&eb) gravitational waves coming from (e) the Bosenova explosion, as well as axion transitions and (e&eb) annihilations in (eb) the gravitational atom. Estimates suggest (eb) that these signals are detectable in (eb) upcoming experiments, such as Advanced LIGO, AGIS, and LISA. Current black hole spin measurements imply (eb) an upper bound on (e&eb) the QCD axion decay constant of (e) 2 x 10^17 GeV, while (e) Advanced LIGO can detect (e&eb) signals from (e) a QCD axion cloud with (e&eb) a decay constant as low as (=)

the GUT scale. Authors finally discuss the possibility of observing the gamma-rays associated with the Bosenova explosion and, perhaps, the radio waves from axion-to-photon conversion for the QCD axion. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Journal reference:Phys.Rev.D83:044026, 2011 DOI: 10.1103/PhysRevD.83.044026 Cite as: arXiv: 1004.3558 [hep-th] (or arXiv: 1004.3558v2 [hep-th] for this version) **Exploring the String Axiverse with Precision Black Hole Physics Asimina Arvanitaki, Sergei Dubovsky (Submitted on 20 Apr 2010 (v1), last revised 16 Aug 2011 (this version, v2)) STOP Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations Gia Dvali, Alexander Gußmann STOP Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust** discuss BMS supertranslations both at (eb) null-infinity and on the horizon for the case of the Schwarzschild black hole. Authors show that both kinds of supertranslations lead to (eb) infinitely many gapless physical excitations. On this basis authors construct a quotient algebra using (e) suited superpositions of (e) both kinds of transformations which cannot be compensated by (e) an ordinary BMS-supertranslation and therefore are (=) intrinsically due to (e) the presence of an event horizon. Authors show that these quotient transformations are (=) physical and generate (eb) gapless excitations on the horizon that can account for (e) the gravitational hair as well as for (e) the black hole entropy. They identify the physics of these modes as associated with (e&eb) Bogolioubov-Goldstone modes due to quantum criticality. Classically the number of these gapless modes is (=) infinite. However, authors show that due to (e) quantum criticality the actual amount of information-carriers becomes (=) finite and consistent with Bekenstein entropy. Although they only consider the case of Schwarzschild geometry, the arguments are extendable to(e&eb) arbitrary space-times containing event horizons. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) Report number: LMU--ASC 05/16, MPP--2016—3 Cite as:

arXiv: 1601.03725 [hep-th] (or arXiv: 1601.03725v1 [hep-th] for this version) **Gravitational Black Hole Hair from Event Horizon Supertranslations Artem Averin, Gia Dvali, Cesar Gomez, Dieter Lust STOP Gia Dvali, Cesar Gomez, Nico Wintergerst** suggest that holography can be formulated in terms of (e&eb) the information capacity of (e) the St\"uckelberg degrees of freedom that maintain gauge invariance of (e) the theory in the presence of (e) an information boundary. These St\"uckelbergs act as (e&eb) qubits that account for a certain fraction of (e) quantum information. Their information capacity is measured by (e) the ratio of the inverse St\"uckelberg energy gap to (e) the size of the system. Systems with (e&eb) the smallest gap are (=) maximally holographic. For massless gauge systems this information measure is (=) universally equal to (=) the inverse coupling evaluated at the systems' length scale. In this language it becomes very transparent why the St\"uckelberg information capacity of black holes saturates (e full) the Bekenstein bound and accounts for (e) the entire information of the system. The physical reason is that the strength of quantum interaction is bounded from below by the gravitational coupling, which scales as (=) area. Observing the striking similarity between the scalings of the energy gap of the boundary St\"uckelberg modes and (e&eb): (AC=DC=0) the Bogoliubov modes of critical many-body systems, authors establish a connection between holography and (e&eb) quantum criticality through (e&eb) the correspondence between these modes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1511.03525 [hep-th] (or arXiv: 1511.03525v1 [hep-th] for this version) **Stückelberg Formulation of Holography Gia Dvali, Cesar Gomez, Nico Wintergerst STOP B**lack hole entropy, denoted by N, in (semi)classical limit is (=) infinite. This scaling reveals (eb) a very important information about the qubit degrees of (e) freedom that carry black hole entropy. Namely, the multiplicity of qubits scales as N, whereas (e) their energy gap and their coupling as 1/N. Such a behavior is indeed exhibited by (e) Bogoliubov-Goldstone degrees of freedom of (e) a quantum-critical state of N soft gravitons (a condensate or a coherent state) describing (eb) the black hole quantum portrait. They can be viewed as (=) the

Goldstone modes of (e) a broken symmetry acting on (e&eb) the graviton condensate. In this picture Minkowski space naturally emerges as (=) a coherent state of infinite-N gravitons of (e) infinite wavelength and it carries (e&eb) an infinite entropy. In this paper authors **Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015)** ask what is the geometric meaning (if any) of (e) the classical limit of this symmetry. Authors argue that the infinite-N limit of Bogoliubov-Goldstone modes of critical graviton condensate is described by (e) recently-discussed classical BMS super-translations broken by the black hole geometry. However, the full black hole information can only be recovered for (e) finite N, since the recovery time becomes (=) infinite in classical limit in which N is (=) infinite. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph) DOI: 10.1016/j.physletb.2015.11.073 Report number: MPP-2015-215, LMU-ASC 59/15 Cite as: arXiv: 1509.02114 [hep-th] (or arXiv: 1509.02114v1 [hep-th] for this version) **Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry Gia Dvali, Cesar Gomez, Dieter Lüst (Submitted on 7 Sep 2015) STOP** Recent ideas about understanding physics of black hole information-processing in terms of (e&eb) quantum criticality allow us to (eb) implement black hole mechanisms of quantum computing within (eb) critical Bose-Einstein systems. The generic feature, uncovered both by analytic and numeric studies, is (=) the emergence at the critical point of (e) gapless weakly-interacting modes, which act as (=) qubits for information-storage at a very low energy cost. These modes can be effectively described in terms of (e&eb) either Bogoliubov or Goldstone degrees of freedom. The ground-state at the critical point is (=) maximally entangled and far from being classical. We confirm this near-critical behavior by a new analytic method. **Gia Dvali, Mischa Panchenko** compute growth of entanglement and show its consistency with (e&eb) black hole type behavior. On the other hand, in the over-critical regime the system develops (eb) a Lyapunov exponent and scrambles quantum information very fast. By, manipulating (e&eb) the system parameters externally, authors can put it in and out of various regimes and in this way control (e&eb) the

sequence of information storage and processing. By using gapless Bogoliubov modes as (=) control qubits they design some simple logic gates. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Cite as: arXiv: 1507.08952 [hep-th] (or arXiv: 1507.08952v1 [hep-th] for this version) **Black Hole Type Quantum Computing in Critical Bose-Einstein Systems Gia Dvali, Mischa Panchenko (Submitted on 31 Jul 2015) STOP Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst** establish a fundamental connection between quantum criticality of a many-body system, such as Bose-Einstein condensates, and (e&eb) its capacity of information-storage and processing. For deriving the effective theory of modes in the vicinity of the quantum critical point they develop a new method by mapping a Bose-Einstein condensate of N-particles onto (e&eb) a sigma model with a continuous global (pseudo)symmetry that mixes (e&eb) bosons of different momenta. The Bogolyubov modes of the condensate are mapped onto (e&eb) the Goldstone modes of the sigma model, which become (=) gapless at the critical point. These gapless Goldstone modes are (=) the quantum carriers of information and entropy. Analyzing their effective theory, authors observe the information-processing properties strikingly similar to (=) the ones predicted by the black hole portrait. The energy cost per qubit of information-storage vanishes in (eb) the large-N limit and the total information-storage capacity increases with (e&eb) N either exponentially or as a power law. The longevity of information-storage also increases with (e&eb) N, whereas (e) the scrambling time in (eb) the over-critical regime is controlled by (e&eb) the Lyapunov exponent and scales logarithmically with (e&eb) N. This connection reveals (eb) that the origin of black hole information storage lies in (eb) the quantum criticality of the graviton Bose-gas, and that much simpler systems that can be manufactured in table-top experiments can exhibit (eb) very similar information-processing dynamics. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Phys. Rev. D 92, 125002 (2015) DOI: 10.1103/PhysRevD.92.125002 Cite as: arXiv:1507.02948 [hep-th]

(or arXiv:1507.02948v1 [hep-th] for this version) **Nambu-Goldstone Effective Theory of Information at Quantum Criticality Gia Dvali, Andre Franca, Cesar Gomez, Nico Wintergerst STOP Gia Dvali, Cesar Gomez** show that a positive cosmological constant is (=) incompatible with the quantum-corpuscular resolution of de Sitter metric in form of (e) a coherent state. The reason is very general and is due to (e) the quantum self-destruction of (e) the coherent state because of (e) the scattering of constituent graviton quanta. This process creates an irreversible quantum clock, which precludes (e) eternal de Sitter. It also eliminates (e) the possibility of Boltzmann brains and Poincare recurrences. This effect is expected to be (=) part of any microscopic theory that takes into account (e) the quantum corpuscular structure of the cosmological background. This observation puts the cosmological constant problem in a very different light, promoting it, from a naturalness problem, into (e&eb) a question of quantum consistency. We are learning that quantum gravity cannot tolerate (e) exceedingly-classical sources. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1412.8077 [hep-th]

(or arXiv:1412.8077v1 [hep-th] for this version) **Quantum Exclusion of Positive Cosmological Constant? Gia Dvali, Cesar Gomez (Submitted on 27 Dec 2014) STOP G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger** establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and (e&eb) unitarization by black hole formation in these scattering processes. **Using as a guideline an explicit microscopic theory in which the black hole represents (eb) a bound-state of many soft gravitons at (eb) the quantum critical point, they were able to identify and compute (e&eb) a set of perturbative amplitudes relevant for (e) black hole formation. These are the tree-level N-graviton scattering S-matrix elements in (eb) a kinematical regime (called classicalization limit) where (e) the two incoming ultra-Planckian gravitons produce (eb) a large number N of soft gravitons. Authors compute these amplitudes by using (e) the Kawai-Lewellen-Tye relations, as well as scattering equations and (e&eb) string theory**

**techniques. Authors discover that this limit reveals (eb) the key features of the microscopic corpuscular black hole N-portrait. In particular, the perturbative suppression factor of a N-graviton final state, derived from (e) the amplitude, matches the non-perturbative black hole entropy when (e) N reaches the quantum criticality value, whereas final states with different value of N are (=) either suppressed or excluded by non-perturbative corpuscular physics. Thus they identify the microscopic reason behind the black hole dominance over other final states including (e) non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing (eb) us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. Authors demonstrate matching and complementarity between the string theory and (e&eb) field theory** in different large-s and large-N regimes. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1016/j.nuclphysb.2015.02.004 Report number: MPP-2014-320, LMU-ASC 52/14 Cite as: arXiv:1409.7405 [hep-th] (or arXiv:1409.7405v2 [hep-th] for this version) **Black Hole Formation and Classicalization in Ultra-Planckian 2 -> N Scattering G. Dvali, C. Gomez, R.S. Isermann, D. Lust, S. Stieberger STOP** Gravitational backgrounds, such as black holes, AdS, de Sitter and inflationary universes, should be viewed as (=) composite of N soft constituent gravitons. It then follows that such systems are close to (e) quantum criticality of graviton Bose-gas to Bose-liquid transition. Generic properties of the ordinary metric description, including (e) geodesic motion or particle-creation in the background metric, emerge as (=) the large-N limit of quantum scattering of (e) constituent longitudinal gravitons. **Gia Dvali, Cesar Gomez** show that this picture correctly accounts for (e) physics of large and small black holes in AdS, as well as reproduces (eb) well-known inflationary predictions for cosmological parameters. However, it anticipates new effects not captured by (e) the standard semi-classical treatment. In particular, **Gia Dvali, Cesar Gomez** predict observable corrections that are sensitive to (e&eb) the inflationary history way beyond

last 60 e-foldings. They derive an absolute upper bound on the number of e-foldings, beyond (e) which neither de Sitter nor inflationary Universe can be approximated by (e) a semi-classical metric. However, they could in principle persist in a new type of (e) it quantum eternity state. Authors discuss implications of this phenomenon for (e&eb) the cosmological constant problem. Subjects: High Energy Physics - Theory (hep-th); Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Quantum Physics (quant-ph) DOI: 10.1088/1475-7516/2014/01/023 Cite as: arXiv: 1312.4795 [hep-th] (or arXiv:1312.4795v1 [hep-th] for this version) **Quantum Compositeness of Gravity: Black Holes, AdS and Inflation Gia Dvali, Cesar Gomez STOP** As quantum information science approaches the goal of constructing quantum computers, understanding loss of information through (e&eb) decoherence becomes (=) increasingly important. The information about a system that can be obtained from (e) its environment can facilitate quantum control and error correction. Moreover, observers gain (eb) most of their information indirectly, by (e) monitoring (primarily photon) environments of (e) the "objects of interest." Exactly how this information is inscribed in the environment is essential for (e) the emergence of "the classical" from (e) the quantum substrate. In this paper, **Robin Blume-Kohout, Wojciech H. Zurek** examine how many-qubit (or many-spin) environments can store (e) information about a single system. The information lost to (e) the environment can be stored redundantly, or it can be encoded in (eb) entangled modes of the environment. Authors go on to show that randomly chosen states of the environment almost always encode (e) the information so that (e) an observer must capture (e) a majority of the environment to deduce (eb) the system's state. Conversely, in the states produced by (e) a typical decoherence process, information about a particular observable of the system is stored redundantly. This selective proliferation of "the fittest information" (known as Quantum Darwinism) plays a key role in (eb) choosing the preferred, effectively classical observables of (e) macroscopic systems. The developing appreciation that the environment functions not just as (=) a garbage dump, but as a communication channel, is extending understanding of the environment's role in

(e&eb) the quantum-classical transition beyond (e) the traditional paradigm of decoherence. Keywords Decoherence quantum Darwinism quantum information quantum-classical transition **Foundations of Physics November 2005, Volume 35, Issue 11, pp 1857-1876 A Simple Example of "Quantum Darwinism": Redundant Information Storage in Many-Spin Environments Robin Blume-Kohout, Wojciech H. Zurek STOP Harold Ollivier, David Poulin, and Wojciech H. Zurek** study the emergence of objective properties in open quantum systems. In our analysis, the environment is promoted from a passive role of a reservoir selectively destroying quantum coherence to an active role of amplifier selectively **proliferating information about the system. We show that only preferred pointer states of the system can leave a redundant and therefore easily detectable imprint on the environment. Observers who—as is almost always the case—discover the state of the system indirectly (by probing a fraction of its environment) will find out only about the corresponding pointer observable. Many observers can act in this fashion independently and without perturbing the system. They will agree about its state. In this operational sense, preferred pointer states exist objectively. Received 24 July 2003 DOI:http://dx.doi.org/10.1103/PhysRevLett.93.220401 Objective Properties from Subjective Quantum States: Environment as a Witness Harold Ollivier, David Poulin, and Wojciech H. Zurek Phys. Rev. Lett. 93, 220401 – Published 22 November 2004 STOP** Quantum Darwinism recognizes that we—the observers—acquire our information about the "systems of interest" indirectly from their imprints on the environment. **Michael Zwolak, H. T. Quan, and Wojciech H. Zurek** show that information about a system can be acquired from a mixed-state, or hazy, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor 1−h, where h is the haziness of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by ~

(1−h)−1 to gain the same information about the system. DOI:http://dx.doi.org/10.1103/PhysRevLett.103.110402 **Quantum Darwinism in a Mixed Environment Michael Zwolak, H. T. Quan, and Wojciech H. Zurek Phys. Rev. Lett. 103, 110402 – Published 8 September 2009** We study quantum Darwinism—the redundant recording of information about the preferred states of a decohering system by its environment—for an object illuminated by a blackbody. In the cases of point-source and isotropic illumination, we calculate the quantum mutual information between the object and its photon environment. We demonstrate that this realistic model exhibits fast and extensive proliferation of information about the object into the environment and results in redundancies orders of magnitude larger than the exactly soluble models considered to date. DOI:http://dx.doi.org/10.1103/PhysRevLett.105.020404 **Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons C. Jess Riedel and Wojciech H. Zurek Phys. Rev. Lett. 105, 020404 – Published 9 July 2010 STOP** Measurements transfer information about a system to the apparatus and then, further on, to observers and (often inadvertently) to the environment. I show that even imperfect copying essential in such situations restricts possible unperturbed outcomes to an orthogonal subset of all possible states of the system, thus breaking the unitary symmetry of its Hilbert space implied by the quantum superposition principle. Preferred outcome states emerge as a result. They provide a framework for "wave-packet collapse," designating terminal points of quantum jumps and defining the measured observable by specifying its eigenstates. In quantum Darwinism, they are the progenitors of multiple copies spread throughout the environment—the fittest quantum states that not only survive decoherence, but subvert the environment into carrying information about them—into becoming a witness. DOI:http://dx.doi.org/10.1103/PhysRevA.76.052110 **Quantum origin of quantum jumps: Breaking of unitary symmetry induced by information transfer in the transition from quantum to classical Wojciech Hubert Zurek Phys. Rev. A 76, 052110 – Published 16 November 2007 STOP** The Born rule, a foundational axiom used to deduce probabilities of events from **wavefunctions, is indispensable in the everyday practice of quantum**

**physics. It is also key in the quest to reconcile the ostensibly inconsistent laws of the quantum and classical realms, as it confers physical significance to reduced density matrices, the essential tools of decoherence theory. Following Bohr's Copenhagen interpretation, textbooks postulate the Born rule outright. However, recent attempts to derive it from (e) other quantum principles have been successful, holding promise for (e) simplifying and clarifying the quantum foundational bedrock. A major family of derivations is based on envariance, a recently discovered symmetry of (e) entangled quantum states. Here, authors identify and experimentally test three premises central to (e) these envariance-based derivations, thus demonstrating, (eb) in the microworld, the symmetries from (e) which the Born rule is derived. Further, they demonstrate envariance in a purely local quantum system, showing (eb) its independence from relativistic causality.Subjects: Quantum Physics (quant-ph)Cite as:arXiv:1604.01471 [quant-ph] (or arXiv:1604.01471v1 [quant-ph] for this version) Quantum probabilities from quantum entanglement: Experimentally unpacking the Born rule Jérémie Harris, Frédéric Bouchard, Enrico Santamato, Wojciech H. Zurek, Robert W. Boyd, Ebrahim Karimi**

Integrating $[m_4]_4(t, a_4)$ with respect to a from 0 to ∞, we have

$$\frac{dM_4(t)}{dt} = \int_0^\infty [mt]_4(t, a_4)da_4$$

$$= -\int_0^\infty [m_a]_4\,(t, a_4)da_4 - \int_0^\infty \mu(a_4)m(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f^{'}{}_{4*}\right]_4 \int_0^\infty p_4^*(a_4)da_4\right)M_4 - [f_{2*}^4]_4 \int_0^\infty p_4^*(a_4)da_4 W_4$$

$$= m_4(t, 0) - \int_0^\infty \mu_4(a_4)\, m_4(t, a_4)da_4$$

$$-\left([f_{4*}]_4 + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 - \left[f_{2*}^{'}\right]_4 P_4^* W_{\cdot 4}$$

From (30), we arrive at the equations

$$\frac{dW_4(t)}{dt} = \int_0^\infty ([h_*]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4$$

$$-\left([f_{4*}] + [f_{2*}]_4 + \left[f_{4*}^{'}\right]_4 P_4^*\right) M_4 + \left(\frac{[h_*^{'}]_4}{[h_*]^4} P_4^*(0) - f_{2*}^{'} P_4^*\right) W_4,$$

(34)

$$\frac{dW_4(t)}{dt}\left[g^{'}{}_*\right]_4 Q_4^* M_{\cdot 4}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_4 \beta_4(a_4) - \mu_4(a_4)) m_4(t, a_4) da_4 = ([h_*]_4 \beta_4(\theta_4) - \mu_4(\theta_4)) M_{,4}$$

Where $0 < \theta_4 = \theta_4(t) < \infty$.

And if $a_4^2 [a_4^2]_4 < 4 [a_2]_4 < 1\, a_2 e$,

$\mathrm{Re}\,\lambda_4 [\lambda_4]_4 = \mathrm{Re}\lambda_2 [\lambda_2]_4 = -\frac{4}{2} a_4 [a2\ ]_4 < 0.$

Let $\Delta = \max_{2<u<\infty} \left|h_* [h_*]_4 \beta_4(a_4) - \mu_4(a_4)\right|$. Then $|n_4(t)| \leq \Delta \forall t \geq 0$. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium$(\rho^*(a), \boldsymbol{Q}^*)$ is asymptotically stable if either $a_4^2 < 4\, a_2 e$ and $[a_4] > 2\Delta$,or $[a_4^2] \geq 4[a_2]e$ and $(a_4 - \sqrt{(a_4^2)_4 - 4(a_2)_4 e}) > 2\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+4}\right)$ and $\beta_4(a_4) = ae^{-ma}$ Then $\Delta=[\mu_0] + 1$ Simple calculation leads to

$$\int_0^{\infty} \beta_4(a_4) e^{-\int_0^a \mu_4(\tau_4) + d\tau_4 da} = \frac{4}{(m + \mu_0 + 1)^2} + \frac{2}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_4 + [\mu_0]$ 0.5. Assume$[f_4](x_4)= K_4 x_4$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_4(x_4)= [0x_4]/(1 + x_4)$; the functional responses $[[f_2](x) = [0]\, x_4/(1 + x_4)$and $h_4(x_4)=(1+\omega_4 x_4)/(1 + x_4)$(See Holling [7] and Rashevsky [13])

Equation (44),now has the form of

$$\frac{m_4 + [\mu_0] + [K][P_0])}{[m_4]_4 + (\mu_0)_4 + [K][P_0])\,{}_4^3\,[K]_4[P_0]_4^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(4 + P_0) < b_4$i e $[P_0] < b/(\delta_4 - b_4)$then the equilibrium$(P_0)$ $[P_0]_4(a), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_4$

$[f_{4*}] = K_4\, P^*\, [f_{2*}] = \frac{\sigma Q_4^*}{4+Q_4^*}$ $[h_*] = \frac{4+\omega_4 Q_4^*}{4+Q_4^*}$

$[f_{4*}] = K_4\, [f_{2*}^4] = \frac{\sigma_4}{(4+Q_4^*)^2} [\, g_*^4] = \frac{\sigma_4}{(4+P_4^*)^2}$ $[h_*^4] = \frac{\omega_4 - 4}{(4+Q_4^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_4}{(m+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma)^2}+\frac{2\omega_4}{([m_4]_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4+b_4}+\sigma_4)^3}<1$$

And

$$\frac{1}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^2}+\frac{2\omega_4}{(m_4+[\mu_0]_4+1+\frac{K_4b_4}{\delta_4-b_4})^3}>1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_4+[\mu_0]_4+1+\tfrac{K_4b_4}{\delta_4-b_4}+(m_4+[\mu_0]_4+\tfrac{K_4b_4}{\delta_4-b_4}+\sigma_4)\,Q_4^*)^3$$

$$=m_4+[\mu_0]_4+3+\tfrac{K_4b_4}{\delta_4-b_4}(4+Q^*)(1+\omega_4Q_4^*$$

Now

$$\left[a_4^2\right]-1\left[a_2\right]e=\left(\frac{2K_4b_4}{\delta_4-b_4}\right)^2+\frac{1K_4b_4\sigma_4\,Q_4^*}{(\delta_4-b_4)(1+Q_4^*)}+\frac{\sigma_4^2Q_4^{*2}}{(1+Q_4^*)^2}$$

$$\frac{1\,b_4\sigma_4\,(\delta_4-b)Q^*}{\delta_4(1+Q_4^*)^2}-\frac{1\,(\delta_4-b_4)^2(1-\omega_4)Q_4^*}{\delta_4(1+\omega_4Q_4^*)(1+Q_4^*)}P^*(0)$$

And

$\Delta = \lim_{0<a<\infty} |[h_*]_4 ae^{mu}_4 \ [\mu_0]_4 - \frac{a_4}{a_4+4}$

Hence if $[a^2_4 > 4(a_2)_4 e$ and

$$[a_4] = \frac{2K_4 b_4}{\delta_4 - b_4} + \frac{\sigma_4 Q^*_4}{1 + Q^*_4} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_4]_4 = [f_{4*}]_4 + [f_{2*}]_4 + [f_{4*}] P^*_4$$

$$[a_2]_4 = \left[f^4_{2*}\right] P^*_4 - \frac{h^4_*}{h_*} P^*(0)$$

$$e = [g^4_*]_4 Q^*_4$$

$$\eta_4(t) = \left[h_*\right]_4 \beta_4 \left(\theta_4(t)\right) - \mu_4(\theta(t)),$$

The system(34)can be written as the following system:

$$[M(t)]\left[a_4\right] M_4 - [a_2]_4 W_4 - [n(t)] M_4$$

$$w_4(t) = eM_4$$

Write

$$A = \begin{pmatrix} -[a_4]_4 - [a_2]_4 \\ e \qquad\quad 0 \end{pmatrix}$$

| |
|---|
| $B=(t)=\begin{pmatrix} [\eta(t)]_4 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_4 \\ W_4 \end{pmatrix}$ |
| The solution of (32)is |
| $X_4(t) = e^{A,t}[X]_4(0) + \int_0 e^{At}$ |
| **Some of the other conceptual and thematic and discursive factorial representationalities to which the model is applicable are:every one-dimensional quantum mechanical Hamiltonian H1 can have (e) a partner H2 such that H1 and H2 taken together may be viewed as (=) the components of a supersymmetric Hamiltonian. The term 'supersymmetric Hamiltonian' is taken to mean (eb) a Hamiltonian defined in terms of (e&eb)charges that obey (eb)the same algebra as that of (e) the generators of supersymmetry in field theory. The consequences of this symmetry for (er) the spectra of H1 and H2 are explored. It is shown how the supersymmetric pairing may be utilised to eliminate (e) the ground state of H1, or add a state below the ground state of H1 or maintain the spectrum of H1. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate (eb) a class of anharmonic potentials with (e&eb) exactly specified spectra. The complete spectrum of an anharmonic potential so generated consists of (e) all the eigenstates of the simple harmonic oscillator and, in addition, a ground state at a specified energy E which lies arbitrarily below (eb) the E=1/2 ground state of the harmonic oscillator. Supersymmetric quantum mechanics of one-dimensional systems C V Sukumar Journal of Physics A: Mathematical and General, Volume 18, Number 15 STOP Authors examine the properties and consequences of (e&eb) pseudo-supersymmetry for (e) quantum** |

**systems admitting (e) a pseudo-Hermitian Hamiltonian. Authors explore the Witten index of (e) pseudo-supersymmetry and show that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with (e&eb) discrete spectra and (e&eb) real or complex-conjugate pairs of (e) eigenvalues are (=) isospectral and have (e) identical degeneracy structure except (e) perhaps for the zero eigenvalue if and only if (e) they are pseudo-supersymmetric partners. This implies (eb) that pseudo-supersymmetry is (=) the basic framework for (e) generating non-Hermitian PT-symmetric and (e&eb) non-PT-symmetric Hamiltonians with (e&eb) a real spectrum via (e&eb) a Darboux transformation, and shows (eb) that every diagonalizable Hamiltonian H with(e&eb) a discrete spectrum and (e&eb) real or complex-conjugate pairs of (e) eigenvalues may be factored as (=) H=L♯L where L is (=) a linear operator with pseudo-adjoint L♯. In particular, this factorization applies to (e&eb) PT-symmetric and Hermitian Hamiltonians. The non-degenerate two-level systems provide (eb) a class of Hamiltonians that are pseudo-Hermitian. Authors demonstrate the implications of general results for (e&eb) this class in some detail. Copyright © 2002 Elsevier Science B.V. All rights reserved. Nuclear Physics B Volume 640, Issue 3, 30 September 2002, Pages 419–434Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians Ali Mostafazadeh STOP Michele Caselle, Alessandro Nada, Marco Panero present a high-precision lattice calculation of (e) the equation of state in the confining phase of (e) SU(2) Yang-Mills theory. Authors show that the results are described very well by (e) a gas of massive, non-interacting glueballs, provided (e) one assumes an exponentially growing Hagedorn spectrum. The latter can be derived within (eb) an effective bosonic closed-string model, leading to (eb) a parameter-free theoretical prediction, which is in perfect agreement with (=) proposed lattice results. Furthermore, when applied to (e&eb) SU (3) Yang-Mills theory, this effective model accurately describes (eb) the lattice results reported by Bors\'anyi et al. in JHEP 07 (2012) 056. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th)**

**Journal reference: JHEP 07 (2015) 143 DOI: 10.1007/JHEP07 (2015)143 Cite as: arXiv: 1505.01106 [hep-lat] (or arXiv: 1505.01106v2 [hep-lat] for this version) Hagedorn spectrum and thermodynamics of SU (2) and SU (3) Yang-Mills theories Michele Caselle, Alessandro Nada, Marco Panero STOP B. de Wit J. Hoppe, H. Nicolai study the quantum-mechanical properties of (e) a supermembrane and examine the nature of (e) its ground state. A supersymmetric gauge theory of (e) area-preserving transformations provides (eb) a convenient framework for this study. The supermembrane can be viewed as (=) a limiting case of a class of models in supersymmetric quantum mechanics. Its mass does not depend on (=) the zero modes and vanishes only if (e) the wave function is (=) a singlet under (e&eb) supersymmetry transformations of (e) the nonzero modes. Authors exhibit (eb) the complexity of (e) the supermembrane ground state and (e&eb) examine various truncations of these models. None of these truncations has massless states. Copyright © 1988 Published by Elsevier B.V. Nuclear Physics B Volume 305, Issue 4, 28 December 1988: Pages 545-581: On the quantum mechanics of supermembranes Author links open the overlay panel. Numbers correspond to the affiliation list which can be exposed by using the show more links. B. de Wit J. Hoppe, H. Nicolai STOP In this note we explain how world-volume geometries of (e) D-branes can be reconstructed within (eb) the microscopic framework where (e) D-branes are described through (e&eb) boundary conformal field theory. Authors extract the (non-commutative) world-volume algebras from (e) the operator product expansions of (e) open string vertex operators. For branes in a flat background with (e&eb) constant non-vanishing B-field, the operator products are computed perturbatively to (e) all orders in the field strength. Resulting series coincides with (=) Kontsevich's presentation of the Moyal product. After extending these considerations to (e) fermionic fields authors conclude with some remarks on the generalization of approach to (e&eb) curved backgrounds. D-branes and deformation quantization Volker Schomerus1,2 Published 25 July 1999 Journal of High Energy Physics, Volume 1999, JHEP06(1999)STOP**

**Authors employ the path integral approach developed in Gangopadhyay S. and Scholtz F. E., Phys. Rev. Lett., 102 (2009) 241602 to discuss the (generalized) harmonic oscillator in (eb) a noncommutative plane. The action for (e) this system is derived in (e&eb) the coherent state basis with (e&eb) additional degrees of freedom. From this the action in the coherent state basis without (e) any additional degrees of freedom is obtained. This gives (eb) the ground-state spectrum of the system. Authors then employ the exact renormalization group approach to show (eb) that equivalence can be constructed between this (noncommutative) system and (e&eb) a commutative system Path integral action and exact renormalization group dualities for quantum systems in noncommutative plane Sunandan Gangopadhyay Published 25 June 2015 • Copyright © EPLA, 2015 STOP Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini present a lattice study of (e)the equation of state in Yang-Mills theory based on (e) the exceptional G(2) gauge group. As is well-known, at zero temperature this theory shares (e&eb) many qualitative features with real-world QCD, including (e) the absence of colored states in the spectrum and dynamical string breaking at (e) large distances. In agreement with previous works, authors show that at finite temperature this theory features (eb) a first-order deconfining phase transition, whose nature can be studied by (e) a semi-classical computation. Authors also show that the equilibrium thermodynamic observables in the deconfined phase bear striking quantitative similarities with (=) those found in SU (N) gauge theories: in particular, these quantities exhibit (eb) nearly perfect proportionality to the number of gluon degrees of freedom, and (e&eb) the trace anomaly reveals (eb) a characteristic quadratic dependence on (e&eb) the temperature, also observed in SU (N) Yang-Mills theories (both in four and in three spacetime dimensions). Authors compare our lattice data with (e&eb) analytical predictions from (e) effective models, and discuss their implications for (e) the deconfinement mechanism and (e&eb) high-temperature properties of (e) strongly interacting, non-supersymmetric gauge theories. Results give (eb) strong evidence for (e) the conjecture that the thermal**

**deconfining transition is governed by (e&eb) a universal mechanism, common to (e) all simple gauge groups. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th) Journal reference: JHEP 03 (2015) 057 DOI: 10.1007/JHEP03 (2015)057 Report number: DESY-14-146; IFT-UAM/CSIC-14-076 Cite as: arXiv: 1409.8305 [hep-lat] (or arXiv: 1409.8305v2 [hep-lat] for this version Exceptional thermodynamics: The equation of state of G(2) gauge theory Mattia Bruno, Michele Caselle, Marco Panero, Roberto Pellegrini STOP In U (1) lattice gauge theory in three spacetime dimensions, the problem of confinement can be studied analytically in (e&eb) a semi-classical approach, in terms of (e&eb) a gas of monopoles with (e&eb) Coulomb-like interactions. In addition, this theory can be mapped to (e&eb) a spin model via (e&eb) an exact duality transformation, which allows (eb) one to perform high-precision numerical studies of (e) the confining potential. Taking advantage of these properties, authors carried out an accurate investigation of (e) the effective string describing (eb) the low-energy properties of flux tubes in (eb) this confining gauge theory. Authors found striking deviations from (e) the expected Nambu-Goto-like behavior, and, for the first time, evidence for (e) contributions that can be described by a term proportional to the extrinsic curvature of the effective string worldsheet. Such term is allowed by (e) Lorentz invariance and its presence in the infrared regime of the U (1) model was indeed predicted by (e) Polyakov several years ago. Results show that this term scales as expected according to (e) Polyakov's solution, and becomes (=) the dominant contribution to (e) the effective string action in the continuum limit. Authors also demonstrate analytically that the corrections to (e) the confining potential induced by (e) the extrinsic curvature term can be related to (e&eb) the partition function of (e) the massive perturbation of (e) a c=1 bosonic conformal field theory. The implications of results for SU (N) Yang-Mills theories in (e&eb) three and in four spacetime dimensions are discussed. Subjects: High Energy Physics - Lattice (hep-lat); High Energy Physics - Theory (hep-th) Journal reference: JHEP 1501 (2015) 105**

**DOI: 10.1007/JHEP01 (2015)105 Report number: IFT-UAM/CSIC-14-052 Cite as: arXiv: 1406.5127 [hep-lat] (or arXiv: 1406.5127v2 [hep-lat] for this version) A different kind of string Michele Caselle, Marco Panero, Roberto Pellegrini, Davide Vadacchino STOP In the past ten years, the ideas of supersymmetry have been profitably applied to (e&eb) many nonrelativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and (e&eb) an array of powerful new approximation methods for (e) handling potentials which are not (e) exactly solvable. In this report, authors review the theoretical formulation of (e) supersymmetric quantum mechanics and discuss many applications. Exactly solvable potentials can be understood in terms of (e&eb) a few basic ideas which include (e) supersymmetric partner potentials (e&eb) shape invariance and (e&eb) operator transformations. Familiar solvable potentials all have (e) the property of shape invariance. Authors describe new exactly solvable shape invariant potentials which include (e) the recently discovered self-similar potentials as (=) a special case. The connection between inverse scattering,(e&eb) isospectral potentials and (e&eb) supersymmetric quantum mechanics is discussed and multisoliton solutions of (e) the K dV equation are constructed. Approximation methods are also discussed within (eb) the framework of supersymmetric quantum mechanics and in particular it is shown that a supersymmetry inspired WKB approximation is (=) exact for a class of shape invariant potentials. Supersymmetry ideas give (eb) particularly nice results for the tunneling rate in (eb) a double well potential and for improving (eb+) large N expansions. Authors also discuss the problem of (e) a charged Dirac particle in (eb) an external magnetic field and other potentials in terms of (e&eb) supersymmetric quantum mechanics. Finally, authors discuss structures more general than supersymmetric quantum mechanics such as(=) parasupersymmetric quantum mechanics in which there is symmetry between a boson and (e&eb) a para-fermion of order p. Copyright © 1995 Published by Elsevier B.V. Physics Reports Volume 251, Issues 5–6, January 1995, Pages 267–385Supersymmetry and quantum**

**mechanics Fred Coopera, Avinash Khareb, Uday Sukhatmec STOP The free energy of (e) the maximally supersymmetric SU (N) gauge theory at temperature T is expected to scale, in the large-N limit, as (=) N2T4 times a function of the 't Hooft coupling, f (gYM2N). In the strong coupling limit the free energy has been deduced from (e) the near-extremal 3-brane geometry, and its normalization has turned out to be Full-size image (<1 K) times that found in (eb) the weak coupling limit. In this paper authors calculate the leading correction to (e) this result in inverse powers of (e) the coupling, which originates from (e) the R4 terms in the tree-level effective action of (e) type IIB string theory. The correction to Full-size image (<1 K) is positive and of (e) order Full-size image (<1 K). Thus, f (gYM2N) increases as (e&eb) the 't Hooft coupling is decreased, in accordance with (e&eb) the expectation that is should be approaching 1 in (eb) the weak coupling limit. Authors also discuss similar corrections for (e) other conformal theories describing (eb) coincident branes. In particular, authors suggest (eb) that the coupling independence of (e) the near extremal entropy for (e) D1-branes bound to (e) D5-branes is related to (e&eb) the vanishing of (e) the Weyl tensor of AdS3 × S3. Copyright © 1998 Published by Elsevier B.V. Nuclear Physics B Volume 534, Issues 1–2, 23 November 1998, Pages 202–222Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory Steven S. Gubsera, Igor R. Klebanovb, 1, Arkady A. Tseytlinc, 2 STOP A new approach to (e) N=2 supersymmetry based on (e) the concept of harmonic superspace is proposed and is used to give (eb) an unconstrained superfield geometric description of N=2 super Yang-Mills and supergravity theories as (=) well as of matter N=2 hypermultiplets. The harmonic N=2 superspace has (e) an independent coordinates, in addition to the usual ones, the isospinor harmonics ui+or- on (e&eb) the sphere SU (2)/U (1). The role of ui+or- is to relate the SU (2) group realised on the component fields to (e&eb) a U (1) group acting on (e&eb) the relevant superfields. Their introduction makes it possible to SU (2)-covariantise (eb (t), e&eb) the notion of Grassmann analyticity. Crucial for the construction is the existence of (e) an analytic subspace of the**

**general harmonic N=2 superspace. The hypermultiplet superfields and (e&eb) the true prepotentials (pre-prepotentials) of N=2 super Yang-Mills and supergravity are (=) unconstrained superfunctions over (e&eb) this analytic subspace. The pre-prepotentials have (e) a clear geometric interpretation as gauge connections with (e&eb) respect to the internal SU (2)/U (1) directions. A radically new feature arises: the number of gauge and (e&eb) auxiliary degrees of freedom becomes (=) infinite while (e) the number of physical degrees of freedom remains (=) finite. Other new results are the massive N=2 Yang-Mills theory and (e&eb) various off-shell self-interactions of hypermultiplets. The propagators for matter and Yang-Mills superfields are given. Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace A Galperin, E Ivanov, S Kalitzin, V Ogievetsky and E Sokatchev STOP By assuming the existence of a novel multipronged string state for (e) D-particles interacting with (e&eb) D-brane intersections in type IIA string theory, authors are able to derive a quantum mechanical description of (e) supersymmetric Reissner-Nordstrom black holes. A supersymmetric index calculation provides (eb) evidence for (e) this conjecture. The quantum mechanical system becomes (=) two decoupled conformal quantum mechanical systems in (eb) the low energy limit. The conformal quantum mechanics has expected properties of (e) a dual description of string theory on (e&eb) AdS2×S2. Subjects: High Energy Physics - Theory (hep-th) Journal reference: JHEP 0001:006, 2000 DOI: 10.1088/1126-6708/2000/01/006 Report number: LBNL-43734 Cite as: arXiv:hep-th/9907101 (or arXiv:hep-th/9907101v5 for this version) Supersymmetric Quantum Mechanical Description of Four Dimensional Black Holes Julie D. BlumSTOP Using recently proposed soliton equations authors derive a basic identity for the scaling violation of (e) N=2 supersymmetric gauge theories Σiai∂F/∂ai−2F=8πib1u. Here F is (=) the prepotential, ai's are (=) the expectation values of (e) the scalar fields in (eb) the vector multiplet, u=1/2 Tr <ϕ2> and b1 is the coefficient of (e) the one-loop β-function. This equation holds in (eb) the Coulomb branch of all N=2 supersymmetric gauge theories coupled**

**with (e&eb) massless matter. TOHRU EGUCHI and SUNG-KIL YANG, Mod Phys. Lett. A 11, 131 (1996). DOI: 10.1142/S0217732396000151 PREPOTENTIALS OF N=2 SUPERSYMMETRIC GAUGE THEORIES AND SOLITON EQUATIONS STOP The Coulomb branch of N = 2 supersymmetric gauge theories in four dimensions is described in general by (e) an integrable Hamiltonian system in (eb) the holomorphic sense. A natural construction of such systems comes from (e) two-dimensional gauge theory and (e&eb) spectral curves. Starting from this point of view, authors propose an integrable system relevant to (e&eb) the N = 2 SU (n) gauge theory with (e&eb) a hypermultiplet in the adjoint representation, and offer (eb) much evidence that it is correct. The model has (e) a Full-size image (<1 K) S-duality group (with the central element −1 of Full-size image (<1 K) acting as (=) charge conjugation); Full-size image (<1 K) permutes the Higgs, confining, and oblique confining phases in the expected fashion. We also study more exotic phases. Copyright © 1996 Published by Elsevier B.V. Nuclear Physics B Volume 460, Issue 2, 5 February 1996, Pages 299–334Supersymmetric Yang-Mills theory and integrable systems Ron Donagia, 1, Edward Wittenb, 2 STOP A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 present a first step towards generalizing (eb) the work of Seiberg and Witten on N = 2 supersymmetric Yang-Mills theory to (e&eb) arbitrary gauge groups. Specifically, authors propose a particular sequence of (e) hyperelliptic genus n − 1 Riemann surfaces to underly (e&eb) the quantum moduli space of SU (n)N = 2 supersymmetric gauge theory. These curves have (e) an obvious generalization to (e) arbitrary simply laced gauge groups, which involves (e&eb) the A-D-E type simple singularities. To support proposal, authors argue that the monodromy in (e&eb) the semiclassical regime is (=) correctly reproduced. We also give some remarks on a possible relation to string theory. Copyright © 1995 Published by Elsevier B.V. Physics Letters B Volume 344, Issues 1–4, 26 January 1995, Pages 169–175Simple singularities and N = 2 supersymmetric Yang-Mills theory A. Klemma, W. Lerchea, S. Yankielowicza, 1, S. Theisenb, 2 STOP Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and**

**Toby Wiseman2 review and extend earlier work that uses (e) the AdS/CFT correspondence to relate the black-hole–black-string transition of gravitational theories on a circle to (e&eb) a phase transition in maximally supersymmetric (1 + 1)-dimensional SU(N) gauge theories at (eb) large N, again compactified on (e&eb) a circle. Authors perform gravity calculations to determine (eb) a likely phase diagram for (e) the strongly coupled gauge theory. Authors then directly study the phase structure of (e) the same gauge theory, now at (eb) weak't Hooft coupling. In the interesting temperature regime for (e) the phase transition, the (1 + 1)-dimensional theory reduces to (e&eb) a (0 + 1)-dimensional bosonic theory, which they solve using (e) Monte Carlo methods. Authors find strong evidence that the weakly coupled gauge theory also exhibits (eb) a black hole–black string-like phase transition in (eb) the large N limit. Authors demonstrate that a simple Landau–Ginzburg-like model describes (eb) the behaviour near the phase transition remarkably well. The weak coupling transition appears to be (=) close to the cusp between a first-order and a second-order transition. Black-hole–black-string phase transitions in thermal (1 + 1)-dimensional supersymmetric Yang–Mills theory on a circle Ofer Aharony1, Joseph Marsano2, Shiraz Minwalla2 and Toby Wiseman2 Published 29 October 2004 • 2004 IOP Publishing Ltd Classical and Quantum Gravity, Volume 21, Number 22STOP Monitoring the orbits of stars around (e&eb) Sgr A* offers(eb)the possibility of detecting (e&eb) the precession of (e) their orbital planes due to (e) frame dragging, of (e) measuring the spin and quadrupole moment of (e) the black hole, and of (e)testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and (e&eb) tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are (=), in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for which an optical interferometer, such as GRAVITY, will detect orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to (e&eb) the timescale of precession**

**due to (e) the quadrupole moment of (e) the black hole. As a result, this non-conservative effect is (=) a potential source of systematic uncertainty in (eb) testing the no-hair theorem with (e&eb) stellar orbits. Subjects: High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI:10.1088/0004-637X/777/1/57 Cite as:arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP Authors perform a phase space analysis of (e) a generalized modified gravity theory with (e&eb) non-minimal coupling between geometry and (e&eb) matter. Authors apply the dynamical system approach to (e&eb) this generalized model and find that (eb) in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, Authors prove that there is (=) an attractor solution to describe (eb) the late time accelerating universe when (e) the modified gravity is chosen in (eb) a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and supernova type Ia data sets, Authors find that (eb) the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as:arXiv: 1512.09281 [gr-qc] (or arXiv: 1512.09281v1 [gr-qc] for this version) Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang study the P−V criticality and phase transition in (eb) the extended phase space of (e) charged Gauss-Bonnet black holes in (eb) anti-de Sitter space, where (e) the cosmological constant appears as (=) a dynamical pressure of the**

**system and (e&eb) its conjugate quantity is (=) the thermodynamic volume of (e) the black hole. The black holes can have (e) a Ricci flat (k=0), spherical (k=1), or hyperbolic (k=−1) horizon. Authors find that for the Ricci flat and hyperbolic Gauss-Bonnet black holes, (eb)no P−V criticality and phase transition appear, while for the black holes with a spherical horizon, even when (e) the charge of the black hole is absent, the P−V criticality and the small black hole/large black hole phase transition (eb) will appear, but it happens only in (eb) d=5 dimensions; when (e) the charge does not vanish, the P−V criticality and (e&eb) the small black hole/large phase transition always appear in (eb) d=5 dimensions; in the case of d≥6, to (e) have the P−V criticality and the small black hole/large black hole phase transition, there exists (eb) an upper bound for (e) the parameter b=α˜|Q|−2/(d−3), where (e) α~ is the Gauss-Bonnet coefficient and Q is the charge of the black hole. Authors calculate the critical exponents at the critical point and find that for all cases, they are (=) the same as those in the van der Waals liquid-gas system. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/JHEP09 (2013)005 Cite as: arXiv: 1306.6233 [gr-qc] (or arXiv: 1306.6233v4 [gr-qc] for this version) P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space Rong-Gen Cai, Li-Ming Cao, Li Li, Run-Qiu Yang STOP paper studies the whole process of (e) gravitational collapse and (e&eb) accretion of (e) a massless scalar field in (eb) asymptotically flat spacetime. Two kinds of initial configurations are considered. One is the initial data without (e) black hole; the other contains (e) a black hole. Under suitable initial conditions, Authors find that multi-horizon will appear which mean (eb) that the initial black hole formed by (e) gravitational collapse or existing at the beginning will instantly expand and suddenly grow (e&eb, eb), rather than grows gradually in the accretion process. A new type of critical behavior is found around (e&eb) the instant expansion. The numerical computation shows (eb) that the critical exponents are (=) universal. Subjects: General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Computational Physics (physics.**

**Comp-ph) Cite as: arXiv: 1604.03363 [gr-qc] (or arXiv: 1604.03363v1 [gr-qc] for this version) Multi-horizon and Critical Behavior in Gravitational Collapse of Massless Scalar Zhoujian Cao, Rong-Gen CAI, Run-Qiu Yang STOP In this paper, Authors present the results on the gravitational collapse of (e) charged massless scalar field in asymptotically flat spacetime with(e&eb) a perfectly reflecting wall. Differing from previous works, Authors study the system in (eb) the double null coordinates, by which we could simulate (e&eb) the system until the black hole forms with (e&eb) higher precision but less performance time. Authors investigate the influence of charge on (e&eb) the black hole formation and the scaling behavior near the critical collapses. The gapless and gapped critical behaviors for (e) black hole mass and charge are studied numerically. Authors find that they satisfy (eb) the scaling laws for critical gravitational collapse but the gapped critical behavior is different from (e) its AdS counterpart. Subjects: General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv: 1602.00112 [gr-qc] (or arXiv: 1602.00112v1 [gr-qc] for this version) Multiple critical gravitational collapse of charged scalar with reflecting wall Rong-Gen CAI, Run-Qiu Yang STOP In this paper Authors investigate the (2+1)-dimensional topological non-Fermi liquid in (eb) strongly correlated electron system, which has (e) a holographic dual description by (e) Einstein gravity in (3+1)-dimensional anti-de Sitter (AdS) space-time. In a dyonic Reissner-Nordstrom black hole background, Authors consider a Dirac fermion coupled to (e&eb) the background U (1) gauge theory and an (e&eb) intrinsic chiral gauge field bM induced by (e) chiral anomaly. UV retarded Green's function of (e) the charged fermion in (eb) the UV boundary from (e) AdS4 gravity is calculated, by imposing (e&eb) in-falling wave condition at the horizon. Authors also obtain IR correlation function of (e) the charged fermion at (eb) the IR boundary arising from (e) the near horizon geometry of (e) the topological black hole with (e&eb) index k=0,±1. By using (e) the UV retarded Green's function and IR correlation function Authors analyze the low frequency behavior of (e) the topological non-Fermi liquid at zero and (e&eb) finite temperatures,**

**especially the relevant non-Fermi liquid behavior near (eb) the quantum critical region. In addition, Authors find that when k=±1, the effective mass of (e) bulk Dirac fermion at IR fixed point is (=) topologically quantized, consequently the Fermi momentum presents (eb) a discrete sparse distributed pattern, in contrary to (e&eb) the Ricci flat case with k=0. As demonstrated examples, Authors calculate the spectral functions of (e) topological non-Fermi liquid due to (e) anomalous Hall effect (AHE) (|b→|≠0,b 0=0) and chiral magnetic effect (CME) (|b→|=0,b0≠0), which can be tested in topological materials such as (=) topological insulator and (e&eb) topological metal with (e&eb) strong couplings. Subjects: High Energy Physics - Theory (hep-th); Strongly Correlated Electrons (cond-mat.str-el); General Relativity and Quantum Cosmology (gr-qc) Cite as:arXiv: 1601.03865 [hep-th] (or arXiv: 1601.03865v1 [hep-th] for this version) Topological Non-Fermi Liquid Rong-Gen Cai, Yong-Hui Qi, Yue-Liang Wu, Yun-Long Zhang STOP The unusual magnetic materials are significant in both science and technology. However, because of (e) the strongly correlated effects, it is difficult to understand their novel properties from (e) theoretical aspects. Holographic duality offers (eb) a new approach to understanding such systems from (e) gravity side. This paper will give a brief review of Authors recent works on the applications of holographic duality in understanding (e&eb) unusual magnetic materials. Some quantitative compare between holographic results and (e&eb) experimental data will be shown and some predictions from (e) holographic duality models will be discussed. Comments: For Second LeCosPA Symposium at Taiwan National University, Taipai, Dec. 14-18, 2015 Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc) Cite as: arXiv:1601.02936 [hep-th] (or arXiv:1601.02936v1 [hep-th] for this version) Understanding strongly coupling magnetism from holographic duality Rong-Gen Cai, Run-Qiu Yang STOP It was shown recently that, without jeopardizing the success of the Λ cold dark matter model on cosmic scales, the modified Newtonian dynamics (MOND) can be derived as (=) an emergent phenomenon when (e) axionlike dark**

**matter particles condense into (e&eb) superfluid on the galactic scales. Authors propose in this paper a Dirac-Born-Infeld (DBI) scalar field conformally coupled to (e&eb) the matter components. To maintain the success of MOND phenomenon of (e) dark matter superfluid on (eb) the galactic scales, the fifth force introduced by (e) the DBI scalar should be screened on (e&eb) the galactic scales. It turns out that the screening effect naturally leads to (eb) a simple explanation for (e) a longstanding puzzle that the MOND critical acceleration coincides with (=) present Hubble scale. This galactic coincidence problem is solved, provided that (e) the screened DBI scalar also plays the role of (e&eb) dark energy on the cosmic scales. Subjects:General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 93, 023515 (2016) DOI: 10.1103/PhysRevD.93.023515 Cite as:arXiv:1511.00627 [gr-qc] (or arXiv:1511.00627v3 [gr-qc] for this version) Dark matter superfluid and DBI dark energy Rong-Gen Cai, Shao-Jiang Wang STOP The tensor-vector-scalar (TeVeS) model is (=) considered a viable theory of (e) gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here Authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (e&eb) the TeVeS model. Furthermore, Authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to Planck data, and find that (eb) it is ruled out by cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference:Phys. Rev. D 92, 083505 (2015) DOI: 10.1103/PhysRevD.92.083505 Cite as: arXiv: 1412.4073 [astro-ph.CO] (or arXiv: 1412.4073v3**

**[astro-ph.CO] for this version) Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang**

# Prescientiał Dynamics And Contential Staticity Of Accentuation Attrition Models

## Module Number Six

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-465) 1990. We demonstrate its applicability cosmology/ quantum computation economics etal. For the first time, physicists couple distant nuclear spins using a single electron July 11, 2016. Some sentences have been deleted due to spatial restrictions. Please excuse me. For the first time, researchers at the University of Basel in Switzerland have coupled the nuclear spins of distant atoms using (e) just a single electron. Three research groups took part in this complex experiment, the results of which have now been published in the journal Nature Nanotechnology. In most materials, the nuclear spins of (e) neighboring atoms have (e) only a very weak effect on one another, as (=) the tiny nuclei are located deep within (e&eb) the atoms. This is not the case in metals, however, where (e) some of the electrons are able to move freely. The electron spins are able to couple nuclear spins at (eb) relatively large distances from one another. Named after four physicists, this RKKY interaction was discovered in the 1950s. Nuclear spins linked by (e&eb) an individual electron spin Now, an experiment by researchers from the Department of Physics at the University of Basel has for the first time succeeded in demonstrating this mechanism with (e&eb) a single electron, describing it using (e) quantum theory. The team led by Prof. Richard Warburton trapped a single electron inside (eb) a quantum dot. With use of a method developed in Basel to measure (eb) nuclear spin resonance, they showed that the electron coupled (e&eb) the spins of nuclei up to five nanometers apart - a huge distance in the world of (e) nuclear spins. The results are particularly relevant to (e&eb) the development of spin qubits: these seek to use (e) electron spins to carry (e&eb) information, but the interaction with (e&eb) the nuclei limits (e) the stability of the quantum information. Concentrated blast of physics More information: Role of the electron spin in determining the coherence of the nuclear spins in a quantum dot, Nature Nanotechnology, DOI: 10.1038/nnano.2016.114 Journal reference: Nature Nanotechnology search and more info website Provided by: University of Basel Read more at: http://phys.org/news/2016-07-physicists-couple-distant-nuclear-electron.html#jCp A new spin on quantum computing: Scientists train electrons with microwaves February 15, 2016 A silicon sample doped with (e&eb) bismuth atoms (left image) that is (=) just 150 nanometers thick is fitted with (e&eb) a superconducting resonator that includes (e) a capacitor (black, in left image; light gray in center image) and an inductive wire (red line … In what may provide (eb) a potential path to processing information in (eb) a quantum computer, researchers have switched an intrinsic property of electrons from an excited state to (e&eb) a relaxed state on demand using (e) a device that served as (=) a microwave "tuning fork." The team's findings could also lead to (eb) enhancements in magnetic resonance techniques, which are (=) widely used to explore (e) the structure of materials and biomolecules, and for medical imaging. The international research team, which included scientists at the Department of Energy's Lawrence Berkeley**

**National Laboratory (Berkeley Lab), demonstrated how to dramatically increase the coupling of (e) microwaves in a specially designed (e&eb) superconducting cavity to (e) a fundamental electron property called spin—which, like a coin, can be flipped. By zapping an exotic silicon material developed at Berkeley Lab with the microwaves, they found that they could rapidly change (e&eb) the electron spins from (e) an excited state to a relaxed, ground state by (e) causing (eb) the electrons to emit (eb) some of their energy in the form of (e) microwave particles known as photons. Left on their own, the electron spins would be extremely unlikely to flip back to (e) a relaxed state and to also emit (eb) a photon - the natural rate for this light-emitting effect, known as the Purcell effect, is about once every 10,000 years. The experiment demonstrated an accelerated, controllable relaxation of (e) electron spins and the release of (e) a microwave photon in about 1 second, said Thomas Schenkel, a physicist in Berkeley Lab's Accelerator Technology and Applied Physics Division who led the design and development of the silicon-bismuth sample used in the experiment. A new spin on quantum computing: Scientists train electrons with microwaves Like the classical tuning fork slung from guitar strings in this photo, scientists used (e) a device that worked like a microwave tuning fork to switch (e&eb) a fundamental property of electrons on (e&eb) demand in a silicon-bismuth sample developed at Berkeley Lab. Credit: Flickr/Pierre Guinoiseau "It's like a juggler who throws the balls up, and the balls come down 1,000 times faster than normal, and they also emit (eb) a microwave flash as they drop," he said. The results were published online Feb. 15 in the journal Nature. "Our results are highly significant for (e&eb) quantum information processing," said Patrice Bertet, a quantum electronics scientist at the French Atomic Energy Commission (CEA) who led the experiment. "Indeed, they are a first step toward (e&eb) the strong coupling of (e) individual electron spins to microwave photons, which could form (eb) the basis of new spin-based quantum computer architecture." John Morton, a professor at the London Center for Nanotechnology and co-author of the study, said, "Our ultimate aim is to find a link between quantum information that is fixed and (e&eb) quantum information that can be transported by photons." In today's computers, information is stored as (=) individual bits, and each bit can either be (=) a one or a zero. Quantum computers, though, could conceivably be (=) exponentially more powerful than modern computers because (e) they would use a different kind of bit, called a qubit, which because of (e) the weird ways of quantum mechanics can simultaneously behave (e&eb) as both a one and a zero. A coupled array of qubits would allow (eb) a quantum computer to perform (e&eb) many, many calculations at (eb) the same time, and electron spins are (=) candidates for qubits in (eb) a quantum computer. The latest study shows how the microwave photons could work in concert with (e&eb) the spins of electrons to move information in a new type of computer. "What we need now is ways to wire up these systems—to couple (e&eb) these spins together," Morton said. "We need to make coupled qubits that can perform (e&eb) computations." In the experiment, conducted at CEA in France, a small sample of a highly purified form of (e) silicon was implanted with (e&eb) a matrix of bismuth atoms, and a superconducting aluminum circuit was deposited on (e&eb) top to create (eb) a high-quality resonant cavity that allowed (eb) precise tuning of the microwaves. The electron spins of the bismuth atoms were then flipped into (e&eb) the excited, "spin-up" state. The microwave cavity was then tuned (e&eb) like a musical tuning fork, to (e) a particular resonance that coaxed the spins into emitting (eb) a photon as they flipped back to (e) a relaxed state. The cavity boosted the number of states into (e&eb) which a photon can be (=) emitted, which greatly increased (eb) the decay rate for (e) the electron spins in (eb) a controllable way. The technique is much like buying more lottery tickets to increase (eb) your chances of winning, Morton said. A new spin on quantum computing: Scientists train electrons with microwaves From left are, Berkeley Lab scientists Thomas Schenkel, Qing Ji and Peter Seidl at the NDCX-II (Neutralized Drift Compression Experiment II), which produces (eb) powerful ion beams. Researchers are exploring how to use NDCX-II to process exotic … The large bismuth atoms embedded in (eb) the silicon sample provided (eb) the electrons with (e&eb) unique spin properties that enabled (eb) the experiment. Schenkel said that implanting the bismuth atoms into (e&eb) the delicate silicon framework, a process known as "doping," was "like squeezing bowling balls into (e&eb) a lattice of ping-pong balls." "We did a new trick with silicon. People wouldn't expect you could squeeze (e) anything new out of (e) silicon,"**

**Schenkel said. "Now we're looking into further improving (eb) bismuth-doped silicon and into (e&eb) tailoring the spin properties of (e) other materials, and using (e) this experimental technique for these materials." To enhance the performance of materials used (e) in future experiments, Schenkel said it will be necessary to improve (eb) the doping process so it is less damaging to (e) the silicon lattice. Also, the implantation process could be designed to produce (eb) regularly spaced arrays of (e) individual electron spins that would be more useful for (e) quantum computing than a concentrated ensemble of (e) electron spins. "We are now doing experiments on processing (e&eb) this and other materials at higher temperature and pressure with nanosecond ion pulses at NDCX-II, one of the accelerators here at Berkeley Lab," Schenkel said. "There are indications that it will improve (eb) the overall spin quality." Researchers said the latest research could potentially prove useful in boosting (eb) the sensitivity of scientific techniques like nuclear magnetic resonance spectroscopy and dynamic nuclear polarization, useful for a range of experiments, and could also shorten (e) experimental times by manipulating (e&eb) spin properties. "You need a way to reset (e&eb) spins—the ability to cause (eb) them to relax on demand to improve (eb) the rate at which you can repeat an experiment," Morton said. Bertet said it may be possible to further accelerate (eb) the electron-flipping behavior to (e) below 1 millisecond, compared to (e) the 1-second rate in the latest results. "This will then open the way to many new applications," he said. Explore further: Bismuth provides perfect dance partners for quantum computing qubits More information: A. Bienfait et al. Controlling spin relaxation with a cavity, Nature (2016). DOI: 10.1038/nature16944 Nature | Letter Controlling spin relaxation with a cavity A. Bienfait, J. J. Pla, Y. Kubo,X. Zhou, M. Stern, C. C. Lo, C. D. Weis, T. Schenkel, D. Vion,D. Esteve, J. J. L. Morton& P. Berte 531, 74–77 (03 March 2016) doi:10.1038/nature16944 Spontaneous emission of radiation is (=) one of the fundamental mechanisms by which an excited quantum system returns to (e) equilibrium. For spins, however, spontaneous emission is (=) generally negligible compared to (e&eb) other non-radiative relaxation processes because of (e) the weak coupling between the magnetic dipole and (e&eb) the electromagnetic field. In 1946, Purcell realized that the rate of spontaneous emission can be greatly enhanced by (e) placing the quantum system in a resonant cavity. This effect has since been used extensively to control (e&eb) the lifetime of atoms and semiconducting heterostructures coupled to (e&eb) microwave2 or optical3, 4 cavities, and is essential for (e) the realization of high-efficiency single-photon sources5. Here authors report the application of this idea to (e&eb) spins in solids. By coupling donor spins in silicon to (e&eb) a superconducting microwave cavity with (e&eb) a high quality factor and a small mode volume, authors reach the regime in which spontaneous emission constitutes (e) the dominant mechanism of spin relaxation. The relaxation rate is increased by (e) three orders of magnitude as the spins are tuned to (e) the cavity resonance, demonstrating that energy relaxation can be controlled (e&eb) on demand. Results provide a general way to initialize (eb) spin systems into their ground state and therefore have applications in (eb) magnetic resonance and (e&eb) quantum information processing6. They also demonstrate that the coupling between the magnetic dipole of a spin and (e&eb) the electromagnetic field can be enhanced up to (e) the point at which quantum fluctuations have (e) a marked effect on (e&eb) the spin dynamics; as such, they represent (eb) an important step towards the coherent magnetic coupling of (e) individual spins to (e) microwave photons Scientists obtain evidence of many-body localization in a closed quantum system July 8, 2016 by Olivia Meyer-Streng Science Cover Vol 352, Issue 6293, 24 June 2016, to highlight the research published in this issue (see original publication). The cover shows an artist's view of the experiment, illustrating the persistence of the density step of (e) an atomic distribution in an optical lattice. During equilibration ordinary many-body systems lose (e) all information about (e&eb) the initial state. Every morning authors experience an example for (e) this behaviour. Milk poured into a cup of coffee mixes (e&eb) perfectly and after some time it is impossible to say how exactly the two fluids were (=) put together. The same behaviour holds for (e) almost all quantum systems. However, recently a new phenomenon called "many-body localization" has been predicted theoretically, which allows (eb) well insulated quantum systems to (e) preserve memory of the initial state forever. Now a team of scientists around Dr. Christian Groß and Professor Immanuel Bloch (Director at MPQ and Chair of**

**Quantum Optics at LMU Munich), in cooperation with David Huse (Princeton University), has obtained evidence of such a behaviour in a two-dimensional quantum system of cold rubidium atoms trapped in (e&eb) an optical lattice. The scientists observed that – beyond a certain degree of disorder imprinted on (eb) the particle ensemble in the beginning – the system would relax into (e&eb) a steady state still containing (e) detailed microscopic information about its past. "We were able to observe the transition from a thermalized state into (e&eb) a many-body localized phase", Christian Groß points out. "It is the first observation of that kind in a regime that is not accessible with (e&eb) state-of-the-art simulations on classical computers." The experiment is not only of fundamental interest; the results might also lead to (eb) new ways for storing quantum information. Motivated by the foundational problem of how interacting particles behave (e&eb) in a disordered system, in the 1950s the American physicist Philip Warren Anderson discovered the famous localization phenomenon for (e) non-interacting particles, now called "Anderson localization". Here, disorder prevents (e) the particles to move and consequently all transport is (=) stopped. But what happens when disorder comes together with (e&eb) interactions? Will interactions lead to (eb) transport and thermalization, or will the localization persist even at (eb) high energies? So far, there is no theoretical model that faithfully predicts (eb) the evolution of a closed quantum system in more than one dimension under (e&eb) these conditions, although, the possibility for (e)localization has been theoretically suggested. In order to investigate these questions experimentally, strict requirements on controllability and perfect isolation of (e) the system have to be fulfilled. In the experiment described here, ultracold rubidium atoms are loaded into (eb) an optical lattice, a microscopic array of light traps formed by interfering (e&eb) laser beams. The disorder is created by projecting (e&eb) a computer-generated random light pattern onto (e&eb) the optical lattice. As a result of this "disorder", the depth of each microscopic trap in the lattice potential varies from site to (e) site. In fact, Prof. Bloch's group has advanced the experimental tools to such perfection that they can steer the position of (e) the atoms in the artificial light crystal and (e&eb) the interaction between (e&eb) them almost at will. With a high resolution microscope that detects the emitted fluorescence light, the position of each atom can be observed with (e&eb) high precision. Additionally, the atomic density distribution of the initial state can be (=) controlled, and for variable time intervals the evolution of the distribution can be (=) measured with high precision. These tools at hand, non-thermalizing behaviour can be probed in (eb) a conception ally very simple way. Any thermalized state of a closed system reflects (e&eb) the symmetry of its container – like, for example, water spilled into (e&eb) a round bowl would immediately cover (e) the whole bottom. In analogy, the scientists generate (eb) a density step in the initially prepared sample by (e) blowing away half of the atomic distribution with laser radiation. Then they watch how the remaining particles migrate into (e&eb) the empty half. For small disorders, the initially prepared density step is smeared out (e) fast and the initially empty and filled halves become (=) indistinguishable. However, when the measurement is repeated for (e) strong disorder, traces of the initial state remain and the system does not (e) relax to a thermal state even for (e) very long times. "We observe a fairly sharp onset of non-thermalizing behaviour above (e) a critical value", says Christian Groß. "This absence of thermalization is remarkable because (e) thermalization persists in a system of interacting particles, even at (eb) the high energies probed in the experiment." The scientists interpret their observation as (=) the onset of many-body localization in (eb) the atomic system. This is of fundamental interest because it means (eb) the breakdown of equilibrium statistical mechanics. On the other hand, the persistence of initial state information could be used as (=) a source for quantum information technologies. "It should also be emphasized that we obtain these results for a system size that is far beyond (e) numerically accessible scales", says Jae-yoon Choi, postdoc at the experiment. More information: J.-y. Choi et al. exploring the many-body localization transition in two dimensions, Science (2016). DOI: 10.1126/science.aaf8834 Journal reference: Science search and more info website Provided by: Max Planck Society search and more info website Bosons refusing to thermalize in 2D Messy, interacting quantum-mechanical systems are difficult to analyze theoretically. In a single spatial dimension, the calculations are (=) still tractable, and experiments have recently confirmed (eb) the prediction that sufficiently strong disorder can disrupt (e&eb) the transport of interacting particles. In two**

**dimensions, however, the theoretical blueprint is missing. Choi et al. used (e) single-site imaging of cold 87Rb atoms in (eb) an optical lattice to show (eb) that similar localization occurs in (eb) two-dimensional (2D) systems. The study highlights the power of quantum simulation to solve problems that are currently inaccessible to classical computing techniques. Science, this issue p. 1547 Abstract A fundamental assumption in statistical physics is that generic closed quantum many-body systems thermalize under (e&eb) their own dynamics. Recently, the emergence of many-body localized systems has questioned (e) this concept and challenged our understanding of the connection between statistical physics and (e&eb) quantum mechanics. Here authors report on the observation of a many-body localization transition between thermal and (e&eb) localized phases for (e) bosons in a two-dimensional disordered optical lattice. With our single-site–resolved measurements, authors track the relaxation dynamics of (e) an initially prepared out-of-equilibrium density pattern and find strong evidence for (e) a diverging length scale when (e) approaching the localization transition. Experiments represent a demonstration and in-depth characterization of many-body localization in (eb) a regime not accessible with (e&eb) state-of-the-art simulations on classical computers. Pelican instrument provides crucial experimental evidence of unusual quantum state July 8, 2016 Artist's illustration. Credit: Australian Nuclear Science and Technology Organisation (ANSTO) Experiments using inelastic neutron scattering at the Australian Centre for Neutron Scattering have found indications (eb) of a possible new quantum spin state in a novel antiferromagnetic material barium ytterbium zinc oxide (Ba3Yb2Zn5O11) which provides (eb) both a challenge and validation of (e) the third law of thermodynamics. The quantum mechanical information gained from (e) the experiment is crucial not only in understanding the quantum state of the specific material but also provides (eb) better understanding of high temperature superconductivity and ultimately offers (eb) guidance in the design of materials for (e) quantum computing. A research team led by Associate Professor Takatsugu Masuda at the Institute for Solid State Physics at the University of Tokyo with collaborators from ANSTO, Osaka University and Taiwan's National Cheng Kung University, have performed a successful experiment using Pelican, time-of-flight cold neutron spectrometer at ANSTO. The results have just been published in Physical Review B as a rapid Communication and an Editors' Suggestion. "The great success of this challenging experiment relied on (e) the excellent performance of the Pelican instrument, wonderful technical support at ANSTO and strong expertise in the relevant field from A/Prof Masuda's group," said Dr Dehong Yu and Dr Richard Mole, co-authors and Pelican instrument scientists. Pelican instrument provides crucial experimental evidence of unusual quantum state INS spectra measured at 1.5 K (a), 6 K (b), 12 K (c), and 40 K (d) using setup I. (e) INS spectrum measured at 1.5 K using the setup II. (f) Q dependence of the integrated intensity obtained from spectrum in (e). Red solid curve is the …….(deleted due to spatial constraints) Barium ytterbium zinc oxide (Ba3Yb2Zn5O11) is a novel antiferromagnetic model system having (e) a "breathing pyrochlore lattice" consisting of (e) arrays of alternating large and (e&eb) small corner-sharing tetrahedra of ytterbium atoms. Earlier research had suggested (eb) the material was (=) an isotropic spin half (S= ½) system. However, four low energy excitations measured on the sample with the Pelican instrument at a temperature of 1.5 K clearly disagree with (e) this designation and the associated theoretical model based on (e) the isotropic spins having simple interactions. "These observations suggested (eb) a much more complex spin state in the system," said Yu. "The experiment is interesting and important because as you start going very low in temperature, quantum mechanics takes over (e) system rather than thermodynamics, and you observe unusual properties," said Mole. "According to low temperature heat capacity measurements, the material does not (e) have zero entropy or a single ground state until you reach 63 milli Kelvin or -272.94 Celsius, before it starts obeying the third law." Pelican instrument provides crucial experimental evidence of (e) unusual quantum state Breathing pyrochlore lattice formed by (e) A3+ ions as determined by Kimura, Nakatsuju and Kimura in previous work The third law of thermodynamics states that for pure crystals as the temperature of a system approaches absolute zero K, (eb) its entropy or disorder approaches zero. "The research clearly indicates (eb) that below this temperature the material may be in (eb) an exotic state, such as (=) a liquid-like spin state, in which the arrangement of spins behaves (e&eb) like a liquid. In this exotic state, novel phenomena may show**

**up and be detectable; for example, charge and spin separation of electrons may be (=) possible," said Yu. The researchers suggest that further investigations, experimentally and theoretically, are required to ultimately determine the true identity of (e) the ground state of the material. "Discovering the existence of regular tetrahedral spin made it well worth coming all the way to Australia to make use of this neutron experimental apparatus," said Associate Professor Masuda. Many Japanese researchers are using ANSTO's neutron scattering instruments through the peer review system, while their nuclear facilities are offline. The outcome of this particular experiment is also part of the collaboration agreement between ANSTO and the University of Tokyo. More information: T. Haku et al. Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet, Physical Review B (2016). DOI: 10.1103/PhysRevB.93.220407 Journal reference: Physical Review B search and more info website: provided by: Australian Nuclear Science and Technology Organisation (ANSTO) search and more info website Low-energy excitations and ground-state selection in the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 T. Haku, K. Kimura, Y. Matsumoto, M. Soda, M. Sera, D. Yu, R. A. Mole, T. Takeuchi, S. Nakatsuji, Y. Kono, T. Sakakibara, L.-J. Chang, and T. Masuda Phys. Rev. B 93, 220407(R) – Published 24 June 2016 Authors study low-energy excitations in (eb) the quantum breathing pyrochlore antiferromagnet Ba3Yb2Zn5O11 by (e) a combination of inelastic neutron scattering (INS) and (e&eb) thermodynamical property measurements. The INS spectra are quantitatively explained by (e) spin-12 single-tetrahedron model having (e) XXZ anisotropy and Dzyaloshinskii-Moriya interactions. This model has a twofold degeneracy of (e) the lowest-energy state per tetrahedron and well reproduces (eb) the magnetization curve at (eb) 0.5 K and heat capacity above 1.5 K. At lower temperatures, however, authors observe a broad maximum in (eb) the heat capacity around 63 mK, demonstrating (eb) that a unique quantum ground state is selected due to (e) extra perturbations with (e&eb) an energy scale smaller than the instrumental resolution of INS. Revised 13 May 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.220407 © 2016 American Physical Society Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor. In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface. As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone. Such planets are all the "right" distance from (e) their star, and hopefully, also have water. But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone. For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement. But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway. But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain. It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine, which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life. Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface. The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of**

**polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113 Abstract The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question. Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry. This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry. Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments. Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy. The electronic and (e&eb) structural variability among them is extraordinary. The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations. The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface. The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers. Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp**

## 102.MODEL FORMULATION

Let $p_2(t, a_2)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and

$P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume

that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_6}{dt} = -b_6 Q_6 + g_6(g_6)Q_6$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit

creations which is assumed to be differentiable and satisfies

$$g_6(0) = 0, \quad g_6(P_6) > 0, \ P_6 \ 0, \qquad \lim_{P_6 \to \infty} g_6(P_6) = (K_6)_6 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_6 + (\rho_a)_6 + D_6(a_6, P_6, Q_6)\rho_6 + 0, \qquad 0 < a_6 \ \infty,$$

$$\rho_6(t,0) = \int_0^{\infty} n_6(a_6, P_6, Q_6)\rho_6(t_6, a_6)da_6$$

Where $D_6$ is the mortality function and $n_6$ is the fertility function.

Put $_6$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_6}{dt}\right]_6 = x_6\left[\frac{dQ_2}{dt}\right]_6 - x_2\left[\frac{dQ_6}{dt}\right]_6$$

$$\left[\frac{dQ_2}{dt}\right]_2 = x_3\left[\frac{dQ_3}{dt}\right]_6 - x_4\left[\frac{dQ_2}{dt}\right]_6$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_2}{dt}\right]_6 - x_6\left[\frac{dQ_3}{dt}\right]_6$$

Where $x_6, x_2, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_6$ and $g_6(P_6)Q_6$

Now when you add the term $(b_6Q_6)$ with $g_6(P_6)(Q_6)$ we are infect see is that $(b_6Q_6)$ is credited incrementally part of $g_6(P_6)(Q_6)$ or instant nearly, when subtracted from $\frac{dQ_6}{dt}$ , it means that $g_6(P_6)(Q_6)$ is slowly eaten by $\frac{dQ_6}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_6 = \mu_6(a_6) + (f_6)_6(P_6)_6 + (f_2)_6(Q)_6 \quad (2)$$

$$n_6 = \beta_6(a_6)h_6(Q_6), \quad (3)$$

Where $\mu_6$the age- dependent debit rate whit is $\mu_6(a_6)$ 0 $\forall a > 0$; $(f_6)_6$ is the density- dependent death rate with $f_6(0) = 0$, $(f_6^6)_6(P_6)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_6)_6(P_6) = (k_6)_6$ ∞; $f_2$ is the functional response of creditors to prey with $[f_2]_6(0) = 0, (f_2^6)_6(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_2)_6(Q_6) = [k_2]$ ∞; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 6, h^6(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_6 + (\rho_a)_6 = -\mu_6(a_6) + (f_6)_6P_6) + \big(f_2)_6(Q_6)\big)(\rho_6)_6 \quad (4)$$

$$\rho_6(t_6, 0) = h_6(Q_6)\int_0^\infty \beta_6(a_6)\rho_6(t, a_6)da, \quad (5)$$

$$\frac{dQ_6}{dt} = -b_6Q_6 + g_6(P_6)Q_6 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_6$ and $Q_6$ of (4-6) satisfy

$$0 \ \rho_6(t, a_6) \ \rho_6^*(t, a_6),$$

$$0 \ Q_6(t) \ Q_6^*(t),$$

Where $\rho_6^*(t, a_6)$ and $Q_6^*(t)$, are solutions of

$$(\rho_t^*)_6 + (\rho_a^*)_6 = -(\mu_6(a_6) + (f_6)_6(P_6^*) + (f_2)_6 Q_6^*))\,\rho_6^*, \quad (7)$$

$$\rho_6^*(t,0) = \int_0^\infty \beta_6(a_6)\,\rho_6^*(t,a_6)da_6 \quad (8)$$

$$\frac{dQ_6^*}{dt} = -b_6\,Q_6^* + g_6(P_6^*)Q_6^*, \quad (9)$$

With $P_6^*(t) = \int_0^\infty \rho_6^*(t,a_6)da_6$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_6^*(t)}{dt} = P_6^*(t)\big(r_6 - (f_6)(P_6^*) - (f_2)_6(Q^*)\big), \quad (10)$$

$$\frac{dQ_6^*(t)}{dt} = Q_6^*(t)(-b_6 + g_6(P_6^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_6(a)e^{-\int_0^a \mu_6(\tau)dt}e^{-ra}\,da_6 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_6^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(t)dt}\,da_6 > 1, \quad (H1)$$

Which guarantees $r_6 > 0$.

*Remark.* If $\beta_6(a) = [\beta_0]_6 a_6 e^{-\alpha_6 a_6}, \mu_6(a_6) \equiv \mu_6$ then the solution of (12) is

$$r_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_6]_6(P_6) = 0, [f_2](Q_6) = \lambda_6 Q_6 \; h_6(Q_6) = 1, \qquad and \;\; g_6(P_6) = c_6 P_6$$

$$[\rho_6]_6 + [\rho_a]_6 = -(\mu_6 + \lambda_6 Q_6)\rho_6$$

$$\rho_6(t,0) = \int_0^\infty [\beta_0]_6 a_6 e^{-\alpha_6 a_6} \rho_6(t, a_6)\, da_6$$

$$\frac{dQ_6}{dt} = -b_6 Q_6 + c_6 P_6 Q_6$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_6}{dt} = \omega_6 P_6 - \lambda_6 P_6 Q_6$$

$$\frac{dQ_6}{dt} = b_6 Q_6 - c_6 P_6 Q_6$$

Where $\omega_6 = \sqrt{\beta_6} - \alpha_6 - \mu_6$

## 103.THE EQUILIBRIUM $(\rho_0)_6(a_6), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_6(a_6), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_6 + (\rho_a)_6 = -\big(\mu_6(a_6) + (f_6)_6(P_6)\big)\rho,$$

$$\rho_6(t,0) = \int_0^\infty \beta_6(a_6)\rho_6(t,a_6)da_6 \qquad (13)$$

The solution of (63) has the form of

$$\rho_6(t,a_6) = Ce_6^{r(t-a_6)} - \int_0^a (\tau)d\tau - \int_0^6 (f_6)_6\big(P_6(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_6(t) = C_6 e^{r_6 t - \int_0^t (f_6)_6(P_6(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists equilibrium $P_0 = \int_6^{-6}(r)$ if $k_6 > r$. This is equivalent to

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{(k_6)_6 a_6} da_6 < 1.$$

In fact, if we define

$$R_6(s_6) = \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau)d\tau} e^{-s_6 a_6}\, da_6$$

$R(s)$ is a decreasing function. $R_6(r_6) = 1$ And $R_6(k_6)_6 < 1$ imply $r_6 < (k_6)_6$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_6(a_6), 0)$ as follows.

**THEOREM 1**. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{k_6 a_6} da_6$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_6 \equiv \int_0^\infty (\rho_0)_{66}(a_6)da_6$ *satisfies*

$$\int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau)d\tau} e^{-(f_6)_6(P_0)a_6} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_6(a_6), 0)$

Let $u_6(t, a_6) = \rho(t, a_6) - \rho_0(a_6)$ Then

$$N_6(t) \equiv \int_0^\infty u_6(t, a_6)da_6 = \int_0^\infty \rho_6(t, a_6)da_6 - \int_0^\infty (\rho_0)_6(a_6)da = P_6(t) - P_0)_6$$

And

$$(u_6)_6 + (u_a)_6 = (\rho_t)_6 + (\rho_a)_6 - (\rho_0^6)_6(a_6)$$

$$= -\mu_6(a_6) + (f_6)_6(P_6) + (f_2)_6(Q_6))\,(u_6 + (\rho_0)_6) + \mu_6(a_6)(\rho_0)_6 + (f_6)_6(P_0)_6(\rho_0)_6$$

$$= -\mu_6(a_6)u_6 - (f_6)_6(P_6) + (f_2)_6(Q_6))\,u_6 - (\rho_0)_6\,(f_6)_6(P_6) - (f_6)_6(P_0)_6$$

$$+ (f_2)_6(Q_6)), \qquad (15)$$

Where $(\rho_0^6)_6(a_6)$ denotes $d(\rho_0)_6(a_6)/da_6$

Expansion $(f_6)_6(P_6)$ and $(f_2)_6(Q_6)$ at $(P_0)_6$ and 0 yield respectively

$$(f_6)_6(P_6) = (f_6)_6((P_0)_6 + N_6)$$

$$= (f_6)_6(P_0)_6 + (f_6^6)_6(P_0)_6N_6 + o_6(N_6), \qquad (16)$$

$$(f_2)_6(Q_6)$$

$$= (f_2^6)_6(0)Q + o_6(Q_6) \qquad (17)$$

Write $f_6(P_0) = f_{60}, f_6'(P_0) = f_{60}'$, and $f_2'(0) = f_{20}'$ and substitute (16) and (17) into (15) to obtain

$$(u_6)_6 + (u_a)_6 \approx -(\mu_6(a_6) + (f_{60})_6)u - (f_{60}^6)_6(\rho_0)_6(N_6) - (f_{20}^6)_6(\rho_0)_6(Q_6)$$

$$u_6(t,0) = \rho_6(t,0)(\rho_0)_6(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a) + (\rho_0)_6(a_6)da_6 - (\rho_0)_6\,(0)$$

$$= h_6(Q_6)\int_0^\infty \beta_6(a_6)(u_6(t,a_6)da_6 + (h_6(Q_6) - 1)(\rho_0)_6\,(0) \qquad (18)$$

Expanding

$$h_6(Q_6) = 1 + (h^6)_6(0)Q_6 + 0(Q_6) \equiv 1 + h_0'Q_6 + 0Q_6$$

And substituting it into (18), we arrive at

$$u_6(t,0) \approx \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6 + (h_{a_6}')(\rho_0)_6Q_6$$

The linearization is the following :

$$[u_6] + [u_6] = -(\mu_6(a_6) + [t_{60}]_6 u_6 - [\alpha_6]_6(a_6)N_6 - [\alpha_2]_6(a_6)Q_6$$

$$u_6(t,0) = \int_0^\infty \beta_6(a_6)u_6(t,a_6)da_6 + [\xi_6]Q_6 \qquad (19)$$

$$\frac{dQ_6}{dt} = -b_6 Q_6$$

where

$$[\alpha_6]_6(a_6) = [t_6^6 a]_6[p,o]_6(a_6)d2(a_6) = [f_{20}^6]_6(\rho\, o\,)_6(a_6)\xi = [\mathrm{h\ o}^6]_6[\rho o]_6(\mathrm{o}) \text{ and } \mathrm{b}_6 = \mathrm{b}_6 - g_6(P_6)$$

If $g_6(P_0) > b$, clearly, $(\rho_0)_6(a_6), 0)$ is unstable and the predator population is persistent. Assume $g_6(P_0)_6 < b$.

Then, $Q_6(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_6 + u_a = -(\mu_6(a_6) + (f_{60})_6)u_6 - (\alpha_6)_6(a_6)N_6$$

$$= -(\mu_6(a_6) + f_{60})u_6 - \alpha_6(a_6)\int_0^\infty u_6(t,a_6)da_6$$

$$u_6(t,0) = \int_0^\infty \beta_6\,(a_6)u_6(t,a_6)da_6$$

Using the method developed by Cushing [2, 3], we suppose

$$u_6(t,a_6) = \tilde{\mathrm{u}}_6(a_6)^{e^{\gamma(t-a_6)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_6}\, da_6$. Then,

A solution of the first differential equation for $\tilde{u}(a_6)$ gives,

$$\tilde{u}_6(a_6) = e^{-\int_0^a \mu_6(\tau_6)d\tau - (f_{60})_6 a}\left(\tilde{u}_6(0) - w\int_0^a e^{\int_0^\tau \mu_6(s_6)ds + (f_{60})}\tau_6\,(\alpha_6)_6(\tau_6)e^{\gamma_6\tau}d\tau\right).$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\right)\tilde{u}_6(0)$$

$$+w\int_0^\infty \beta_6(a_6)e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a (\alpha_6)_6(\tau_6)e^{-\gamma_6\tau_6}e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds}d\tau_6\,da_6$$

$$= 0 \qquad (22)$$

Substituting (21) into the definition of of *w* yields

$$\left(\int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}da\right)(\tilde{u}_6(0))_6$$

$$+\left(-6 - \int_0^\infty e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6)d\tau}\int_0^a e^{(f_{60})_6\tau_6 + \int_0^\tau \mu_6(s_6)ds + y\tau}\alpha_6(\tau)d\tau\,da\right)w = 0.$$

(23)

Define $Y(\gamma, a) = e^{-\gamma_6 a_6 - (f_{60})_6 a_6 - \int_0^a \mu_6(t_6)d\tau}$. Then

$$\left(1 - \int_0^\infty \beta_6\,(a_6)Y_6(\gamma_6, a_6)da_6\right)u_6(0)$$

$$+w\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6=0,$$

(24)

$$\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\tilde{u}_6(0)$$

$$+\left(-1-\int_0^{\infty}Y_6(\gamma_6,a_6)\int_0^{a}(\alpha_6)_6(\tau)_6(Y^{-6})_6(\gamma_6,\tau_6)d\tau_6\,da_6\right)w=0.$$

Equqtion (20) has a solution of the form of $u_6(t,a_6)=u_6(a_6)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_6(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_6(\gamma_6,a_6)=\int_0^a(\rho_0)_6(\tau_6)Y^{-6}(\gamma_6,a_6)d\tau_6$. We obtain the characteristic equation

$$f_{60}^{l}{}_6\left(\int_0^{\infty}Y_6(\gamma_6,a_6)da_6\right)\left(\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$+\left(1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6,a_6)da\right)\left(1+(f_{60}^{l})_6\int_0^{\infty}Y_6(\gamma_6,a_6)I_6(\gamma_6,a_6)da_6\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_6(a_6)=(\rho_0)_6(0)e^{-\int_0^a\mu_6(\tau_6)d\tau_6-(f_{60})_6a_6}$$

$$I_6(\gamma_6,a_6)=\int_0^{a}(\rho_0)_6(\tau_6)e^{\gamma_6\tau_6+(f_{60})_6\tau+\int_0^{\tau}\mu_6(s_6)ds_6}d\tau$$

$$= \frac{(\rho_0)_6(0)}{\gamma_6}(e^{\gamma_6 a_6} - 1)$$

Let

$$E_6(a_6) = e^{-(f_{60})_6 a_6 - \int_0^a \mu_6(\tau_6) d\tau_6}$$

This relationship is cardinal one. It is useful in quantum computation and other black hole dynamics problems

**Quantum space and quantum time**

(5732) **Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.**

(5733) **In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.**

(5734) **They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.**

(5735) **When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.**

(5736) **Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.**

**The Planck scale**

(5737) **Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".**

(5738) **Here ħ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.**

(5739) **The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.**

(5740) **Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)**

(5741) **Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort**

**of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.**

## Physical applications

**Some phenomena where quantum gravity plays a major role are the following.**

**The microscopic structure of spacetime**

(5742) **If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.**

(5743) **Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length**

(5744) **Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.**

(5745) **A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.**

**Early cosmology**

(5746) **According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.**

(5747) **However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.**

(5748) **A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.**

(5749) **Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".**

(5750) **One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.**

**Black holes**

(5751) **Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.**

(5752) **First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.**

(5753) **Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.**

(5754) **What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.**

(5755) **Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.**

## Astrophysical effects

(5756) **Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.**

(5757) **An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.**

(5758) **The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.**

## Main approaches

(5759) **The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.**

(5760) **Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.**

(5761) **The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.**

## String theory

(5762) **String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.**

(5763) **Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.**

(5764) **String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.**

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| (5765) **Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.** |
| (5766) **However, when summed the entire series appears to be (=) divergent.** |
| (5767) **In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.** |
| (5768) **Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.** |
| (5769) **Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.** |
| (5770) **In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.** |

**Loop quantum gravity and spin foams**

| |
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| (5771) **Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.** |
| (5772) **The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.** |
| (5773) **Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.** |
| (5774) **The granular properties of space can be (=) explicitly computed.** |
| (5775) **In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.** |
| (5776) **The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.** |
| (5777) **Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.** |
| (5778) **The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.** |
| (5779) **Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.** |
| (5780) **This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.** |

**Noncommutative geometry**

| |
|---|
| (5781) **Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.** |
| (5782) **Quantum mechanics teaches us that dynamical quantities are (=) noncommutative -- meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.** |
| (5783) **It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.** |
| (5784) **A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.** |
| (5785) **Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.** |
| (5786) **This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.** |

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| **Others** |

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| (5787) **Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays** |
| (5788) **. Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.** |
| (5789) **Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.** |
| (5790) **The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.** |

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| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (5791) **These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field.** |

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| (5792) **The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets.** |
| (5793) **Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state.** |
| (5794) **Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands.** |
| (5795) **The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI** |
| (5796) **The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9].** |
| (5797) **This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive.** |
| (5798) **More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X.** |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

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| (5799) **Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.** |
| (5800) **Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.** |
| (5801) **The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.** |
| (5802) **This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.** |
| (5803) **The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.** |
| (5804) **This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.** |
| (5805) **For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.** |
| (5806) **The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society** |

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption**

**expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

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| (5807) | **Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.** |
| (5808) | **We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.** |
| (5809) | **When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called thermalization.** |
| (5810) | **However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.** |
| (5811) | **Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).** |
| (5812) | **Bringing them to thermal equilibrium requires (e) an external bath.** |
| (5813) | **But so far, there has been no rigorous proof of (e) the existence of such many-body localization. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.** |
| (5814) | **The concept of localization was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).** |
| (5815) | **The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.** |
| (5816) | **But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.** |
| (5817) | **Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.** |
| (5818) | **For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.** |
| (5819) | **It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].** |
| (5820) | **Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.** |
| (5821) | **One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.** |
| (5822) | **However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only** |

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| **be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].** |
| (5823) **Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.** |
| (5824) **Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.** |
| (5825) **As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.** |
| (5826) **His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.** |
| (5827) **An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.** |
| (5828) **This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).** |
| (5829) **This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.** |
| (5830) **However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime:** |
| (5831) **These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects.** |
| (5832) **It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies).** |
| (5833) **This is a realistic scenario for (e) all known systems.** |
| (5834) **On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects.** |
| (5835) **On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other.** |
| (5836) **No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true.** |
| (5837) **His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found.** |
| (5838) **Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization?** |
| (5839) **Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects?** |

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| (5840) **Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions.** |
| (5841) **So in those systems, the quantum phase transition does indeed exist.** |
| (5842) **But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension.** |
| (5843) **Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters.** |

**Focus: Biological Cells Form Electric Circuits July 1, 2016• Physics 9, 75 Cells that are electrically active and that also produce light for easy voltage monitoring could lead to new studies of heart arrhythmias and possibly bio-computing. Figure caption expand figure H. M. McNamara et al., Phys. Rev. X (2016) Cell circuit. Waves of electrical activity propagate in a loop within a collection of cells engineered with electrical and optical properties (see video below).Some sentences, pictures and videos are deleted due to constraints on space.**

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| (5844) **The human heartbeat is produced by (e) electrical pulses that propagate through (e&eb) cardiac tissue, causing (eb) rhythmic muscle contraction.** |
| (5845) **Researchers have previously engineered cells to create (eb) an artificial tissue capable of producing (eb) coordinated electrical activity, and now a team has added the ability to monitor (e&eb) their electrical state by (e) detecting fluorescent emission.** |
| (5846) **They have also fashioned the cells into (e&eb) "living circuits" that might act as (=) model systems for studying heart behavior.** |
| (5847) **Constructing electrically active tissues from their basic building blocks by (e) genetically engineering cells allows (eb) researchers to study a "clean" system similar to (e) the more complicated cardiac tissue.** |
| (5848) **These systems let them investigate, for example, the electrical signals that lead to (eb) the potentially fatal irregular heartbeat called arrhythmia.** |
| (5849) **Previously researchers have introduced (eb) ion channels—proteins that conduct ions through (e&eb) cell membranes—into human kidney cells to make (eb) electrically "excitable" tissues [1].** |
| (5850) **Adam E. Cohen of Harvard University and his co-workers decided to add more ingredients to such cells, so that the electrical activity could be (=) light-activated and easily visualized.** |
| (5851) **They incorporated four additional gene products into (e&eb) human embryonic kidney (HEK) cells.** |
| (5852) **Two were ordinary ion channels; the other two were (=) a blue-light-activated ion channel (a light-sensitive switch) and (e&eb) a protein that fluoresces with (e&eb) an intensity that depends on (=) the voltage across the membrane in which it is embedded.** |
| (5853) **Light activation of electrically excitable cells has been shown before [2]. But Cohen and colleagues have now included a convenient "readout" mechanism (the fluorescent protein) and have used patterning methods to shape (e&eb) the cell cultures into "tissue circuits."** |
| (5854) **They grew rings of cells a few millimeters in diameter that could act as (e&eb) oscillators that support (eb) circulating waves of electrical excitation.** |

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| (5855) **These waves arise from (e) the collective behavior of the cells: a pulse produced by (e) one cell can trigger another from (e) its neighbors, eventually producing (eb) a coherent wave of excitation traveling through (e&eb) the tissue.** |
| (5856) **It's this coordination, in a repeating cycle of excitation and quiescence, which gives rise to (eb) rhythmic activity in cardiac tissue.** |

**H. M. McNamara et al., Phys. Rev. X (2016)**

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| (5857) **A pattern of blue light (not visible) illuminates these cells, defining (eb) a rectangular loop around which electrical waves can propagate.** |
| (5858) **At about 0:07 the rectangle begins to shrink in (eb) the vertical direction, causing t(eb) he cycle time to gradually decrease. (In other experiments, the cells were grown in (eb) fixed, ring-shaped patterns.)** |
| (5859) **The ring of engineered HEK cells acted like an electronic "ring oscillator," a device with a characteristic frequency to (e&eb) its circulating electrical signal.** |
| (5860) **The electrical waves in a ring oscillator can propagate (e&eb) in either direction around the ring, enabling (eb) it to serve as a binary memory unit.** |
| (5861) **Cohen and colleagues found that in their HEK tissues, these circulating waves remained (eb) stable for more than two hours, with (e&eb) the pulses completing a few circuits every second.** |
| (5862) **The electrical signals were easily visualized as (=) waves of fluorescent emission.** |
| (5863) **The researchers say that these tissues could be used to (e) understand and predict the behavior of real cardiac cells, especially arrhythmias.** |
| (5864) **"In real heart, if you pace the tissue fast enough, *(eb) it starts skipping beats and eventually becomes (=) arrhythmic," says Cohen.** |
| (5865) **This state is a well-known precursor to (e) sudden cardiac death and is associated with (e&eb) the breakup of the usual regular waves of electrical excitation.** |
| (5866) **Some drugs also have the dangerous side effect of inducing (eb) arrhythmia, and Cohen says that the new system might offer a quick and convenient way of screening for such hazards. The work "is really very elegant and deserves attention," says physiologist Gil Bub of Oxford University in the UK. "It opens new research questions for people like me who are interested in wave dynamics in excitable systems."** |
| (5867) **There might also be practical uses for information processing in living tissues. "By combining patterns of cell growth and (e&eb) patterns of ion channel expression, we think we can build diodes, amplifiers, and other simple logic elements," says Cohen.** |
| (5868) **Andrew Adamatzky of the University of the West of England in Bristol, a specialist in "unconventional computing," foresees circuits that perform computations through (e&eb) the interactions of electrical waves. "This research might lead to fabrication of implantable bio-computers," he says. This research is published in Physical Review X. –Philip Ball Philip Ball is a freelance science writer in London and author of Invisible: The Dangerous Allure of the Unseen (University of Chicago Press, 2014).** |

**Optically Controlled Oscillators in an Engineered Bioelectric Tissue Harold M. McNamara, Hongkang Zhang, Christopher A. Werley, and Adam E. Cohen Phys. Rev. X 6, 031001 – Published 1 July 2016**

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| (5869) **Complex electrical dynamics in excitable tissues occur throughout (e&eb) biology, but the roles of individual ion channels can be difficult to determine due to (e) the complex nonlinear interactions in (eb) native tissue.** |
| (5870) **Here, authors ask whether they can engineer a tissue capable of (e) basic information storage and processing, where (e) all functional components are known and well understood.** |
| (5871) **Authors develop a cell line with (e&eb) four transgenic components: two to enable (eb) collective propagation of electrical waves and (e&eb) two to enable optical perturbation and (e&eb) optical readout of membrane potential.** |
| (5872) **They pattern the cell growth to define (eb) simple cellular ring oscillators that run stably for >2 h (∼104 cycles) and that can store data encoded in (eb) the direction of electrical circulation.** |
| (5873) **Using (e) patterned optogenetic stimulation authors probe the biophysical attributes of (e) this synthetic excitable tissue in detail, including (e) dispersion relations, (e&eb) curvature-dependent wave front propagation, (e&eb) electrotonic coupling, and (e&eb) boundary effects.** |
| (5874) **Authors then apply the biophysical characterization to (e) develop an optically reconfigurable bioelectric oscillator.** |
| (5875) **These results demonstrate the feasibility of engineering bioelectric tissues capable of (e&eb) complex information processing with optical input and output. Received 12 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevX.6.031001 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society** |

**Scientists detect radio emission from a nearby brown dwarf July 6, 2016 by Tomasz Nowakowski report**

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| (5876) **K2 and Spitzer images of W0607+24: The position of W0607+24 is marked by (e) a red circle. SDSS J060738.06+242939.5, marked by (e) an orange circle, are the brightest star in the K2 image.** |
| (5877) **The three faint (r = 18.3 − 19.6) stars marked by cyan circles, contribute significant flux to (e) the K2 measurement but are (-) negligible at Spitzer's longer wavelength and (e &êb) higher resolution. Other stars with r < 20 in SDSS are marked with blue circles. Credit: Gizis et al., 2016.** |
| (5878) **(Phys.org)—Located some 23 light years away, the brown dwarf designated WISEP J060738.65+242953.4 or W0607+24 for short, turns out to be (=) a source of radio emission.** |
| (5879) **According to a research paper published July 4, this substellar object showcases (eb) quiescent radio emission, making it one of (e) the most radio-faint, ultra-cool dwarfs yet detected.** |
| (5880) **Brown dwarfs like W0607+24 are objects that are too large to be called planets and too small to be stars. With a mass below that necessary to maintain hydrogen-burning nuclear fusion reactions, they are much cooler and dimmer than (eb) main sequence stars.** |
| (5881) **These objects could be also sources of (e) radio emission, but the cause of this process is still not completely understood.** |
| (5882) **Due to its proximity, W0607+24 is (=) an excellent target for studying the nature of (e) brown dwarfs.** |
| (5883) **It is the nearest known late-L dwarf in the northern hemisphere, and the third-nearest in (=) the whole sky, (eb) allowing researchers to study phenomena occurring in brown dwarfs.** |

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| (5884) | **That's why a team of astronomers, led by John Gizis of the University of (e)Delaware, Newark, carried out a series of mid-infrared, radio, and optical observations of W0607+24 to study any activity in this curious substellar object.** |
| (5885) | **Motivated by the results from NASA's prolonged Kepler mission, named K2, which obtained (eb) long time-series photometry of W0607+24; they observed this object simultaneously with the Karl G. Jansky Very Large Array (VLA) in New Mexico and NASA's Spitzer Space Telescope.** |
| (5886) | **The most important results were provided by VLA, as the team found that W0607+24 is (=) a source of quiescent, non-bursting, radio emission.** |
| (5887) | **The scientists noted that the discovery of this process in this nearby object could be (=) crucial for our understanding of this process in (eb) ultracool dwarfs in general.** |
| (5888) | **"W0607+24 is a valuable test case for (e) understanding the mechanisms of radio emission in ultracool dwarfs.** |
| (5889) | **Some of the radio-detected objects emit (eb) rapid, intense bursts with high circular polarization, often found to occur periodically and attributed to coherent emission due to (e) the electron cyclotron maser instability," the paper reads.** |
| (5890) | **The astronomers also suggested that the relative radio faintness of W0607+24 may be due to (e) its orientation if (e) the emission is concentrated at the poles.** |
| (5891) | **However, follow-up observations are needed to better characterize (eb) the detected radio emission and to eliminate (e) all the uncertainties.** |
| (5892) | **The team also hopes for more significant findings when (e) studying this nearby substellar object in the future.** |
| (5893) | **"As the nearest known northern hemisphere late-L dwarf, this offers many possibilities for future studies. For example, it could serve as (=) an useful standard for (e) line-broadening and (e&eb) polarization studies.** |
| (5894) | **If there is a planetary system aligned with (e&eb) (the primary's rotation axis, then we would expect very little radial velocity signal but a strong astrometric signal," the researchers concluded.** |
| (5895) | **Moreover, the team plans to monitor (e&eb) additional late-M and L dwarfs in other K2 campaigns, expecting even more promising results.** |
| (5896) | **They are more distant than W0607+24, but because they are warmer, (eb) some are brighter and higher signal-to-noise than that described in this paper.** |
| (5897) | **In addition to detecting the radio emission, (eb)the scientists also estimated the age, mass and radius of W0607+24** |
| (5898) | **. According to the study, the object is less than two billion years old, has (e) a maximum mass of about 0.055 solar masses and a radius approximately of 0.1 solar radii.** |
| (5899) | **More information: WISEP J060738.65+242953.4: A Nearby. Pole-On L8 Brown Dwarf with Radio Emission, arXiv: 1607.00943 [astro-ph.SR] arxiv.org/abs/1607.00943** |
| (5900) | **Authors present a simultaneous, multi-wavelength campaign targeting the nearby (7.2 pc) L8/L9 (optical/near-infrared) dwarf WISEP J060738.65+242953.4 in (eb) the mid-infrared, radio, and optical.** |
| (5901) | **Spitzer Space Telescope observations show no (e) variability at the 0.2% level over 10 hours each in the 3.6 and 4.5 micron bands.** |
| (5902) | **Kepler K2 monitoring over 36 days in Campaign 0 rules out (e) stable periodic signals in (eb) the optical with amplitudes great than 1.5% and periods between 1.5 hours and 2 days.** |

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| (5903) **Non-simultaneous Gemini optical spectroscopy detects lithium, constraining (e) this L dwarf to be less than ~2 Gyr old, but no Balmer emission is observed.** |
| (5904) **The low measured projected rotation velocity (v sin i < 6 km/s) and lack of variability are (=) very unusual compared to other brown dwarfs, and authors argue that this substellar object is (=) likely viewed pole-on.** |
| (5905) **Authors detect quiescent (non-bursting) radio emission with (e&eb) the VLA.** |
| (5906) **Amongst radio detected L and T dwarfs, it has (e) the lowest observed L_nu and the lowest v sin i.** |
| (5907) **They discuss the implications of pole-on detection for (e) various proposed radio emission scenarios. © 2016 Phys.org** |

**A new look at the galaxy-shaping power of black holes July 6, 2016**

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| (5908) **It wasn't until the advent of X-ray astronomy that the full picture of (e) the galaxy-shaping power of black holes began to emerge (eb) with the ability to see plasma.** |
| (5909) **Data from a now-defunct X-ray satellite is providing (eb) new insights into the complex tug-of-war between galaxies, the hot plasma that surrounds them, and the giant black holes that lurk in their centres.** |
| (5910) **Launched from Japan on February 17, 2016, the Japanese space agency (JAXA) Hitomi X-ray Observatory functioned for just over a month before contact was lost and (e&eb) the craft disintegrated.** |
| (5911) **But the data obtained during those few weeks was enough to paint (eb) a startling new picture of the dynamic forces at work within galaxies.** |
| (5912) **New research, published in the journal Nature today, reveals data that shows just how important the giant black holes in galactic centres are to the evolution (eb) of the galaxies as a whole.** |
| (5913) **"We think that supermassive black holes act(e&eb) like thermostats," said Brian McNamara, University Research Chair in Astrophysics at the University of Waterloo. "They regulate (e&eb) the growth of galaxies."** |

**Champagne bubbles of plasma**

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| (5914) **During its brief life, the Hitomi satellite collected X-ray data from (e) the core of the Perseus cluster, an enormous gravitationally-bound grouping of (e) hundreds of galaxies.** |
| (5915) **Located some 240 million light years from earth, the Perseus cluster is one of (e) the largest known structures in the universe.** |
| (5916) **The cluster includes not only the ordinary matter that makes up the galaxies, but an (e) "atmosphere" of hot plasma with a temperature of tens of millions of degrees, as well as a halo of (e) invisible dark matter.** |
| (5917) **Earlier studies, going back to the 1960s, have shown that each of the galaxies in the cluster - and indeed most galaxies - likely contains (e) a supermassive black hole in its centre, an object 100 million to more than ten billion times as massive as our sun** |

**New look at the galaxy-shaping power of black holes**

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| (5918) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of the plasma are moving.** |
| (5919) **"These giant black holes are (=) among the universe's most efficient energy generators, a hundred times more efficient than a nuclear reactor," said McNamara from Waterloo's Department of Physics and Astronomy in the Faculty of Science.** |
| (5920) **"Matter falling into the black hole is ripped apart, releasing (eb) vast amounts of energy in the form of high speed particles and thermal energy."** |
| (5921) **This heat is released from (e) just outside the black hole's event horizon (=) the boundary of no return.** |
| (5922) **The remaining matter gets absorbed into the black hole, adding to (e) its mass.** |
| (5923) **The released energy heats up (eb) the surrounding gas, creating (eb) bubbles of hot plasma that ripple through (e&eb) the cluster, just as bubbles of air rise up in (eb) a glass of champagne.** |
| (5924) **The research is shedding light on the crucial role that this hot plasma plays (e&eb) in galactic evolution.** |
| (5925) **Researchers are now tackling the foremost issue in the formation of (e) structure in the universe and asking: why don't most of the gas cool down, and form (eb) stars and galaxies? The answer seems to be that bubbles created by blasts of energy from (e) the black holes keep temperatures too high for (e) such structures to form.** |
| (5926) **"Any time a little bit of gas falls into the black hole, it releases (eb) an enormous amount of energy," said McNamara.** |
| (5927) **"It creates (eb) these bubbles, and the bubbles keep the plasma hot.** |
| (5928) **That's what prevents galaxies from (e) becoming even bigger than they are now."** |
| (5929) **Because plasma is invisible to the eye, and to optical telescopes, it wasn't until the advent of X-ray astronomy that the full picture began to emerge. In visible light, the Perseus cluster appears to contain (e) many individual galaxies, separated by (e) seemingly-empty space.** |
| (5930) **In an X-ray image, however, the individual galaxies are invisible, and the plasma atmosphere, centered on the cluster's largest galaxy, known as NGC 1275, dominates (e) the scene.** |
| (5931) **Although the black hole at the heart of NGC 1275 has (e) only one-thousandth of the mass of its host galaxy, and has (e) a much smaller volume, it seems to have (e) a huge influence on (e&eb) how the galaxy and how the surrounding hot plasma atmosphere evolve.** |
| (5932) **"It's as though the galaxy somehow knows (e) about this black hole sitting at the centre," said McNamara.** |
| (5933) **"It's like nature's thermostat, which keeps these galaxies from (e) growing.** |
| (5934) **If the galaxy tries to grow too fast, (eb) matter falls into the black hole, releasing (eb) an enormous amount of energy, which (eb) drives out the matter and prevents (e) it from forming new stars."** |

**A new look at the galaxy-shaping power of black holes**

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| (5935) **An X-ray image reveals (eb) the hot plasma that envelope (e) the Perseus cluster. Located some 240 million light years from earth, the** |
| (5936) **McNamara notes that the actual event horizon of the black hole is (=) about the same size as our solar system, making it as small compared to (e) its host galaxy as a grape is to the Earth. "What's going on in this tiny region is affecting a (e&eb) vast volume of space," he said.** |

(5937) **Thanks to the black hole's regulatory effect, (eb) the gas that would have formed (eb) new stars instead remains hot plasma - whose properties Hitomi was designed to measure.**

**Doomed satellite missions**

(5938) **Hitomi employed an X-ray spectrometer which measures (eb) the Doppler shifts in emissions from (e) the plasma; those shifts can then be used to (e) calculate the speed at which different parts of (e) the plasma are moving.**

(5939) **At the heart of the spectrometer is a microcalorimeter; cooled to just one-twentieth of a degree above absolute zero, the device records (eb) the precise energy of each incoming X-ray photon.**

(5940) **Getting an X-ray satellite equipped with a microcalorimeter into space has proved daunting: McNamara was deeply involved with NASA's Chandra X-ray Observatory, launched in 1999, that was initially set to include a microcalorimeter, but the project was scaled back due to budget constraints, and the calorimeter was dropped. Another mission with the Japanese space agency known as ASTRO-E was equipped with a microcalorimeter; it was set for launch in 2000, but the rocket exploded shortly after liftoff. A third effort, Japan's Suzaku satellite, launched in 2005, but a leak in the cooling system destroyed the calorimeter. Hitomi launched and deployed perfectly, but a series of problems with the attitude control system caused the satellite to spin out of control and break up.**

(5941) **The data from Hitomi, limited as it is, is enough to make astronomers re-think the role of plasma in galactic evolution, according to McNamara. "The plasma can be thought of forming an enormous atmosphere those envelopes (e) whole clusters of galaxies.**

(5942) **These hot atmospheres represent (eb) the failure of the past—the failure of the universe to create (eb) bigger galaxies," he said.**

(5943) **"But it's also the hope for the future. This is the raw material for the future growth of galaxies - which is (=) everything: stars, planets, and people**

(5944) **. It's the raw material that in the next several billion years is going to make (eb) the next generation of suns and solar systems. And how rapidly that happens is governed by (e&eb) the black hole."**

(5945) **The observations give researchers, for the first time, a direct measurement of (e) the turbulent speed of the hot plasma.**

(5946) **"This measurement tells us how the enormous energy released by (e) supermassive black holes regulates (e&eb) the growth of the galaxy and the black hole itself," said McNamara. More information: The quiescent intracluster medium in the core of the Perseus cluster, Nature, DOI: 10.1038/nature18627 Journal reference: Nature search and more info website provided by: University of Waterloo search and more info website**

**The quiescent intracluster medium in the core of the Perseus cluster Nature 535, 117–121 (07 July 2016) doi: 10.1038/nature18627 Published online 06 July 2016**

(5947) **Clusters of galaxies are (=) the most massive gravitationally bound objects in the Universe and are (=) still forming.**

(5948) **They are thus important probes1 of (e) cosmological parameters and (e&eb) many astrophysical processes.**

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| (5949) **However, knowledge of the dynamics of (e) the pervasive hot gas, the mass of which is much larger than *(e) the combined mass of all the stars in the cluster, is lacking.** |
| (5950) **Such knowledge would enable (eb) insights into the injection of (e) mechanical energy by (e) the central supermassive black hole and the use of (e) hydrostatic equilibrium for (e) determining cluster masses.** |
| (5951) **X-rays from the core of the Perseus cluster are emitted by (e) the 50-million-kelvin diffuse hot plasma filling (e) its gravitational potential well.** |
| (5952) **The active galactic nucleus of the central galaxy NGC 1275 is pumping jetted energy into (e&eb) the surrounding intracluster medium, creating (eb) buoyant bubbles filled with (e) relativistic plasma.** |
| (5953) **These bubbles probably induce *(eb) motions in the intracluster medium and heat the inner gas, preventing (e) runaway radiative cooling—a process known as (=) active galactic nucleus feedback2, 3, 4, 5, 6.** |
| (5954) **Here authors report X-ray observations of (e) the core of the Perseus cluster, which reveal (eb) a remarkably quiescent atmosphere in which the gas has (e) a line-of-sight velocity dispersion of 164 ± 10 kilometres per second in the region 30–60 kiloparsecs from the central nucleus.** |
| (5955) **A gradient in the line-of-sight velocity of 150 ± 70 kilometres per second is found across the 60-kiloparsec image of the cluster core. Turbulent pressure support in the gas is (=) four per cent of the thermodynamic pressure, with (e&eb) large-scale shear at most doubling this estimate.** |
| (5956) **Authors infer that a total cluster mass determined from (e) hydrostatic equilibrium in (eb) a central region would require little correction for (e) turbulent pressure.** |

**Dark matter guides growth of supermassive black holes February 18, 2015**

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| (5957) **This illustration (deleted attributable and ascribable to spatial restraints) shows two spiral galaxies - each with supermassive black holes at (eb) their center - as they are about to collide and form an elliptical galaxy.** |
| (5958) **Every massive galaxy has (e) a black hole at its center, and the heftier the galaxy, (eb) the bigger its black hole.** |
| (5959) **But why are the two related? After all, the black hole is millions of times smaller and less massive than (eb) its home galaxy.** |
| (5960) **A new study of football-shaped collections of stars called elliptical galaxies provides (eb) new insights into the connection between a galaxy and (e&eb) its black hole.** |
| (5961) **It finds that the invisible hand of dark matter somehow influences (e&eb) black hole growth.** |
| (5962) **"There seems to be a mysterious link between the amount of dark matter a galaxy holds and (e&eb) the size of its central black hole, even though the two operate on (e&eb) vastly different scales," says lead author Akos Bogdan of the Harvard-Smithsonian Center for Astrophysics (CfA).** |
| (5963) **This new research was designed to address a controversy in the field. Previous observations had found a relationship between the mass of the central black hole and (e&eb) the total mass of stars in elliptical galaxies.** |
| (5964) **However, more recent studies have suggested (eb) a tight correlation between the masses of the black hole and (e&eb) the galaxy's dark matter halo. It wasn't clear which relationship dominated.** |

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| (5965) **In our universe, dark matter outweighs (e) normal matter - the everyday stuff we see all around us - by a factor of 6 to 1.** |
| (5966) **We know dark matter exists only from (e) its gravitational effects.** |
| (5967) **It holds together galaxies and (e&eb) galaxy clusters.** |
| (5968) **Every galaxy is surrounded by (e&eb) a halo of dark matter that weighs as (=) much as a trillion suns and extends for (e) hundreds of thousands of light-years.** |
| (5969) **To investigate the link between dark matter halos and (e&eb) supermassive black holes, Bogdan and his colleague Andy Goulding (Princeton University) studied more than 3,000 elliptical galaxies.** |
| (5970) **They used star motions as tracers to (e) weigh the galaxies' central black holes.** |
| (5971) **X-ray measurements of hot gas surrounding (e&eb) the galaxies helped weigh the dark matter halo, because (e) the more dark matter a galaxy has, (eb) the more hot gas it can hold onto.** |
| (5972) **They found a distinct relationship between the mass of the dark matter halo and (e&eb) the black hole mass - a relationship stronger than (e) that between a black hole and (e&eb) the galaxy's stars alone.** |
| (5973) **This connection is likely to be related to (e&eb) how elliptical galaxies grow.** |
| (5974) **An elliptical galaxy is formed when (e) smaller galaxies merge, their stars and dark matter mingling and (e&eb) mixing together.** |
| (5975) **Because the dark matter outweighs (e) everything else, it molds (e&eb) the newly formed elliptical galaxy and guides (e&eb) the growth of the central black hole.** |
| (5976) **"In effect, the act of merging creates a gravitational blueprint that the galaxy, the stars and the black hole will follow in order to build themselves," explains Bogdan.The paper describing this work has been accepted for publication in the Astrophysical Journal. This result relied on data from the Sloan Digital Sky Survey and the ROSAT X-ray satellite's all-sky survey. Journal reference: Astrophysical Journal search and more info website Provided by: Harvard-Smithsonian Center for Astrophysics search and more info website Read more at: http://phys.org/news/2015-02-dark-growth-supermassive-black-holes.html#jCp** |

**Network theory sheds new light on origins of consciousness by Melanie Moran (APS site)**

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| (5977) **The black dots correspond to (e&eb) the 264 areas of the cerebral cortex that the researchers probed, and the lines correspond to (e&eb) the increased strength of the functional connections between each of these brain areas when subjects consciously perceive the …** |
| (5978) **Where in (eb) your brain do you exist? Is your awareness of the world around you and of yourself as an individual the result of (e) specific, focused changes in (eb) your brain, or does that awareness come from (e) a broad network of neural activity? How does your brain produce (eb) awareness?** |
| (5979) **Vanderbilt University researchers took a significant step toward answering these longstanding questions with a recent brain imaging study, in which they discovered global changes in how brain areas communicate with (e&eb) one another during awareness. Their findings, which were published March 9 in the Proceedings of the National Academy of Sciences, challenge previous theories that hypothesized much more restricted changes were responsible for (e) producing awareness.** |
| (5980) **"Identifying the fingerprints of consciousness in humans would be (=) a significant advancement for basic and medical research, let alone its philosophical implications on (e&eb) the** |

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| | **underpinnings of the human experience," said René Marois, professor and chair of psychology at Vanderbilt University and senior author of the study. "Many of the cognitive deficits observed in various neurological diseases may ultimately stem from (e) changes in how information is (=) communicated throughout the brain."** |
| (5981) | **Using graph theory, a branch of mathematics concerned with explaining the interactive links between (e&eb) members of a complex network, such as (=) social networks or flight routes, the researchers aimed to characterize (eb) how connections between the various parts of the brain were related to (e&eb) awareness.** |
| (5982) | **"With graph theory, one can ask questions about how efficiently the transportation networks in the United States and Europe are connected via (e&eb) transportation hubs like LaGuardia Airport in New York," Douglass Godwin, graduate student and lead author on the research, said. "We can ask those same questions about (e&eb) brain networks and hubs of neural communication."** |
| (5983) | **Modern theories of the neural basis of consciousness fall generally into two camps: focal and global. Focal theories contend there are specific areas of the brain that are critical for generating (eb) consciousness, while global theories argue consciousness arise from (e) large-scale brain changes in activity. This study applied graph theory analysis to (e&eb) adjudicates between these theories.** |
| (5984) | **The researchers recruited 24 members of the university community to participate in a functional magnetic resonance imaging (fMRI) experiment. While in the fMRI scanner, participants were asked to detect (e&eb) a disk that was briefly flashed on a screen.** |
| (5985) | **In each trial, participants responded whether they were able to detect (e&eb) the target disk and how much confidence they had in their answer.** |
| (5986) | **Experimenters then compared the results of the high-confidence trials during which the target was detected to the trials when (e) it was missed by participants.** |
| (5987) | **These were treated as (=) "aware" and "unaware" trials, respectively.** |
| (5988) | **Comparison of aware and unaware trials using (e) conventional fMRI analyses that assess the amplitude of brain activity showed (eb) a pattern of results typical of similar studies, with only a few areas of the brain showing (eb) more activity during detection of the target than when participants missed seeing it.** |
| (5989) | **The present study, however, was interested not simply in what regions might be more activated with (e&eb) awareness, but how they communicate with (e&eb) one another.** |
| (5990) | **Unlike the focal results seen using (e) more conventional analysis methods, the results via (e&eb) this network approach pointed toward (e&eb) a different conclusion.** |
| (5991) | **No one area or network of areas of the brain stood out as (=) particularly more connected during awareness of the target; the whole brain appeared to become (=) functionally more connected following reports of awareness.** |
| (5992) | **"We know there are numerous brain networks that control (e&eb) distinct cognitive functions such as (=) attention, language and control, with each node of a network densely interconnected with (e&eb) other nodes of the same network, but not (e) with other networks," Marois said.** |
| (5993) | **"Consciousness appears to break down (e) the modularity of these networks, as we observed (e) a broad increase in functional connectivity between these networks with awareness."** |

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| (5994) | **The research suggests that consciousness is likely a product of (e) this widespread communication, and that we can only report things that we have seen once they are being represented in (eb) the brain in this manner.** |
| (5995) | **Thus, no one part of the brain is (=) truly the "seat of the soul," as René Descartes once wrote in a hypothesis about the pineal gland, but rather, consciousness appears to be (=) an emergent property of how information that needs to be acted upon (e&eb) gets propagated throughout the brain.** |
| (5996) | **"We take for granted how unified our experience of the world is. We don't (e) experience separate visual and auditory worlds, it's all integrated into (e&eb) a single conscious experience," Godwin said.** |
| (5997) | **"This widespread cross-network communication makes sense as (=) a mechanism by which consciousness gets integrated into (eb) that singular world." More information: Breakdown of the brain's functional network modularity with awareness, Douglass Godwin, DOI: 10.1073/pnas.1414466112 Journal reference: Proceedings of the National Academy of Sciences search and more info website: Provided by Vanderbilt University search and more info website Scientists find 'hidden brain signatures' of consciousness in vegetative state patients** |

**Satellite studies x-ray emission from matter falling into the black hole at the center of a galaxy December 5, 2014 Figure 1: Photograph of galaxy NGC 3227. Credit: Ken Crawford (deleted due to spatial restrictions. Please excuse me. )**

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| (5998) | **Most galaxies are assumed to have (e) at their heart a supermassive black hole that draws in (e) vast amounts of surrounding matter.** |
| (5999) | **As this matter is sucked in,(eb)it releases energy in the form of intense x-ray emissions that in some cases can be more intense than (e) the emission from all the stars in the galaxy combined.** |
| (6000) | **Through detailed study of the x-ray emissions from (e) the center of the galaxy known as NGC 3227, Hirofumi Noda from the RIKEN Nishina Center for (e) Accelerator-Based Science and colleagues have now revealed (eb) the multiple processes responsible for (e) these emissions.** |
| (6001) | **"Our results show the presence of multiple mechanisms of energy conversion of matter near (eb) a central black hole," explains Noda.** |
| (6002) | **The x-ray emissions from the centers of galaxies provide (eb) valuable information on the properties of (e) their central black holes, as well as (=) the surrounding matter and ultimately the history of galaxies in (eb) general.** |
| (6003) | **However, it can be difficult to separate (e&eb) the original radiation emitted by (e) matter close to (e) the black hole from (e) secondary processes caused by (e) other matter around (e&eb) the black hole.** |
| (6004) | **The typical spectral analysis applied to (e&eb) images of galactic centers is (=) unable to clearly distinguish the different emission processes.** |
| (6005) | **Noda and his colleagues instead studied the change in emissions over (e&eb) time.** |
| (6006) | **By analyzing the different radiation patterns and (e&eb) their correlations, they were able to separate primary emission from (e) the emission spectra arising from (e) secondary processes such as reflection.** |
| (6007) | **Using this approach and wide-range x-ray data obtained over (e&eb) the course of a few weeks by (e) the Japanese research satellite Suzaku, the team identified (e&eb) three distinct spectral components from (e) NGC 3227—a typical type I Seyfert galaxy that lies (eb) about 80 million light years from Earth (Fig. 1).** |

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| (6008) **The intensity of the source varied by (e) almost an order of magnitude over (e&eb) the course of the observations, and by analyzing these variations, the researchers could correlate the spectral components with (e&eb) three different mass accretion flows into (e&eb) the galaxy's black hole.** |
| (6009) **Noda believes that studying the ultraviolet emissions from (e) matter at a greater distance from (e) the black hole could also provide (eb) further insight into the detailed physics of (e) supermassive black holes. X-ray emissions, however, are still expected to reveal (eb) much more information if (e) the spectral resolution of observations can be improved.** |
| (6010) **"The soon-to-be-launched Japanese x-ray satellite ASTRO-H will achieve an unprecedentedly fine energy resolution, which will enable (eb) us to understand the properties of black holes, their evolution and their interaction with (e&eb) the galaxy around them." More information: Noda, H., Makishima, K., Yamada, S., Nakazawa, K., Sakurai, S. & Miyake, K. "Suzaku studies of the central engine in the typical type I Seyfert NGC 3227: Detection of multiple primary x-ray continua with distinct properties." Astrophysical Journal 794, 2 (2014). DOI: 10.1088/0004-637X/794/1/2Read more at: http://phys.org/news/2014-12-satellite-x-ray-emission-falling-black.html#jCp** |

**Researchers chip away at Smale's 7th unsolved problem in mathematics July 15, 2016 by Lisa Zyga feature Thompson's problem**

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| (6011) **The disconnectivity graph for (e) 147 charges on (e&eb) a sphere has a structure-seeking "palm tree" organization.** |
| (6012) **The five lowest minimum energy configurations are shown. Credit: Mehta et al. ©2016 American Physical Society** |
| (6013) **(Phys.org)—How do you arrange a group of points on (e&eb) the surface of a sphere so that all the points are as far apart from (e) each other as possible?** |
| (6014) **With two points, the answer is easy: place them on opposite sides of the sphere, as if they are (=)endpoints of the diameter. With three points, make them the vertices of an equilateral triangle, and so on. But as the number of points increases, so does the difficulty of the problem.** |
| (6015) **This puzzle is the essence of Thomson's problem, which asks how to arrange equal charges (such as electrons) on (e&eb)the surface of a sphere in (e) a way that minimizes their electrostatic potential energy—the energy caused by (e) all of the electrons repelling each other.** |
| (6016) **According to Coulomb's law, the total energy is inversely related to (e&eb) the sum of the distances between all possible pairs of charges, so (e) the goal is to spread the charges as far apart as (=) possible.** |
| (6017) **This task is more difficult than it sounds—Thomson's problem has been rigorously solved only for (e) numbers of 2, 3, 4, 6, and 12 charges.** |
| (6018) **In 1998, mathematician Steven Smale identified the problem of how to choose starting points close to (e) the lowest-energy state (which makes it easier to solve Thomson's problem) as (=) the seventh problem on his list of 18 unsolved problems for the 21st century.** |
| (6019) **Part of the reason why Thomson's problem is so important is because (e) its applications are so far-reaching.** |
| (6020) **In 1904, J.J. Thomson originally proposed the model of charges on a sphere to describe (eb) the structure of an atom.** |
| (6021) **Even though experiments disproved this "plum pudding model" long ago, the Thomson problem still has notable applications in (e&eb) chemistry (for understanding how electrons fill (e)** |

**electron shells in atoms), biology (for determining (eb) the arrangements of proteins on the shells of spherical viruses), as well as in physics, computer science, and such practical applications as determining (eb) the optimal placement of communication satellites around the Earth.**

**Spheres on trees**

(6022) **Now in a new paper published in Physical Review Letters, a team of mathematicians, engineers, and scientists from the US, the UK, and Australia has (e) taken a new approach to Thomson's problem that makes it much easier to determine (eb) the lowest-energy configuration.**

(6023) **They have constructed for seven numbers of charges (every third number from 132 to 150), tree-shaped disconnectivity graphs, where (e) the vertical axis or "trunk" corresponds to (e&eb) the energy of a particular charge arrangement.**

(6024) **Each "branch" terminates at (eb) a local minimum, which are (=) the states that have (e) lower energies than all of their neighboring states, and so they are (=) candidates for the ultimate lowest energy state, the global minimum.**

(6025) **By visualizing the problem in this way, the researchers noticed that these particular graphs don't (e) have lots of branches extending from lots of (e) other branches.**

(6026) **Instead, every branch connects to (e&eb) only a few other branches and then to (e&eb) the trunk at regularly spaced energy thresholds, so that (e) the graph resembles a palm tree or single funnel structure.**

(6027) **The researchers found that this "funneled potential energy landscape" is (=) characteristic of a highly ordered structure and displays (eb) characteristics of a small-world network.**

(6028) **As a result, it provides (eb) an important clue for finding the global minimum.**

(6029) **It tells the researchers to start their optimization algorithms using (e) the local minima because, (e) in these networks, it turns out that every local minimum are (=) always within 5-7 steps (branches) of the global minimum.**

(6030) **This is true even for (e) local minima that have (e) much higher energies than the global minimum, and even when (e) the total number of local minima is very large.**

**Thompson's problem 2**

(6031) **By visualizing the problem from a new perspective, the researchers found (eb) that lower-energy configurations have (e) more connections than (e) higher-energy configurations do. Credit: Mehta et al. ©2016 American Physical Society**

(6032) **The disconnectivity graph reveals (eb) other information, such as those lower-energy local minima have (e) more connections to (e&eb) other states than do higher-energy local minima.**

(6033) **The researchers also discovered that the global minimum is (=) always the most highly connected node in the entire network, making it the network's central node.**

**Implications for Thomson's and Smale's problems**

(6034) **Using this insight from the network's single-funneled, small-world structure, finding the global minimum for Thomson's problem becomes (=) much easier than before.**

(6035) **"Our work looks at Smale's seventh problem from (e) a completely different perspective and (e&eb) sheds novel light on it," coauthor Dhagash Mehta, at the University of Notre Dame and the University of Adelaide, told Phys.org. "In this work, methods developed by the theoretical chemistry**

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| | **community have helped understand (eb) a deep mathematical problem. Often it is the other way around."** |
| (6036) | **As the researchers explain, it's easier to solve Thomson's problem in these particular cases than it is to solve Smale's problem (of choosing good starting points). So although the results will likely be useful, they do not go very far toward solving Smale's seventh problem.** |
| (6037) | **"I think 'chip away' is about right," said coauthor David Wales at University Chemical Laboratories in Cambridge, UK. "There is (=) no rigorous mathematical progress on the problem from (e) an analytic point of view."** |
| (6038) | **In the future, the researchers plan to extend this approach to (e&eb) larger numbers of charges.** |
| (6039) | **From earlier work, they expect that landscapes with (e&eb) more than 400 charges will start to display (eb) multiple funnels, so (e) the small-world structure may disappear.** |
| (6040) | **"While we have only shown data for seven numbers, we have strong reasons to believe that the single funnel is (=) a feature for numbers less than 150," said coauthor Halim Kusumaatmaja at Durham University in Durham, UK.** |
| (6041) | **"For larger numbers, there will likely be (=) multiple funnels.** |
| (6042) | **Nonetheless, the network analysis could still be exploited to (e) help us quickly identify candidates for (e) the global minimum."** |
| (6043) | **Other lines of work include exploiting the small-world properties discovered here to improve other optimization algorithms and develop novel algorithms, as well as to incorporate weight and direction into these networks, which may provide additional insight into the Thomson problem.** |
| (6044) | **"The social network analogy for networks of minima of (e) the Thomson problem will go further when (e) we analyze other network properties of these networks of minima," Mehta said.** |
| (6045) | **"Our results will also help in constructing novel algorithms to find (eb) the global minimum more efficiently by (e) exploiting these network properties." More information: Dhagash Mehta et al. "Kinetic Transition Networks for the Thomson Problem and Smale's Seventh Problem." Physical Review Letters DOI: 10.1103/PhysRevLett.117.028301, Also at arXiv: 1605.08459 [cond-mat.soft] Journal reference: Physical Review Letters search and more info website © 2016 Phys.org** |

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**Introduction:**

**Black holes Teviet Creighton and Richard H. Price (2008), Scholarpedia, 3(1):4277. doi:10.4249/scholarpedia.4277 revision #135523 [link to/cite this article] Post-publication activity Curator: Teviet Creighton Dr. Teviet Creighton, University of Texas, Brownsville, TX Richard H. Price, University of Texas, Brownsville, TX Figure 1: Artist's conception of an accretion disk around a black hole. Credit: NASA/CXC/M.Weiss.**

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| **(1057)** | **Black Holes are regions of (e) space in which gravitational fields are so strong that no particle or signal can escape (eb) the pull of gravity.** |
| **(1058)** | **The boundary of this no-escape region is (=) called the event horizon, since distant observers outside the black hole cannot (e) see (cannot get light from) events inside.** |

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| **(1059) Although the fundamental possibility of such an object exists within (eb) Newton's classical theory of gravitation, Einstein's theory of gravity makes (eb)black holes inevitable under some circumstances.** |
| **(1060) Prior to the early 1960s, black holes seemed to be (=) only an interesting theoretical concept with no (e) astrophysical plausibility, but with (e&eb) the discovery of quasars in 1963 it became clear that very exotic astrophysical objects could exist.** |
| **(1061) Nowadays it is taken for granted that black holes do exist in (eb) at least two different forms.** |
| **(1062) Stellar mass black holes are (=) the endpoint of the death of some stars, and supermassive black holes are the result of (e) coalescences in the centers of most galaxies, including our own.** |
| **(1063) No signal can propagate from (e) inside a black hole, but the gravitational influence of a black hole is (=) always present.** |
| **(1064) This influence does not (e) propagate out of the hole; it is permanently present outside, and depends only on (=) the total amount of mass, angular momentum, and electric charge that have gone into forming (eb) the hole.** |
| **(1065) Black holes can be detected through the influence of this (e&eb) strong gravity on the surroundings just outside the hole.** |
| **(1066) In this way, stellar mass holes produce (eb) detectable X-rays, supermassive black holes produce *(e&eb) a wide spectrum of electromagnetic signals, and both types can be inferred from (e) the orbital motion of luminous stars and (e&eb) matter around them.** |
| **(1067) Phenomena involving black holes of any mass can produce (eb) strong gravitational waves, and are of interest as (=) sources for present and future gravitational wave detectors.** |

**Classical vs. relativistic black holes**

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| **(1068) Something like a black hole exists within (eb) Newton's classical theory of gravity.** |
| **(1069) In that theory, an energy argument tells us that there is (=) an escape velocity vesc=2GM/R−−−−−−−√ from (e) the surface of any spherical object of (e) mass M and radius R** |
| **(1070) If this velocity is greater than (e) the speed of light c then (eb) light from this object cannot escape to infinity.** |
| **(1071) Thus the condition for such an "unseeable" object is R<2GM/c2** |
| **(1072) In the classical theory, a particle could overcome (e) this gravity with strong enough engines to provide (eb) the energy needed for escape.** |
| **(1073) This is not so in general relativity, Einstein's theory of gravitation. In that theory, escaping the black hole is (=) equivalent to moving faster than light, impossibility in relativity.** |
| **(1074) To understand the relativistic black hole it is useful to think of (e) space being dragged inward towards (e&eb) a gravitational center, at a faster rate near (eb) the center than far from it.** |
| **(1075) The distance at which space is moving inward at the speed of light represents (eb) the location of the event horizon, since (e) no signal can progress outward through (e&eb) space faster than c .** |
| **(1076) This comparison is more than a metaphor; black hole analog experiments with (e&eb) accelerating gas flows and (e&eb) other phenomena are being designed.** |
| **(1077) An important difference from Newton's theory is that Einstein's, and other relativistic theories of gravitation are (=) nonlinear in the sense that gravitation (as well as mass) can be (=) a source of gravity.** |

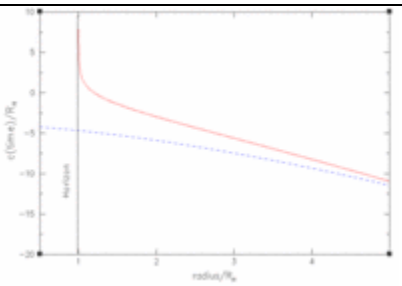
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| **(1078) Thus when a massive object collapses (eb) small enough, the tendency to continue the collapse and form (eb) a black hole can become unstoppable.** |

**Stationary black holes**

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| **(1079) In the Newtonian theory, gravity is described by (e) the potential Φ.** |
| **(1080) Inside a spherical object the form of (e) Φ(r) depends on (=) the interior structure, but in the vacuum outside (e) matter the potential −GM/r depends only on (=) the interior mass.** |
| **(1081) Similarly, in Einstein's theory the stationary (time-independent) spherically symmetric exterior solution, called the Schwarzschild spacetime, depends only on (=) the mass of the interior object.** |
| **(1082) If the interior object is small enough, then the Schwarzschild exterior extends to (e&eb) small enough radius that there is (=) a horizon, a surface across which light cannot (e) move outward.** |
| **(1083) This horizon radius RH=2GM/c2 is, coincidentally, (=) the same as the critical radius for (e) "unseeable" objects in Newton's theory.** |
| **(1084) The meaning of "radius" as distance to (e) the center is not straightforward for (e) the Schwarzschild solution.** |
| **(1085) Radius RHhere actually means that the area of the event horizon is (=) 4πR2H.** |
| **(1086) In Einstein's theory, the "exterior" solution can be taken to apply with no (e) interior solution.** |
| **(1087) In this case it is gravity itself, rather than matter, that acts as (=) the source of gravity.** |
| **(1088) The inward-extended exterior solution does not (e) reach a center, but rather is connected via (e&eb) a spacetime bridge to (e&eb) another universe, or another section of our own.** |
| **(1089) For an astrophysical black hole, formed from (e) the collapse of matter, a physical solution for (e) the matter distribution replaces (e&eb) the pure vacuum Schwarzschild solution in (eb) the interior of the black hole.** |
| **(1090) This physical solution lacks (e) the Spacetime Bridge of ideal mathematical black holes, but contains (e) a central "singularity" where (e) matter is compressed to infinite density.** |
| **(1091) Very close to this singularity it is expected that the laws of general relativity will no longer apply, and as-yet unknown laws of quantum gravity are needed.** |
| **(1092) A more general stationary black hole solution of (e) Einstein's theory is the Kerr solution, a vacuum spacetime with (e&eb) both mass and angular momentum, and taken to represent (eb) a rotating black hole.** |
| **(1093) In its pure mathematical form the Kerr hole contains (e) a spacetime bridge, but as in the case of the Schwarzschild black hole this bridge is absent in (eb) realistic black holes that form by (e) the collapse of matter.** |
| **(1094) Unlike the Schwarzschild spacetime, the Kerr solution is not the exterior spacetime of (e) a material object with (e&eb) angular momentum.** |
| **(1095) In fact no realistic solution has been found to join a Kerr exterior to (e&eb) a material interior.** |
| **(1096) The Kerr solution only becomes (=) the exterior spacetime asymptotically at (eb) very late times after (e) the collapse of an object.** |
| **(1097) Two other exact mathematical black hole solutions are (=) the Reissner-Nordström spacetime, representing (eb) a hole with mass and electrical charge, and the Kerr-Newman spacetime, representing (eb) a hole with mass, electrical charge, and angular momentum.** |

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| **(1098) These spacetimes are not astrophysically relevant, since (e) astrophysical bodies have (e) negligible net electrical charge. [For a detailed description of these spacetimes see, e.g., Misner et al. (1973), Part VII; or Wald (1984), Chap. 12.]** |
| **(1099) All of these spacetimes, including (e) the ones with angular momentum, are stationary: that is, they are (=) independent of time.** |
| **(1100) But in relativity there is (=) no unique meaning to time, so an important question is: "Just what 'time' is it of (e) which the stationary black holes are independent?"** |
| **(1101) The answer lies in the fact that one can assign (e&eb) every spacetime point four coordinates, four labels that uniquely identify (e&eb) the location of each point.** |
| **(1102) One of these coordinates is (=) called the "coordinate time."** |
| **(1103) Spacetimes that are said to be stationary, like the spacetime of a Kerr hole, have (e) a special property: the time coordinate may be chosen so that (e) the spacetime geometry is (=) the same at any moment of this time coordinate.** |



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| **(1104) Figure 2: Coordinate time vs. (e&eb) proper time for the radial fall of (e) a particle towards a Schwarzschild black hole** |
| **(1105) At large distances from a stationary black hole, where (e) spacetime curvatures are weak, this stationary time coordinate can be chosen also to have (e) another important property: to agree with (=) the "proper time," or ordinary clock time, of (e) an observer at rest with respect to (e&eb) the hole.** |
| **(1106) Since we ourselves are more-or-less at rest (or are at (eb) nonrelativistic velocities) very far from black holes, this kind of stationary coordinate time is (=) the time used in astronomical observations.** |
| **(1107) For observers near the hole, however, proper time and (e&eb) stationary coordinate time will not agree** |
| **(1108) An interval of proper time between two events is (=) shorter (near the horizon, much shorter) than the interval of stationary coordinate time between (e&eb) those two events.** |
| **(1109) The relationship between stationary coordinate and (e&eb) proper time is further complicated by (e) special relativistic time dilation for (e&eb) observers who are rapidly moving.** |
| **(1110) Figure 2 illustrates this by showing the coordinate time vs. radius for a particle falling into (eb) a black hole, and comparing it with (e&eb) the proper time measured by (e) an observer riding along with (e&eb) the infalling particle.** |
| **(1111) The progress as measured with proper time is in no way special at (eb) the horizon: falling observers will not notice anything strange as they pass (e&eb) the point of no return.** |
| **(1112) But as described in coordinate time, the observer (and likewise, the surface of a collapsing star) takes an infinite amount of time to reach (eb) the horizon.** |

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| **(1113) Since coordinate time is the proper time of (e) distant observers, astronomers will see the particle reach (eb) the horizon only in the infinite future.** |
| **(1114) The two (or more) types of time are sometimes a source of (e) confusion in discussions of black hole phenomena, since they often give totally different answers to the question "how long does it take?"** |

**Black hole parameters**

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| **(1115) Astrophysical black holes are characterized by (e) two parameters: their mass and their angular momentum (or spin).** |
| **(1116) The mass parameter M is equivalent to (=) a characteristic length GM/c2=1.48km (M/Mo) , or a characteristic timescale GM/c3=4.93×10−6sec(M/Mo) , where (e) Mo denotes the mass of the Sun.** |
| **(1117) These scales, for example, give (eb) the order of magnitude of the radii and (e&eb) periods of near-hole orbits.** |
| **(1118) The timescale also applies to (e&eb) the process in which a developing horizon settles into (e&eb) its asymptotically stationary form.** |
| **(1119) For a stellar mass hole this is of order 10−5sec, while for (e) a supermassive hole of 108Mo, it is thousands of seconds.** |
| **(1120) For Schwarzschild holes, and approximately for Kerr holes, the horizon is (=) at radius RH=2GM/c2.** |
| **(1121) At the horizon, the "acceleration of gravity" has no meaning, since (e) a falling observer cannot stop at the horizon to be weighed.** |
| **(1122) What are relevant at the horizon are (e) the tidal stresses that stretch and distort (e&eb) the falling observer.** |
| **(1123) This tidal stretching is given by (e) the same expression, the gradient of the gravitational acceleration, as in Newtonian theory 2GM/R3H=c6/(4G2M2)** |
| **(1124) In the case of a solar mass black hole the tidal stress (acceleration per unit length) is (=) enormous at the horizon, on the order of 3×109(Mo/M)2s−2 : that is, a person would experience (e) a differential gravitational field of about 109 Earth gravities, enough to rip apart (e&eb) ordinary materials.** |
| **(1125) For a supermassive hole, by contrast, the tidal force at the horizon is smaller by (e) a typical factor 1010--16 and would be easily survivable** |
| **(1126) However, at the central singularity, deep inside the event horizon, the tidal stress is (=) infinite.** |
| **(1127) In addition to its mass M, the Kerr spacetime is described with (e&eb) a spin parameter a defined by (e) the dimensionless expression aM=cJGM2, where J is the angular momentum of the hole.** |
| **(1128) For the Sun (based on (e) surface rotation) this number is about 0.2, and is much larger for many stars.** |
| **(1129) Since angular momentum is ubiquitous in astrophysics, and since it is expected to be (=) approximately conserved during (e&eb) collapse and black hole formation, astrophysical holes are expected to have (e) significant values of a/M, from (e) several tenths up to and approaching unity.** |
| **(1130) The value of a/M can be (=) unity (an "extreme" Kerr hole), but it cannot be (=) greater than unity.** |

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| **(1131) In the mathematics of general relativity, exceeding this limit replaces the event horizon with (e&eb) an inner boundary on the spacetime where tidal forces become (=) infinite.** |
| **(1132) Because this singularity is (=) "visible" to observers, rather than (eb) hidden behind a horizon, as in a black hole, it is called a naked singularity.** |
| **(1133) Toy models and heuristic arguments suggest (eb) that as a/M approaches unity it becomes (=) more and more difficult to add angular momentum.** |
| **(1134) The conjecture that such mechanisms will always keep a/M below (eb) unity is called cosmic censorship.** |
| **(1135) The inclusion of angular momentum changes details of the description of the horizon, so that, for example, the horizon area becomes (=) Horizon area= (4πG2/c4)[(M+M2−a2−−−−−−−√)2+a2].** |
| **(1136) This modification of the Schwarzschild (a=0) result is not significant until (e) a/M becomes very close to (e) unity. For this reason, good estimates can be made in many astrophysical scenarios with a ignored.** |

**Dynamical black holes**

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| **(1137) The event horizon is (=) defined as the outer boundary of the region from which there is (=) no escape.** |
| **(1138) For stationary black holes this surface is at (eb) a fixed location in space, but more generally the horizon is (=) dynamical; it can grow, change shape, and oscillate. In particular, a horizon can be born.** |
| **(1139) The birth of a horizon is a change first studied by Oppenheimer and Snyder for the collapse of spherical pressure less fluid.** |
| **(1140) The general scenario is for a horizon to be (=) born deep inside the collapsing matter and to spread outward.** |
| **(1141) Small changes in a horizon can be treated as (=) perturbations, greatly simplifying the mathematics of Einstein's equations.** |
| **(1142) In this way, it has been found that black holes have (e) characteristic patterns of oscillations, called quasinormal modes.** |
| **(1143) These modes are (=) like mechanical resonances, but are (=) highly damped by the emission of (e) gravitational waves, and have (e) both periods and damping times on the order of (e) the characteristic time GM/c3.** |
| **(1144) Figure 3: Distorted horizons of a black hole binary near (eb) the point of merger: still frame from (e) an inspiral movie. Numerical simulation and animation produced by Frans Pretorius, http://physics.princeton.edu/~fpretori/** |
| **(1145) For large nonspherical changes in (eb) a dynamical horizon, only supercomputer simulations can give (eb) quantitative answers.** |
| **(1146) The focus of such work has been the merger of (e) two black holes in binary orbits around (e&eb) each other, a scenario of special interest as (=) a source of strong gravitational wave emission.** |
| **(1147) Only in 2005 were technical problems first overcome [Pretorius (2005) (See Fig. 3.), Campanelli et al. (2006), Baker et al. (2006)] so that an accurate picture could emerge of how two horizons join to become (=) a single final horizon.** |
| **(1148) There is one change that a horizon cannot make according to classical (nonquantum) general relativity: it cannot (e) decrease its area.** |

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| **(1149) But considerations by Stephen Hawking and others of quantum effects in black hole spacetimes suggest that radiation arising in (eb) the close exterior of the black hole can carry off (e&eb) energy, and decreases (e) the mass (and hence horizon area) of the hole.** |
| **(1150) Although no quantum theory of relativistic gravitation currently exists, it is generally accepted that this Hawking radiation will be (=) a feature of any such theory.** |
| **(1151) The radiation behaves as if the horizon were (=) a blackbody (perfect thermal emitter) at a temperature 6×10−8 (Mo/M) K.** |
| **(1152) Thus for astrophysical black holes, mass loss by (e) Hawking radiation is much less important than the mass increase due to (e) absorption of the 3K cosmic microwave background, and that mass increase is (=) itself negligible.** |

Then $[Y_6]_6[\gamma_6]_6, a_6) = E_6(a_6)e^{-\gamma_6 a_6}$. by a simple calculation,

$$\int_0^\infty \beta_6(a_6)Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma}\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}\ da\right);$$

$$\int_0^\infty Y_6(\gamma_6, a_6)I(\gamma_6, a_6)da = \frac{[\rho_0]_6(0)}{\gamma_6}\left(\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}\ da_6\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_6(0)}{\gamma_6}[f_{60}^6]\int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\ (\int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6)$$

$$+\left(1 - \int_0^\infty \beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right).$$

$$\left(1 + \frac{(\rho_0)_6(0)}{\gamma_6}(f_{60}^{'})_6\left(\int_0^\infty E_6(a_6)da_6 - \int_0^\infty E_6(a_6)e^{-\gamma_6 a_6}da_6\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6\right)(1+\frac{(f_{60}^{'})_6}{\gamma_6}(\rho_0)_6(0)\int_0^{\infty}E_6(a_6)da_6)=0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)da_6 = 1-\int_0^{\infty}\beta_6(a_6)E_6(a_6)e^{-\gamma_6 a_6}da_6 \neq 0,$$

Since $\int_0^{\infty}\beta_6(a_6)Y_6(\gamma_6 a_6)\ \int_0^{a}\alpha_6(\tau_6)Y^{-6}(\gamma_6,\tau_6)d\tau_6 da_6 > 0$. Therefore, the characteristic solution is

$$\gamma_6 = -(f_{60}^{'})_6(\rho_0)_6\int_0^{\infty}E_6(a_6)da_6.$$

Since $f_6^{'}(P_0)_6 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_6),0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT TYPES AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_6),0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a),Q^*)$. It must satisfy the equations:

$$[p_a^*]_6+(\mu_6(a_6)+[f_6]_6(P_6^*)+[f_2]_6(Q_6^*))P_6^*=0, \qquad (26)$$

$p_6^*(0) = h_6(Q_6^*) \int_0^\infty \beta_6(a_6) p_6^*(a_J) p_6^*(a_6) da_6$ (27)

$g_6(P_6^*) - b_6 = 0,$ (28)

Where $P_6^* = \int_0^\infty p_6^*(a_6)\, da_6$

Equation (28) yields $P^* = g^{-6}(b),$

From (26),

$p_6^*(a) = p_6^*(0) e^{\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6}$

Substituting into (27), we need to solve

$$\left(1 - h_6(Q_6^*) \int_0^\infty \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(Q_6^*)a_6} da_6\right) p_6^*(0) = 0$$

For $Q_6^*$

Fix $p_6^*(0)$ and $P_6^*$ Let

$$H_6(z_6) = 1 - h_6(z_6) \int_0^\infty \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6) d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6} da_{,6}$$

And assume that

$$[k_3]_6 \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(g_6^{-6}(b_6))a_6 - [k_2]_6 a_6}\, da_6 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_6(z_6)] > 0.$$

Clearly,

$$\frac{dH_6}{dz_6} = h_6(z)(\,[f_2^{'}]_6(z_6)\,) \int_0^{\infty} a_6\beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - f_6(P_6^*)a_6 - f_2(z_6)a_6} da_6$$

$-h^{'}(z_6)\int_0^{\infty} \beta_6(a_6)\, e^{\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6 - [f_2]_6(z_6)a_6}\ da_6$

By the assumptions on $\left[f_2^6\right]_6 and\ [h^6]^6 dH_6/dz > 0$. Hence, if

$$H_6(0) = 1 - \int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6(P_6^*)a_6} da_6 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_6(0) < 0$ follows if $g_6^-(b_6) < [f_6^{-6}]_6(r_6)$, where $r_6$ is the solution of (12).

Indeed, since

$$R_6(s_6) = \int_0^{\infty} \beta_6(a_6)\, e^{-\int_0^a \mu_6(\tau_6)d\tau_6 e^{-y_6^a} da_6}$$

Is a decreasing function of $s_6$ when $P_6^* = [g^{-6}]_6(b_6) < [f_6^{-6}]_6(r_6)$, that is $[f_6]_6(P_6^*) < r$, $R\big(f_6(P^*)\big) > 1$which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_6(g^{-6}(b))) > 6, i.e.,$

$$\int_0^{\infty} \beta_6(a_6) e^{-\int_0^a \mu_6(\tau_6)d\tau_6 - [f_6]_6\left(g_6^{-6}(b)\right)a_6} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_6^*(a_6), Q_6^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_6^*(a_6), Q_6^*)$ is more complicated. We again consider perturbations $m_6(t_6 a_6) = p_6(t_6, a_6) - p_6^*(a_6)$ and $W_6(t) = Q_6(t) - Q^*$.Then

$$M_6(t) \equiv \int_0^{\infty} m_6(t, a_6) da_6 = P_6(t) - P_6^*,$$

$$[m_6]_6 + [m_a]_6 = [p_6]_6 + [p_a]_6 - \left[p^*\right]_6(a_6) \approx -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6$$

$$-[f_{6*}^6]_6 p_6^*(a_6)M - [f_{2*}^6]_6 p_6^*(\mathrm{a})W_6,$$

$$m_6(t,0)p_6(t,0) - p_6^*(0) \approx h_* \int_0^{\infty} \beta_6(a_6) m_6(t, a_6)\, da + \frac{[h_*^6]_6}{[h_*]_6}\, p_6^*(0) W_6$$

$$\frac{dW_6}{dt} = \frac{dQ_6}{dt} \approx [g_*^6]_6 Q_6^* M_{,6}$$

Where $[f_{6*}]_6$ denoting $[f_6(P^*)]_6$, $[f_{6*}]_6$ denoting $[f_2(P^*)]_6$, $\left[f_{6*}^{'}\right]_6$ denoting $\left[f^{'}{}_6(P^*)\right]_6 \left[f_{2*}^{'}\right]_6$ denoting $[f_2]_6$ $(Q_6^*,$ $h_{*}_6$ denoting $h_*^{'}$ denoting $h_6^{'}$ $(Q_6^*)$, and $\left[g_*^{'}\right]_6$

Denoting $\left[g^{'}\right]_6(P_6^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_6]_6 + [m_a]_6 = -(\mu_6(a_6) + [f_{6*}]_6 + [f_{2*}]_6)m_6 - [f'_{6*}]_6[p_6^*](a_6)M_6 - [f_{2*}']_6 p_6^*(a_6)W_6$$

$$m_6(t,0) = [h_*]_6 \int_0^\infty \beta_6(a_6)m_6(t,a_6)da_6 + \frac{[h_*^6]_6}{[h^*]_6}[p^*]_6(0)\, W_{,6} \qquad (30)$$

$$\frac{dW_6}{dt} = [g_*^6]_6[Q_6^*]M_6$$

Integrating $[m_6]_6(t,a_6)$ with respect to a from 0 to ∞, we have

$$\frac{dM_6(t)}{dt} = \int_0^\infty [mt]_6(t,a_6)da_6$$

$$= -\int_0^\infty [m_a]_6\,(t,a_6)da_6 - \int_0^\infty \mu(a_6)m(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f'_{6*}]_6 \int_0^\infty p_6^*(a_6)da_6\right)M_6 - [f_{2*}^6]_6 \int_0^\infty p_6^*(a_6)da_6 W_6$$

$$= m_6(t,0) - \int_0^\infty \mu_6(a_6)\, m_6(t,a_6)da_6$$

$$-\left([f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 - [f_{2*}']_6 P_6^* W_{\cdot 6}$$

From (30), we arrive at the equations

$$\frac{dW_6(t)}{dt} = \int_0^\infty ([h_*]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6$$

$$-\left([f_{6*}] + [f_{2*}]_6 + [f_{6*}']_6 P_6^*\right)M_6 + \left(\frac{[h_*']_6}{[h_*]^6}\, P_6^*(0) - f_{2*}' P_6^*\right)W_6, \qquad (36)$$

$$\frac{dW_6(t)}{dt}[g^{'}{}_{*}]_6 Q_6^* M_{\cdot 6}$$

By the mean value theorem,

$$\int_0^\infty ([h_{*}]_6\beta_6(a_6) - \mu_6(a_6))m_6(t,a_6)da_6 = ([h_*]_6\beta_6(\theta_6) - \mu_6(\theta_6))M_{\cdot 6}$$

Where $0< \theta_6 = \theta_6(t) < \infty$.

And if $a_6^2\,[a_6^2]_6$<4 $[a_2]_6$<4 $a_2$e,

Re $\lambda_6[\lambda_6]_6 = \mathrm{Re}\lambda_2[\lambda_2]_6 = -\frac{6}{2}a_6[a2\ ]_6 < 0$.

Let ⊿=max$_{2<u<\infty}$ $\left|h_*[h_*]_6\beta_6(a_6) - \mu_6(a_6)\right|$. Then$|n_6(t)|$ ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_6^2$ <4 $a_2$ e and $[a_6] > 2\Delta$,or $[a_6^2]$ [$a_2$]e and $(a_6 - \sqrt{(a_6^2)_6 - 4(a_2)_6 e) > 2\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+6}\right)$ and $\beta_6(a_6) = ae^{-ma}$ Then ⊿=$[\mu_0] + 1$ Simple calculation leads to

$$\int_0^\infty \beta_6(a_6)e^{-\int_0^a \mu_6(\tau_6) + d\tau_6 da} = \frac{6}{(m+\mu_0+1)^2} + \frac{2}{(m+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H1), we assume $m_6 + [\mu_0]$ 0.5. Assume$[f_6](x_6)= K_6x_6$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_6(x_6)= [0x_6]/(1 + x_6)$; the functional responses $[[f_2](x) = [0]\, x_6/(1 + x_6)$and $h_6(x_6)$=(1+$\omega_6 x_6$)/$(1 + x_6)$(See Holling [7] and Rashevsky [13])

Equation (64),now has the form of

$$\frac{m_6 + [\mu_0] + [\mathrm{K}][P_0])}{[m_6]_{\,6} + (\mu_0)_{\,6} + [\mathrm{K}][P_0])\,{}^{3}_{6}\ [\mathrm{K}]_{\,6}[P_0]\,{}^{3}_{6}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(6 + P_0) < b_6$i e $[P_0] < b/(\delta_6 - b_6)$then the equilibrium$(P_0)$ $[P_0]_{\,6}(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_6$

$[f_{6*}] = K_6\ \mathrm{P}^{*}\,[f_{2*}] = \frac{\sigma \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$ $[h_*] = \frac{6+\omega_6 \mathrm{Q}_6^*}{6+\mathrm{Q}_6^*}$

$[f_{6*}] = K_6\ [\mathrm{f}\,{}^{6}_{2*}] = \frac{\sigma_6}{(6+\mathrm{Q}_6^*)^2}[\ \mathrm{g}\,{}^{6}_{*}] = \frac{\sigma_6}{(6+\mathrm{P}_6^*)^2}$ $[\mathrm{h}\,{}^{6}_{*}] = \frac{\omega_6 - 6}{(6+\mathrm{Q}_6^*)^2}$

In order to have (H3) and (H4) hold,$m, \mu_0$, K, b, $\delta$, and $\omega$ need to satisfy

$$\frac{\omega_6}{(m + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma)^{\,2}} + \frac{2\omega_6}{([m_6]_{\,6} + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 + b_6} + \sigma_6)^{\,3}} < 1$$

And

$$\frac{1}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,2}} + \frac{2\omega_6}{(m_6 + [\mu_0]_{\,6} + 1 + \frac{K_6 b_6}{\delta_6 - b_6})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$$(m_6 + [\mu_0]_6 + 1 + \frac{K_6 b_6}{\delta_6 - b_6} + (m_6 + [\mu_0]_6 + \frac{K_6 b_6}{\delta_6 - b_6} + \sigma_6)\, Q_6^*)^3$$

$$= m_6 + [\mu_0]_6 + 3 + \frac{K_6 b_6}{\delta_6 - b_6}(1 + Q^*)(1 + \omega_6 Q_6^*$$

Now

$$\left[a_6^2\right] - 4\left[a_2\right] e = \left(\frac{2K_6 b_6}{\delta_6 - b_6}\right)^2 + \frac{4K_6 b_6 \sigma_6 Q_6^*}{(\delta_6 - b_6)(1 + Q_6^*)} + \frac{\sigma_6^2 Q_6^{*2}}{(1 + Q_6^*)^2}$$

$$\frac{4\, b_6 \sigma_6 (\delta_6 - b) Q^*}{\delta_6 (1 + Q_6^*)^2} - \frac{4\, (\delta_6 - b_6)^2 (1 - \omega_6) Q_6^*}{\delta_6 \left(1 + \omega_6 Q_6^*\right) 1 + Q_6^*)} P * (0)$$

And

$$\Delta = \lim_{0 < a < \infty} |[h_*]_6\, a e_6^{mu}\ [\mu_0]_6 - \frac{a_6}{a_6 + 6}$$

Hence if $[a_6^2 > 4(a_2)_6 e$ and

$$[a_6] = \frac{2K_6 b_6}{\delta_6 - b_6} + \frac{\sigma_6 Q_6^*}{6 + Q_6^*} > 2\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_6]_6 = [f_{6*}]_6 + [f_{2*}]_6 + [f_{6*}]P^*_6$$

$$[a_2]_6 = \left[f^6_{2*}\right]P^*_6 - \frac{h^6_*}{h_*}P^*(0)$$

$$e=[g^6_*]_6 Q^*_6$$

$$\eta_6(t) = \left[h_*\right]_6 \beta_6\left(\theta_6(t)\right) - \mu_6(\theta(t)),$$

The system(36)can be written as the following system:

$$[M(t)]\left[a_6\right]M_6 - [a_2]_6 W_6 - [n(t)]M_6$$

$$w_6(t) = eM_6$$

Write

$$A=\begin{pmatrix} -[a_6]_6 - [a_2]_6 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_6 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_6 \\ W_6 \end{pmatrix}$$

The solution of (32)is

$$X_6(t) = e^{A,t}[X]_6(0) + \int_0 e^{At}$$

Other problems to which the model is applied are: To understand the effect of third order Lovelock gravity, P−V criticality of (e) topological AdS black holes in Lovelock-Born-Infeld gravity is investigated. The thermodynamics is further explored with some more extensions and details than the former literature. A detailed analysis of the limit case $\beta\rightarrow\infty$ is performed for (e) the seven-dimensional black holes. It is shown that for the spherical topology, P−V criticality exists for (e) both the uncharged and charged cases. Results demonstrate again that the charge is (=) not the indispensable condition of P−V criticality. It may be attributed to (e) the effect of higher derivative terms of curvature because (e) similar phenomenon was also found for (e) Gauss-Bonnet black holes. For k=0, there would be (=) no P−V criticality. Interesting findings occur in (eb) the case k=−1, in which positive solutions of critical points are found for (e) both the uncharged and (e&eb) charged cases. However, the P−v diagram is (=) quite strange. To check whether these findings are physical, authors give the analysis on the non-negative definiteness condition of (e) entropy. It is shown that for any nontrivial value of α, the entropy is (=) always positive for any specific volume v. Since no P−V criticality exists for (e) k=−1 in Einstein gravity and Gauss-Bonnet gravity, findings can be related with (e&eb) the peculiar property of (e) third order Lovelock gravity. The entropy in third order Lovelock gravity consists of (e) extra terms which is absent in the Gauss-Bonnet black holes, which makes

(eb) the critical points satisfy (eb)the constraint of non-negative definiteness condition of (e) entropy. Authors also check the Gibbs free energy graph and (e&eb) the "swallow tail" behavior can be observed. Moreover, the effect of nonlinear electrodynamics is (=) also included in research. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: Eur. Phys. J. C (2014) 74:2836 DOI: 10.1140/epjc/s10052-014-2836-0 Cite as: arXiv: 1401.0785 [gr-qc] (or arXiv: 1401.0785v1 [gr-qc] for this version) **P-V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity Jie-Xiong Mo, Wen-Biao Liu STOP** Authors investigate the phase transitions of (e) black holes with (e&eb) conformal anomaly in canonical ensemble from (e) different perspectives. Some interesting and novel phase transition phenomena have been discovered. Firstly, authors discuss the behavior of the specific heat and (e&eb) the inverse of the isothermal compressibility. It is shown that there are (=)striking differences in Hawking temperature and (e&eb) phase structure between black holes with (e&eb)conformal anomaly and those without (e) it. In the case with conformal anomaly, there exists (eb) local minimum temperature corresponding to (e&eb) the phase transition point. Phase transitions take place not only from (e) an unstable large black hole to (e) a locally stable medium black hole but also from (e) an unstable medium black hole to (e) a locally stable small black hole. Secondly, authors probe in detail the dependence of phase transitions on (=) the choice of parameters. The results show (eb) that black holes with (e&eb) conformal anomaly have much richer phase structure than (e) those without it. There would be two, only one or no phase transition points depending on the parameters we have chosen. The corresponding parameter region is derived both numerically and graphically. Thirdly, geometrothermodynamics are built up to (e) examine the phase structure authors have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges (e&eb) exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by (e) calculating the relevant critical exponents. It is proved that these critical exponents satisfy (eb) the thermodynamic scaling laws, leading to the conclusion that critical exponents and the scaling laws can reserve even when we consider conformal

anomaly. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc)Journal reference: Advances in High Energy Physics (2014), Article ID 739454DOI:10.1155/2014/739454 Cite as: arXiv:1312.0679 [hep-th] (or arXiv:1312.0679v3 [hep-th] for this version) **Phase transitions, geometrothermodynamics and critical exponents of black holes with conformal anomaly Jie-Xiong Mo, Wen-Biao Liu STOP** Adopting the strong field limit approach, we investigated the strong gravitational lensing in a Brane-World black hole, which means (eb) that the strong field limit coefficients and (e&eb) the deflection angle in this gravitational field are obtained. With this result, it can be said with certainly that the strong gravitational lensing is related to (e&eb) the metric of gravitational fields closely, the cosmology parameter {\alpha} and the dark matter parameter \b{eta} come from (e) the Brane-World black hole exerts a great influence on it. Comparing with (e&eb) the Schwarzschild-AdS spacetime and (e&eb) the Schwarzschild-XCMD spacetime, the parameters {\alpha}, \b{eta} of (e) black holes have (e) the similar effects on (e&eb) the gravitational lensing. In some way, authors infer that the real gravitational fields in our universe can be described by (e) this metric, so the results of (e) the strong gravitational lensing in this spacetime will be more reasonable for (e) us to observe. Finally, it has to be noticed that the influence which the parameters {\alpha}, \b{eta} exerted on (e&eb) the main observable quantities of (e) this gravitational field is (=) discussed. Subjects: General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1007/s10773-015-2545-y Cite as: arXiv:1506.08410 [gr-qc] (or arXiv:1506.08410v1 [gr-qc] for this version) **Strong Gravitational Lensing in a Brane-World Black Hole GuoPing Li, Biao Cao, Zhongwen Feng, Xiaotao Zu STOP** In this Letter, authors investigate the effects of dark energy on P−V criticality of charged AdS black holes by considering the case of the RN-AdS black holes surrounded by quintessence. By treating the cosmological constant as (=) thermodynamic pressure, authors study its thermodynamics in (eb) the extended phase space. It is shown that quintessence dark energy does not (e) affect the existence of small/large black hole phase transition. For the case ωq=−2/3 authors derive analytic expressions of (e) critical physical quantities, while (e) for

cases $\omega_q \neq -2/3$ appeal to (e) numerical method for help. It is shown that quintessence dark energy affects (e&eb) the critical physical quantities near (eb) the critical point. Critical exponents are also calculated. They are exactly the same as (=) those obtained before for (e) arbitrary other AdS black holes, which implies (eb) that quintessence dark energy does not (e) change the critical exponents. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO) Journal reference: Physics Letters B (2014), pp.256-260 DOI: 10.1016/j.physletb.2014.06.047 Cite as:arXiv:1407.0011 [gr-qc] (or arXiv:1407.0011v1 [gr-qc] for this version) **Effects of dark energy on P-V criticality of charged AdS black holes Gu-Qiang Li STOP** Monitoring the orbits of stars around Sgr A* offers the possibility of detecting the precession of (e) their orbital planes due to frame dragging, of measuring the spin and quadrupole moment of the black hole, and of testing the no-hair theorem. Here authors investigate whether the deviations of stellar orbits from (e) test-particle trajectories due to (e) wind mass loss and tidal dissipation of (e) the orbital energy compromise such measurements. Authors find that the effects of stellar winds are, in general, negligible. On the other hand, for the most eccentric orbits (e>0.96) for (e) which an optical interferometer, such as GRAVITY, will detect (e&eb) orbital plane precession due to (e) frame dragging, the tidal dissipation of orbital energy occurs at (eb) timescales comparable to the timescale of precession due to (e) the quadrupole moment of the black hole. As a result, this non-conservative effect is a potential source of systematic uncertainty in testing the no-hair theorem with stellar orbits.Subjects:High Energy Astrophysical Phenomena (astro-ph.HE); Earth and Planetary Astrophysics (astro-ph.EP); General Relativity and Quantum Cosmology (gr-qc) DOI: 10.1088/0004-637X/777/1/57 Cite as: arXiv:1212.3342 [astro-ph.HE] (or arXiv:1212.3342v1 [astro-ph.HE] for this version) **Deviation of Stellar Orbits from Test Particle Trajectories Around Sgr A* Due to Tides and Winds Dimitrios Psaltis (Arizona), Gongjie Li (Harvard), Abraham Loeb (Harvard) STOP** Authors perform a phase space analysis of a generalized modified gravity theory with non-minimal coupling between geometry and matter. Authors apply the dynamical system approach to (e&eb)

this generalized model and find that in the cosmological context, different choices of (e) Lagrangian density will apparently result in (eb) different phases of the universe. By carefully choosing the variables, authors prove that there is an attractor solution to describe (eb)the late time accelerating universe when (e) the modified gravity is(=) chosen in a simple power-law form of (e) the curvature scalar. Authors further examine the temperature evolution based on (e) the thermodynamic understanding of the model. Confronting the model with (e&eb) temperature-redshift and (e&eb) supernova type Ia data sets; authors find that the non-minimally coupled theory of (e) gravity is (=) a viable model to describe (eb) the late time universe acceleration. Subjects: General Relativity and Quantum Cosmology (gr-qc); Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1512.09281 [gr-qc] (or arXiv:1512.09281v1 [gr-qc] for this version) **Dynamical analysis of modified gravity with non-minimal gravitational coupling to matter Rui An, Xiaodong Xu, Bin Wang, Yungui Gong STOP** The tensor-vector-scalar (TeVeS) model is considered a viable theory of gravity. It produces (eb) the Milgrom's modified Newtonian dynamics in (eb) the nonrelativistic weak field limit and is (=) free from ghosts. This model has been tested against (e&eb) various cosmological observations. Here authors investigate whether new observations such as the galaxy velocity power spectrum measured by (e) 6dF and the kinetic Sunyaev Zel'dovich effect power spectrum measured by (e) ACT/SPT can put further constraints on (eb) the TeVeS model. Furthermore, authors perform the test of TeVeS cosmology with (e&eb) a sterile neutrino by (e) confronting to (e) Planck data, and find that (eb) it is ruled out by (e) cosmic microwave background measurements from (e) the Planck mission. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Journal reference: Phys. Rev. D 92, 083505 (2015) DOI:10.1103/PhysRevD.92.083505 Cite as: arXiv:1412.4073 [astro-ph.CO] (or arXiv:1412.4073v3 [astro-ph.CO] for this version) **Testing the tensor-vector-scalar Theory with the latest cosmological observations Xiao-dong Xu, Bin Wang, Pengjie Zhang STOP** Using the anti–de Sitter/conformal field theory correspondence,

authors relate the shear viscosity η of the finite-temperature N=4 supersymmetric Yang-Mills theory in the large N, strong-coupling regime with (e&eb) the absorption cross section of (e) low-energy gravitons by (e) a near-extremal black three-brane. **G. Policastro, D. T. Son, and A. O. Starinets** show that in the limit of zero frequency this cross section coincides with (=) the area of the horizon. From this result they find η=π8N2T3. They conjecture that for finite't Hooft coupling g2YMN the shear viscosity is (=) η=f(g2YMN)N2T3, where f(x) is a monotonic function that decreases from O(x−2ln−1(1/x)) at small x to π/8 when x→∞. Received 14 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevLett.87.081601 **Shear Viscosity of Strongly Coupled N=4 Supersymmetric Yang-Mills Plasma G. Policastro, D. T. Son, and A. O. Starinets Phys. Rev. Lett. 87, 081601 – Published 2 August 2001 STOP** Authors use the AdS/CFT correspondence to determine the rate of energy loss of a heavy quark moving through (e&eb) Script N = 4 SU (Nc) supersymmetric Yang-Mills plasma at large't Hooft coupling. Using the dual description of the quark as a classical string ending on a D7-brane, **Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1** use a complementary combination of analytic and numerical techniques to determine (eb) the friction coefficient as a function of quark mass. Provided strongly coupled Script N = 4 Yang-Mills plasma is (=) a good model for hot, strongly coupled QCD, results may be relevant for (e) charm and bottom physics at RHIC. **Energy loss of a heavy quark moving through Script N = 4 supersymmetric Yang-Mills plasma Christopher P. Herzog1, Andreas Karch1, Pavel Kovtun2, Can Kozcaz1 and Laurence G. Yaffe1 Published 11 July 2006 Journal of High Energy Physics, Volume 2006, JHEP07(2006) STOP Benjamin Svetitsky** calculates the classical drag and diffusion coefficients for (e) a charmed quark propagating in the quark-gluon plasma. Both coefficients turn out rather large, so that (1) a charmed quark created when (e) the plasma is hot will be (=) stopped before propagating 1 fm and (2) subsequent diffusion will be fast. The first effect should serve to (e) increase the yield of J/ψ mesons in (eb) relativistic heavy-ion collisions, while the second should work in the opposite direction. In any case, the two effects should dominate (e) the

dynamics of a $c\bar{c}$ pair. Received 16 November 1987 DOI: http://dx.doi.org/10.1103/PhysRevD.37.2484 **Diffusion of charmed quarks in the quark-gluon plasma Benjamin Svetitsky Phys. Rev. D 37, 2484 – Published 1 May 1988 STOP** Using holographic duality, authors present results for (e) both head-on and off-center collisions of Gaussian shock waves in (eb) strongly coupled N=4 supersymmetric Yang-Mills theory. The shock waves superficially resemble (e&eb) Lorentz contracted colliding protons. The collisions results in (eb) the formation of a plasma whose evolution is well described by (e) viscous hydrodynamics. The size of the produced droplet is R~1/Teff where (e) Teff is the effective temperature, which is (=) the characteristic microscopic scale in strongly coupled plasma. These results demonstrate (eb) the applicability of hydrodynamics to microscopically small systems and bolster (eb) the notion that hydrodynamics can be applied to (e&eb) heavy-light ion collisions as well as some proton-proton collisions. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1601.01583 [hep-th] (or arXiv: 1601.01583v2 [hep-th] for this version) **How big are the smallest drops of quark-gluon plasma? Paul M. Chesler STOP Paul M. Chesler, Krishna Rajagopal** calculate how the energy and (e&eb) the opening angle of jets in N=4 SYM theory evolve as they propagate through (e&eb) the strongly coupled plasma of that theory. **Paul M. Chesler, Krishna Rajagopal** define the rate of energy loss dEjet/dx and (e&eb) the jet opening angle in a straightforward fashion directly in the gauge theory before (e) calculating both holographically, in (eb) the dual gravitational description. In this way, authors rederive the previously known result for (e) dEjet/dx without (e) the need to introduce (e&eb) a finite slab of plasma. They obtain a striking relationship between the initial opening angle of the jet, which is to say the opening angle that it would have had if it had found itself in vacuum instead of in plasma, and (e&eb) the thermalization distance of the jet. Via this relationship, they show that N=4 SYM jets with (e&eb) any initial energy that have the same initial opening angle and the same trajectory through (e&eb) the plasma experience the same fractional energy loss. They

also provide an expansion that describes how the opening angle of the N=4 SYM jets increases slowly as they lose (e) energy, over (e&eb) the fraction of their lifetime when (e) their fractional energy loss is (=) not yet large. Authors close by looking ahead toward potential qualitative lessons from (e) results for QCD jets produced in (eb) heavy collisions and propagating through (e&eb) quark-gluon plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1511.07567 [hep-th] (or arXiv:1511.07567v2 [hep-th] for this version) **On the Evolution of Jet Energy and Opening Angle in Strongly Coupled Plasma Paul M. Chesler, Krishna Rajagopal STOP Paul M. Chesler, Niki Kilbertus, Wilke van der Schee study** the collision of planar shock waves in (eb) AdS5 as a function of (e) shock profile. In the dual field theory the shock waves describe (eb) planar sheets of energy whose collision results in (eb) the formation of a plasma which behaves (e&eb) hydrodynamically at late times. They find that the post-collision stress tensor near (eb) the light cone exhibits (eb) transient non-universal behavior which depends on (=) both the shock width and (e&eb) the precise functional form of (e) the shock profile. However, over a large range of shock widths, including those which yield (eb) qualitative different behavior near (eb) the future light cone, and for (e) different shock profiles, authors find universal behavior in (eb) the subsequent hydrodynamic evolution. Additionally, authors compute the rapidity distribution of (e) produced particles and find it to be well described by (e) a Gaussian. Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) DOI: 10.1007/JHEP11 (2015)135 Cite as: arXiv: 1507.02548 [hep-th] (or arXiv: 1507.02548v2 [hep-th] for this version) **Universal hydrodynamic flow in holographic planar shock collisions Paul M. Chesler, Niki Kilbertus, Wilke van der Schee STOP** Using numerical holography, authors study the collision, at non-zero impact parameter, of bounded, localized distributions of energy density chosen to (e) mimic relativistic heavy ion collisions, in (eb) strongly coupled N = 4 supersymmetric Yang-Mills theory. Both longitudinal and transverse dynamics in the dual

field theory are (=) properly described. Using the gravitational description we solve 5D Einstein equations, without dimensionality reducing symmetry restrictions, to find the asymptotically anti-de Sitter spacetime geometry. Implications of **Paul M. Chesler, Laurence G. Yaffe** results on the understanding of early stages of heavy ion collisions, including the development of transverse radial flow, are discussed. Comments: Enlarged JHEP version, 14 pages Subjects: High Energy Physics - Theory (hep-th); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Cite as: arXiv: 1501.04644 [hep-th] (or arXiv:1501.04644v2 [hep-th] for this version) **Holography and off-center collisions of localized shock waves Paul M. Chesler, Laurence G. Yaffe STOP Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** study the dynamic after a smooth quench across a continuous transition from the disordered phase to the ordered phase. Based on scaling ideas, linear response and (e&eb) the spectrum of unstable modes, authors develop a theoretical framework, valid for (e) any second order phase transition, for the early-time evolution of the condensate in the broken phase. Their analysis unveils (eb) a novel period of non-adiabatic evolution after the system passes through (e&eb) the phase transition, where (e) a parametrically large amount of coarsening occurs before (e) a well-defined condensate forms. Formalism predicts (eb) a rate of defect formation parametrically smaller than (eb) the Kibble-Zurek prediction and yields (eb) a criterion for (e) the break-down of Kibble-Zurek scaling for sufficiently fast quenches. **Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu** numerically test formalism for (e) a thermal quench in (eb) a 2 + 1 dimensional holographic superfluid. These findings, of (e) direct relevance in a broad range of fields including cold atom, condensed matter, statistical mechanism and cosmology, ar (=)e an important step towards a more quantitative understanding of dynamical phase transitions. Subjects: High Energy Physics - Theory (hep-th); Quantum Gases (cond-mat.quant-gas); Statistical Mechanics (cond-mat.stat-mech) Journal reference: Phys. Rev. X 5, 021015 (2015) DOI: 10.1103/PhysRevX.5.021015 Cite as: arXiv:1407.1862 [hep-th] (or arXiv:1407.1862v2

[hep-th] for this version) **Defect formation beyond Kibble-Zurek mechanism and holography Paul M. Chesler, Antonio M. Garcia-Garcia, Hong Liu STOP Paul M. Chesler, Krishna Rajagopal** present calculations in which an energetic light quark shoots through (e&eb) a finite slab of strongly coupled N=4 supersymmetric Yang-Mills (SYM) plasma, with thickness L, focussing on what comes out on (eb) the other side. Authors find that even when the "jets" that emerge from (e) the plasma have lost a substantial fraction of (e) their energy they look in almost all respects like "jets" in (eb) vacuum with (e&eb) the same reduced energy. The one possible exception is (=) that the opening angle of the "jet" is (=) larger after passage through (e&eb) the slab of plasma than before. Along the way, authors obtain (eb) a fully geometric characterization of energy loss in (eb) the strongly coupled plasma and show that (eb) dE out/dL∝L2/x2stop−L2−−−−−−−−√, where Eout is the energy of the "jet" that emerges from (e) the slab of plasma and x stop is (=) the (previously known) stopping distance for (e) the light quark in an infinite volume of (e) plasma. Subjects: High Energy Physics - Theory (hep-th); High Energy Physics - Phenomenology (hep-ph); Nuclear Theory (nucl-th) Journal reference: Phys. Rev. D 90, 025033 (2014) DOI: 10.1103/PhysRevD.90.025033 Cite as: arXiv: 1402.6756 [hep-th] (or arXiv: 1402.6756v1 [hep-th] for this version) **Jet quenching in strongly coupled plasma Paul M. Chesler, Krishna Rajagopal STOP** Quasinormal modes are (=) eigenmodes of dissipative systems. Perturbations of classical gravitational backgrounds involving (e&eb) black holes or branes naturally lead to (eb) quasinormal modes. The analysis and classification of (e) the quasinormal spectra require solving (e) non-Hermitian eigenvalue problems for (e) the associated linear differential equations. Within the recently developed gauge-gravity duality, these modes serve as (=) an important tool for (e) determining the near-equilibrium properties of (e) strongly coupled quantum field theories, in particular their transport coefficients, such as (=) viscosity, conductivity and diffusion constants. In astrophysics, the detection (e&eb) of quasinormal modes in gravitational wave experiments would allow (eb) precise measurements of (e) the mass and spin of black holes as well as new tests of general relativity. This review is meant as an introduction to the subject, with a focus on the

recent developments in the field. **Quasinormal modes of black holes and black branes Emanuele Berti1,2, Vitor Cardoso1,3 and Andrei O Starinets4 Published 24 July 2009 • 2009 IOP Publishing Ltd Classical and Quantum Gravity, Volume 26, Number 16 STOP Gary T. Horowitz and Veronika E. Hubeny** investigate the decay of a scalar field outside (e) a Schwarzschild anti–de Sitter black hole. This is determined by computing (e&eb) the complex frequencies associated with (e&eb) quasinormal modes. There are qualitative differences from (e) the asymptotically flat case, even in (eb) the limit of small black holes. In particular, for a given angular dependence, the decay is (=) always exponential—there are (=) no power law tails at late times. In terms of the AdS-CFT correspondence, a large black hole corresponds to (e&eb) an approximately thermal state in the field theory, and the decay of the scalar field corresponds to (e&eb) the decay of a perturbation of this state. Thus one obtains the time scale for the approach to (e) thermal equilibrium. **Gary T. Horowitz and Veronika E. Hubeny** compute (e&eb) these time scales for the strongly coupled field theories in three, four, and six dimensions, which are (=)dual to string theory in asymptotically AdS spacetimes. Received 27 September 1999 DOI: http://dx.doi.org/10.1103/PhysRevD.62.024027 **Quasinormal modes of AdS black holes and the approach to thermal equilibrium Gary T. Horowitz and Veronika E. Hubeny Phys. Rev. D 62, 024027 – Published 27 June 2000 STOP** Gravitational waves emitted by (e) perturbed black holes or relativistic stars are dominated by (e) `quasinormal ringing', damped oscillations at (eb) single frequencies which are (=) characteristic of the underlying system. These quasinormal modes have been studied for (e) a long time, often with the intent of describing the time evolution of (e) a perturbation in terms of (e&eb) these modes in a way very similar to a normal-mode analysis. In this review, authors summarize how quasinormal modes are defined and computed. **Hans-Peter Nollert** finds why they have been regarded as closely analogous to normal modes, and discovers why they are actually quite different. Authors also discuss how quasinormal modes can be used in (eb) the analysis of a gravitational wave signal, such as will hopefully be detected in (e&eb) the near future. **Quasinormal modes: the characteristic `sound' of black**

**holes and neutron stars Hans-Peter Nollert STOP** Perturbations of black holes, initially considered in the context of possible observations of astrophysical effects, have been studied for the past 10 years in string theory, brane-world models, and quantum gravity. Through the famous gauge/gravity duality, proper oscillations of (e) perturbed black holes, called quasinormal modes, allow (eb) for the description of (e) the hydrodynamic regime in (eb) the dual finite temperature field theory at (eb) strong coupling, which can be used to (e) predict the behavior of quark-gluon plasmas in (eb) the nonperturbative regime. On the other hand, the brane-world scenarios assume (eb) the existence of extra dimensions in nature, so that (e) multidimensional black holes can be formed in (eb) a laboratory experiment. All this stimulated active research in the field of perturbations of (e) higher-dimensional black holes and branes during recent years. In this review recent achievements on various aspects of black hole perturbations are discussed such as decoupling of variables in the perturbation equations, quasinormal modes (with special emphasis on various numerical and analytical methods of calculations), late-time tails, (e&eb) gravitational stability, (e&eb) anti–de Sitter/conformal field theory interpretation of (e) quasinormal modes, and holographic superconductors. Authors also touch on state-of-the-art observational possibilities for (e) detecting quasinormal modes of black holes. DOI: http://dx.doi.org/10.1103/RevModPhys.83.793 © 2011 American Physical Society **Quasinormal modes of black holes: From astrophysics to string theory R. A. Konoplya and Alexander Zhidenko Rev. Mod. Phys. 83, 793 – Published 11 July 2011 STOP Vitor Cardoso and José P. S. Lemos** study the quasinormal modes (QNM) of electromagnetic and gravitational perturbations of a Schwarzschild black hole in (eb) an asymptotically anti–de Sitter (AdS) spacetime. Some of the electromagnetic modes do not (e) oscillate; they only decay, since they have (e) pure imaginary frequencies. The gravitational modes show (eb) peculiar features: the odd and even gravitational perturbations no longer have (e) the same characteristic quasinormal frequencies. There is a special mode for odd perturbations whose behavior differs completely from (e) the usual one in scalar and electromagnetic perturbations in AdS spacetime, but has a similar behavior to the Schwarzschild black hole in an asymptotically flat spacetime: the

imaginary part of the frequency goes as 1/r+, where r+ is the horizon radius. **Vitor Cardoso and José P. S. Lemos** also investigate the small black-hole limit showing that the imaginary part of the frequency goes as r2+. These results are important to the AdS/CFT conjecture since, according to it, the QNM's describe the approach to equilibrium in the conformal field theory. Received 12 April 2001 DOI: http://dx.doi.org/10.1103/PhysRevD.64.084017 **STOP Gianluca Calcagni** studies the ultraviolet complete non-relativistic theory recently proposed by Hořava. After introducing a Lifshitz scalar for a general background, authors analyze the cosmology of (e) the model in Lorentzian and Euclidean signature. Vacuum solutions are found and it is argued the existence of (e) non-singular bouncing profiles. They find a general qualitative agreement with both the picture of Causal Dynamical Triangulations and (e&eb) Quantum Einstein Gravity. However, inflation driven by (e) a Lifshitz scalar field on a classical background might not (e) produce a scale-invariant spectrum when (e) the principle of detailed balance is assumed. **Cosmology of the Lifshitz universe Gianluca Calcagni Published 25 September 2009 • Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Petr Hořavaa, , Edward Wittenb** propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as (=) the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. **Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Petr Hořava1** proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves

(e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2. This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point" with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized ground states, correlated across all general. **Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009)STOP Petr Hořava** presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the **Cotton tensor**. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. **Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009STOP** Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise

mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of (e) quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from (e) rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. **Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova,c** address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. **On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** identify signatures of chaos in the dynamics of discord in a multiqubit system collectively modeled as (=) a quantum kicked top.

Evolution of discord between any two qubits is quasiperiodic in regular regions, while (e) in chaotic regions the quasiperiodicity is lost. As the initial wave function is varied from (e) the regular regions to the chaotic sea, a contour plot of the time-averaged discord remarkably reproduces (eb) the structures of the classical stroboscopic map. **Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose** also find surprisingly **opposite behavior** of two-qubit discord versus (e&eb) entanglement of the two qubits as measured by the concurrence. Results provide evidence of signatures of (e) chaos in dynamically generated discord. http://dx.doi.org/10.1103/PhysRevE.91.032906©2015 American Physical Society **Signatures of chaos in the dynamics of quantum discord Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trottier, and Shohini Ghose Phys. Rev. E 91, 032906 – Published 10 March 2015 STOP.** How do closed quantum many-body systems driven out of (e) equilibrium eventually achieve equilibration? And how do these systems thermalize, given that they comprise so many degrees of freedom? Progress in answering these—and related—questions has accelerated in recent years—a trend that can be partially attributed to success with experiments performing quantum simulations using ultracold atoms and trapped ions. **J. Eisert, M. Friesdorf & C. Gogolin** provide (eb) an overview of this progress, specifically in studies probing dynamical equilibration and (e&eb) thermalization of systems driven out of (e) equilibrium by quenches, ramps and periodic driving. In doing so, they also address (e&eb) topics such as the eigenstates thermalization hypothesis, typicality, transport, many-body localization and universality near phase transitions, as well as future prospects for quantum simulation. **Quantum many-body systems out of equilibrium J. Eisert, M. Friesdorf& C. Gogolin Nature Physics 11, 124–130 (2015) doi: 10.1038/nphys3215 Received 11 August 2014 Accepted 03 December 2014 published online 03 February 2015 STOP.** Quantum teleportation is (=) one of the most important protocols in quantum information. By exploiting the physical resource of entanglement, quantum teleportation serves as a key primitive in a variety of quantum information tasks and represents an important building block for quantum technologies, with a pivotal role in the continuing progress of

quantum communication, quantum computing and quantum networks. **Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein** review the basic theoretical ideas behind quantum teleportation and its variant protocols. They focus on the main experiments, together with (e&eb) the technical advantages and disadvantages associated with (e&eb) the use of the various technologies, from photonic qubits and (e&eb) optical modes to atomic ensembles, trapped atoms, and (e&eb) solid-state systems. Analysing the current state-of-the-art, authors finish by discussing open issues, challenges and potential future implementations. Cite as: arXiv: 1505.07831 [quant-ph] (or arXiv: 1505.07831v1 [quant-ph] for this version) **Advances in Quantum Teleportation Stefano Pirandola, Jens Eisert, Christian Weedbrook, Akira Furusawa, and Samuel L. Braunstein (Submitted on 28 May 2015) STOP. Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are channels for which the coherent information is zero, but which nonetheless have capacity. Transmitting data reliably over (e&eb) noisy communication channels is one of the most important applications of (e) information theory, and is well understood for (e) channels modeled by classical physics. However, when quantum effects are involved, (eb) one do not know how to compute channel capacities. This is because the formula for (e) the quantum capacity involves (e&eb) maximizing the coherent information over (e&eb) an unbounded number of (e) channel uses. In fact, entanglement across channel uses can even increase the coherent information from zero to non-zero. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** study the number of channel uses necessary to detect positive coherent information. In all previous known examples, two channel uses already sufficed. It might be that only a finite number of channel uses are always sufficient. **Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk** show that this is not the case: for any number of uses, there are (=) channels for which the coherent information is (=) zero, but which nonetheless have (e) capacity.

**Unbounded number of channel uses may be required to detect quantum capacity Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García & Sergii Strelchuk Nature Communications 6, Article number: 6739 doi:10.1038/ncomms7739 Published 31 March 2015 STOP.** Characterizing the behaviour of strongly coupled quantum systems out of (e) equilibrium is a cardinal challenge for (e) both theory and experiment. With diverse applications ranging from the dynamics of the quark–gluon plasma to transport in novel states of quantum matter, establishing universal results and organizing principles out of equilibrium is crucial. **M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm** present a universal description of energy transport between quantum critical heat baths in arbitrary dimension. Current-carrying non-equilibrium steady state (NESS) is a Lorentz-boosted thermal state. In the context of gauge/gravity duality this reveals an intimate correspondence between far-from-equilibrium transport and black hole uniqueness theorems. Authors provide analytical expressions for (e) the energy current and the generating function of (e) energy current fluctuations, together with (e&eb) predictions for experiment. **Energy flow in (eb) quantum critical systems far from (e) equilibrium M. J. Bhaseen, Benjamin Doyon, Andrew Lucas& Koenraad Schalm Nature Physics 11, 509–514 (2015) doi: 10.1038/nphys3320: Published online 04 May 2015 STOP.** Properties of systems near quantum critical points (QCPs) have been studied extensively. A QCP is (=) a point across which the symmetry of (e) the ground state of a quantum system changes in (eb) a fundamental way; such a point can be accessed by (e) changing some parameter, say λ, in the Hamiltonian governing the system. The change in the ground state across (e&eb) a QCP is mediated by (e) quantum fluctuations. Unlike conventional thermal critical points, thermal fluctuations do not (e) play a crucial role in such transitions. Similar to its thermal counterparts, the low-energy physics near (e&eb) a QCP is associated with a number of critical exponents which characterize the universality class of such a transition. Among these exponents, the dynamical critical exponent z provides (eb) the signature of the relative scaling of space and time at (eb) the transition and has (e) no counterpart in thermal phase transitions.

The other exponent which is going to be important for (e) the purpose of this review is (=) the well-known correlation length exponent ν. These exponents are formally defined as follows. As we approach the critical point at λ=λc, the correlation length diverges as ξ∼|λ−λc|−ν, while (e) the gap between the ground state and first excited state vanishes as (e) ΔE∼ξ−z∼|λ−λc|zν. Exactly at the critical point λ=λc, the energy of the low-lying excitations vanishes at (eb) some wave number k 0 as ω∼|k−k0|z. The critical exponents are independent of (e) the details of the microscopic Hamiltonian; they depend only on a few parameters such as the dimensionality of the system and the symmetry of the order parameter. These features render (eb) the low-energy equilibrium physics of a quantum system near a QCP truly universal. **Quantum Quenching, Annealing and Computation Lecture Notes in Physics Volume 802, 2010, pp 21-56 Date: 08 Mar 2010 Non-equilibrium Dynamics of Quantum Systems: Order Parameter Evolution, Defect Generation, and Qubit Transfer S. Mondal, D. Sen, K. Sengupta STOP Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl** propose a class of qubit networks that admit (e) the perfect state transfer of (e) any quantum state in a fixed period of time. Unlike many other schemes for (e) quantum computation and communication, these networks do not (e) require qubit couplings to be switched on and off. When restricted to (e) N-qubit spin networks of identical qubit couplings, they show that (eb) 2log3N is the maximal perfect communication distance for (e) hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then (eb) perfect state transfer can be achieved over (e&eb) arbitrarily long distances in a linear chain. **Perfect State Transfer in Quantum Spin Networks Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl Phys. Rev. Lett. 92, 187902 – Published 4 May 2004 STOP.** As with classical information processing, a quantum information processor requires (e) bits (qubits) that can be independently addressed (e&eb) and read out, long-term memory elements to store (e) arbitrary quantum states1, 2, and the ability to transfer (e&eb) quantum information through (e&eb) a coherent communication bus accessible to (e) a large number of qubits3, 4. Superconducting qubits made with (e&eb) scalable microfabrication techniques

are (=) a promising candidate for the realization of (e) a large-scale quantum information processor5, 6, 7, 8, and 9. (For references please see the paper). Although these systems have successfully passed tests of (e) coherent coupling for (e) up to four qubits10, 11, 12, 13, communication of individual quantum states between (e&eb) superconducting qubits via (e&eb) a quantum bus has not yet been realized. Here, authors perform an experiment demonstrating the ability to coherently transfer quantum states between (e&eb) two superconducting Josephson phase qubits through a quantum bus. This quantum bus is (=) a resonant cavity formed by an open-ended superconducting transmission line of length 7 mm. After preparing an initial quantum state with (e&eb) the first qubit, this quantum information is transferred and stored (e&eb) as a nonclassical photon state of (e) the resonant cavity, then retrieved later by (e) the second qubit connected to (e&eb) the opposite end of the cavity. Beyond simple state transfer, these results suggest (eb) that a high-quality-factor superconducting cavity could also function as (=) a useful short-term memory element. The basic architecture presented here can be expanded, offering (eb) the possibility for the coherent interaction of (e&eb) a large number of superconducting qubits. **Nature 449, 438-442 (27 September 2007) | doi:10.1038/nature06124; Received 18 April 2007; Accepted 25 July 2007 Coherent quantum state storage and transfer between two phase qubits via a resonant cavity Mika A. Sillanpää1, Jae I. Park1 & Raymond W. Simmonds1 STOP. Chui-Ping Yang, Shih-I Chu, and Siyuan** present a scheme to achieve maximally entangled states, controlled (e&eb) phase-shift gate, and SWAP gate for (e) two superconducting-quantum-interference-devices (SQUID) qubits, by (e) placing SQUIDs in a microwave cavity. **Chui-Ping Yang, Shih-I Chu, and Siyuan** also show how to transfer quantum information from one SQUID qubit to (e&eb) another. In this scheme, no transfer of quantum information between (e&eb) the SQUIDs and the cavity is required, the cavity field is (=) only virtually excited and thus the requirement on the quality factor of the cavity is (=) greatly relaxed. http://dx.doi.org/10.1103/PhysRevA.67.042311 **Possible realization of entanglement, logical gates, and quantum-information transfer with (e&eb) superconducting-quantum-interference-device qubits in cavity**

**QED Chui-Ping Yang, Shih-I Chu, and Siyuan Han Phys. Rev. A 67, 042311 – Published 17 April 2003 STOP. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** explore the capability of spin-1/2 chains to act (e&eb) as quantum channels for (e) both teleportation and transfer of (e) qubits. Exploiting (e) the emergence of (e) long-distance entanglement in (eb) low-dimensional systems [Phys. Rev. Lett. 96, 247206 (2006)], here **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** show (eb) how to obtain high communication fidelities between (e&eb) distant parties. An investigation of protocols of teleportation and (e&eb) state transfer is presented, in (eb) the realistic situation where (e) temperature is included. Basing setup on (eb) antiferromagnetic rotationally invariant systems, both protocols are represented by (e) pure depolarizing channels. **L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia** propose a scheme where channel fidelity close to 1 can be achieved on (e&eb) very long chains at moderately small temperature. **Qubit Teleportation and (e&eb) Transfer across Antiferromagnetic Spin Chains L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia Phys. Rev. Lett. 99, 060401 – Published 6 August 2007**

# PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MOODULE NUMBER SEVEN

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Systems to which have applied the model are as follows: Quantum computers have (e) the capability of out-performing their classical counterparts for (e) certain computational problems1. Several scalable quantum-computing architectures have been proposed. An attractive architecture is a large set of (e) physically independent qubits arranged in three spatial regions where (1) the initialized qubits are (=) stored in (eb) a register, (2) two qubits are brought together to (e) realize a gate and (3) the readout of (e) the qubits is carried out2, 3. For a neutral-**

**atom-based architecture, a natural way to connect these regions is to use (e) optical tweezers to move qubits within (e&eb) the system. Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier demonstrate the coherent transport of (e) a qubit, encoded on (e&eb) an atom trapped in (eb) a submicrometre tweezer, over (e&eb) a distance typical of the separation between atoms in (eb) an array of optical traps4, 5, 6. Furthermore, authors transfer a qubit between (e&eb) two tweezers, and show that this manipulation also preserves (e&eb) the coherence of the qubit. Nature Physics 3, 696 - 699 (2007) Published online: 12 August 2007 | doi: 10.1038/nphys698 Subject Categories: Atomic and molecular physics | Optical physics | Quantum physics | Information theory and (e&eb) computation Two-dimensional transport and transfer of (e) a single atomic qubit in (eb) optical tweezers Jérôme Beugnon, Charles Tuchendler, Harold Marion, Alpha Gaëtan, Yevhen Miroshnychenko, Yvan R. P. Sortais, Andrew M. Lance, Matthew P. A. Jones, Gaétan Messin, Antoine Browaeys & Philippe Grangier STOP. In optically controlled quantum computers it may be favorable to address (e&eb) different qubits using (e) light with different frequencies, since (e) the optical diffraction does not then limit the distance between (e&eb) qubits. Using qubits that are close to each other enable (erb) qubit-qubit interactions and (e&eb) gate operations that are strong and fast in comparison to (e&eb) qubit-environment interactions and decoherence rates. However, as qubits are addressed in frequency space, great care has to be taken when designing the laser pulses, so that they perform (e&eb) the desired operation on (eb) one qubit, without (e) affecting other qubits. Complex hyperbolic secant pulses have theoretically been shown to be excellent for (e) such frequency-addressed quantum computing [I. Roos and K. Molmer, Phys. Rev. A 69, 022321 (2004)]—e.g., for use in quantum computers based on optical interactions in rare-earth-metal-ion-doped crystals. The optical transition lines of (e) the rare-earth-metal-ions are (=) inhomogeneously broadened and therefore the frequency of (e) the excitation pulses can be used to (e) selectively address qubit ions that are (=) spatially separated by (e&eb) a distance much less than a wavelength. , Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter demonstrate experimentally frequency-selective transfer of (e) qubit ions between (e&eb) qubit states using (e) complex hyperbolic secant pulses. Transfer efficiencies better than 90% were obtained. Using (e) the complex hyperbolic secant pulses it was also possible to create (eb) two groups of ions, absorbing (e) at specific frequencies, where (e) 85% of the ions at one of the frequencies were shifted out of (e) resonance with (e&eb) the field when (e) ions in the other frequency group were excited. This procedure of selecting (e&eb) interacting ions, called qubit distillation, was carried out (e&eb) in preparation for two-qubit gate operations in (eb) the rare-earth-metal-ion-doped crystals. The techniques for frequency-selective state-to-state transfer developed here may be also useful also for (e) other quantum optics and quantum information experiments in (eb) these long-coherence-time solid-state systems. Experimental demonstration of efficient and selective population transfer and qubit distillation in a rare-earth-metal-ion-doped crystal Lars Rippe, Mattias Nilsson, Stefan Kröll, Robert Klieber, and Dieter Suter Phys. Rev. A 71, 062328 – Published 23 June 2005 STOP. Martin Leijnse and Karsten Flensberg propose a method to coherently transfer (e&eb) quantum information, and to create (eb) entanglement, between topological qubits and (e&eb) conventional spin qubits. Suggestion uses (e) gated control to transfer an electron (spin qubit) between (e&eb) a quantum dot and edge Majorana modes in (eb) adjacent topological superconductors. Because of the spin polarization of (e) the Majorana modes, the electron transfer translates spin superposition states into (e&eb) superposition states of the Majorana system, and vice versa. It is also shown how a topological superconductor can be used to (e) facilitate long-distance quantum information transfer (e&eb) and entanglement between (e&eb) spatially separated spin qubits. Quantum Information Transfer between Topological and (e&eb) Spin Qubit Systems Martin Leijnse and Karsten Flensberg Phys. Rev. Lett. 107, 210502 – Published 18 November 2011 http://dx.doi.org/10.1103/PhysRevLett.107.210502 STOP. Gogyan proposes a method that enables efficient conversion of the quantum information frequency between (e&eb) different regions of a spectrum of light based on (e) recently demonstrated (eb) strong parametric coupling between (e&eb) two narrow-band single-photon pulses propagating in (eb) a slow-light atomic medium [N. Sisakyan and Yu. Malakyan, Phys. Rev. A,**

**75, 063831 (2007)]. It is shown that an input qubit at (eb) telecom wavelength is transformed into (e&eb) another at a visible domain in (eb) a lossless and shape-conserving manner while (e) keeping the initial quantum coherence and (e&eb) entanglement. These transformations can be realized with quantum efficiency close to (e) its maximum value. http://dx.doi.org/10.1103/PhysRevA.81.024304 Qubit transfer between (e&eb) photons at telecom and visible wavelengths in (eb) a slow-light atomic medium A. Gogyan Phys. Rev. A 81, 024304 – Published 25 February 2010 STOP. N. Cleland and M. R. Geller propose a quantum computing architecture based on (e) the integration of nanomechanical resonators with (e&eb) Josephson-junction phase qubits. Resonators are GHz-frequency, dilatational disk resonators, which couple to (e&eb) the junctions through (e&eb) a piezoelectric interaction. System is analogous to (e) a collection of tunable few-level atoms (the Josephson junctions) coupled to (e&eb) one or more electromagnetic cavities (the resonators). Architecture combines desirable features of solid-state and optical approaches and may make (eb) quantum computing possible in a scalable, solid-state environment. Superconducting Qubit Storage and Entanglement with (e&eb) Nanomechanical Resonators A. N. Cleland and M. R. Geller Phys. Rev. Lett 93, 070501 – Published 10 August 2004 STOP. D. N. Matsukevich, A. Kuzmich report on the coherent quantum state transfer from (e) a two-level atomic system to a single photon Entanglement between (e&eb) a single photon (signal) and a two-component ensemble of (e) cold rubidium atoms is used to (e) project the quantum memory element (the atomic ensemble) onto (e&eb) any desired state by measuring (eb) the signal in a suitable basis. Atomic qubit is read out by (e) stimulating directional emission of (e) a single photon (idler) from (e) the (entangled) collective state of (e) the ensemble. Faithful atomic memory preparation and readout are verified by (e) the observed correlations between (e&eb) the signal and (e&eb) the idler photons. These results enable (eb) implementation of (e) distributed quantum networking. Science 22 October 2004: Vol. 306 no. 5696 pp. 663-666 DOI: 10.1126/science.1103346 Quantum State Transfer Between Matter and (e&eb) Light D. N. Matsukevich, A. Kuzmich STOP. Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi investigate the most general conditions under (e&eb) which a finite ferromagnetic long-range interacting (e&eb) spin chain achieves (eb) unitary fidelity and the shortest transfer time in transmitting (e&eb) an unknown input qubit. A deeper insight into system dynamics, allows (eb) to identify an ideal system involving (e&eb) sender and receiver only. However, this two-spin ideal chain is unpractical due to (e) the rapid decrease of (e) the coupling strength with (e&eb) the distance. Therefore authors propose an optimization scheme for (e) approaching the ideal behavior, while keeping the interaction strength still reasonably high. The procedure is (=) scalable with the size of the system and straightforward to implement. Perfect state transfer in (eb) long-range interacting spin chains Giulia Gualdi, Vojtech Kostak, Irene Marzoli, and Paolo Tombesi Phys. Rev. A 78, 022325 – Published 18 August 2008 STOP. L Fedichkin, M Yanchenko and K. A Valiev investigated the use of (e) quantum bits (qubits) - semiconductor quantum dots containing (e) one electron and each consisting of (e) two tunnel-connected parts - as basic elements of the quantum computer. Numerical solution of a Schrödinger equation taking account of (e) the Coulomb field of (e) adjacent electrons shows (eb) that in such structures (e&eb)the realization of (e) a full set of basic logic operations, which are necessary for (e) fulfillment of quantum computations, is (=) possible. Durations of one- and two-qubit operations versus (e&eb) qubit geometry are obtained. Decoherence rates due to (e) spontaneous emission of (e) phonons and acoustic phonons (both piezoelectric and deformation) are evaluated. Analysis of these rates shows (eb) the proposed qubit to be (eb) coherent enough to work for (e) an unlimited time. L Fedichkin et al 2000 Nanotechnology 11 387 doi:10.1088/0957-4484/11/4/339 Coherent charge qubits based on GaAs quantum dots with a built-in barrier L Fedichkin, M Yanchenko and K A Valiev ©2015 American Physical Society STOP. In recent years S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P have investigated the fidelity of (e) the quantum state transfer (QST) of two qubits by means of (e) an arbitrary spin-12 network, on (e&eb) a lattice of any dimensionality. Under the assumptions that the network Hamiltonian preserves (eb) the magnetization and that a fully polarized initial state is taken for (e) the lattice, they obtain (eb) a general formula for the average fidelity of (e) the two qubits QST, linking it to (e&eb) the one- and two-particle transfer amplitudes of (e) the spin excitations among the**

**sites of (e&eb) the lattice. S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina P apply this formalism to (e&eb) a 1D spin chain with (e&eb) XX-Heisenberg type nearest-neighbour interactions adopting (e&eb) a protocol that is a generalization of the single qubit one proposed in Paganelli et al. [Phys. Rev. A 87, 062309 (2013)]. They report that a high-quality two qubit QST can be achieved provided one can control (e&eb) the local fields at sites near the sender and receiver. Under such conditions, they obtain (eb) an almost perfect transfer in a time that scales (e&eb) either linearly or, depending on the spin number, quadratically with (e&eb) the length of the chain. Transfer of arbitrary two-qubit states via (e&eb) a spin chain S. Lorenzo, T. J. G. Apollaro, S. Paganelli, G. M. Palma, and F. Plastina Phys. Rev. A 91, 042321 – Published 16 April 2015 http://dx.doi.org/10.1103/PhysRevA.91.042321 ©2015 American Physical Society STOP. Combining techniques of cavity quantum electrodynamics (e&eb) quantum measurement, and (e&eb) quantum feedback, Norbert Kalb, Andreas Reiserer, Stephan Ritter, and Gerhard Rempe realize the heralded transfer of (e) a polarization qubit from a photon onto (e&eb) a single atom with 39% efficiency and 86% fidelity. The reverse process, namely, qubit transfer from (e) the atom onto a given photon, is demonstrated with 88% fidelity and an estimated efficiency of up to 69%. In contrast to previous work based on (e) two-photon interference, scheme is (=) robust against photon arrival-time jitter and achieves (eb) much higher efficiencies. Thus, it constitutes (eb) a key step toward the implementation of (e) a long-distance quantum network. Journal reference: Phys. Rev. Lett. 114, 220501 (2015) DOI:10.1103/PhysRevLett.114.220501 Cite as:arXiv:1503.06709 [quant-ph] (or arXiv:1503.06709v2 [quant-ph] for this version) Heralded Storage of a Photonic Quantum Bit in a Single Atom Norbert Kalb, Andreas Reiserer, Stephan Ritter, Gerhard Rempe 2 Jun 2015 STOP. Current high-throughput DNA sequencing technologies enable (eb) acquisition of billions of data points through (e&eb) which myriad biological processes can be interrogated, including (e) genetic variation (e&eb) chromatin structure (e&eb) gene expression patterns, small RNAs and protein–DNA interactions. Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker describe the MethylC-sequencing (MethylC-seq) library preparation method, a 2-d protocol that enables (eb) the genome-wide identification of cytosine DNA methylation states at (eb) single-base resolution. The technique involves (e&eb) fragmentation of genomic DNA followed by (e) adapter ligation, bisulfite conversion and (e&eb) limited amplification using (e) adapter-specific PCR primers in preparation for (e) sequencing. To date, this protocol has been successfully applied to (e&eb) genomic DNA isolated from (e) primary cell culture, (e&eb) sorted cells and (e&eb) fresh tissue from over (e&eb) a thousand plant and animal samples. NATURE PROTOCOLS | PROTOCOL MethylC-seq library preparation for base-resolution whole-genome bisulfite sequencing Mark A Urich, Joseph R Nery, Ryan Lister, Robert J Schmitz & Joseph R Ecker Nature Protocols 10, 475–483 (2015) doi:10.1038/nprot.2014.114 Published online 18 February 2015 STOP. Starting from a product initial state, equal-time correlations in (eb) nonrelativistic quantum lattice models propagate (e&eb) within (eb) a lightcone-like causal region. The presence of entanglement in (eb) the initial state can modify (e&eb) this behavior, enhancing and accelerating (eb, eb+) the growth of correlations. Michael Kastner gives a quantitative description, in the form of Lieb-Robinson-type bounds on (e&eb) equal-time correlation functions, of (e) the interplay of dynamics vs.(e&eb) initial entanglement in quantum lattice models out of (e) equilibrium. Michael Kastner tests the bounds against (e&eb) model calculations, and also Cite as: arXiv: 1507.00529 [quant-ph] (or arXiv: 1507.00529v1 [quant-ph] for this version) Entanglement-enhanced spreading of (e) correlations Michael Kastner (Submitted on 2 Jul 2015) STOP. A scheme based on (e) coherent tunneling by (e) adiabatic passage (CTAP) of exchange-only spin qubit quantum states in a linearly arranged double quantum dot chain is demonstrated. Logical states for the qubit are defined by adopting the spin state of three electrons confined in a double quantum dot. The possibility to obtain gate operations entirely with electrical manipulations makes this qubit a valuable architecture in the field of quantum computing for the implementation of quantum algorithms. Effect of the external control parameters as well as the effect of the dephasing on the coherent tunneling in the chain is studied. During adiabatic transport, within a constant energy degenerate eigenspace, the states in the double quantum dots internal to the chain**

**are not populated, while transient populations of the mixed states in the external ones are predicted. Coherent tunneling by adiabatic passage of an exchange-only spin qubit in a double quantum dot chain E. Ferraro, M. De Michielis, M. Fanciulli, and E. Prati Phys. Rev. B 91, 075435 – Published 26 February 2015 STOP. Black holes appear to conform to (e) a very straightforward generalisation of standard laboratory thermodynamics. This generalised theory is examined in detail, and some concrete results are presented. The thermodynamic connection is based on (e) Hawking's application of quantum theory to (e) black holes, and the quantum aspects are described in detail from (e) several standpoints, both heuristic and otherwise. The precise mechanism by which the black hole produces thermal radiation, its nature and origin, and the energetics of back-reaction on the hole are reviewed. The thermal states of quantum holes are also treated using (e) the theory of thermal Green functions, and the entropy of the hole is shown to be related to (e&eb) the loss of information about the quantum states hidden (e) behind the event horizon. Some related topics such as accelerated mirrors and (e&eb) observers in Minkowski space; (e&eb) super-radiance from rotating holes and the thermodynamics of general self-gravitating systems are also briefly discussed. Thermodynamics of black holes P C W Davies Reports on Progress in Physics, Volume 41, and Number STOP D. Blasa, O. Pujolàsb and S. Sibiryakova, c address the consistency of Hořava's proposal for a theory of quantum gravity from the low-energy perspective. They uncover the additional scalar degree of freedom arising from (e) the explicit breaking of the general covariance and study its properties. The analysis is performed both in (eb) the original formulation of the theory and in (eb) the Stückelberg picture. A peculiarity of the new mode is that it satisfies (eb) an equation of motion that is of (e) first order in time derivatives. At linear level the mode is Manifest only around (e&eb) spatially inhomogeneous and time-dependent backgrounds. Authors find two serious problems associated with (e&eb) this mode. First, the mode develops (eb) very fast exponential instabilities at (eb) short distances. Second, it becomes (eb) strongly coupled at (eb) an extremely low cutoff scale. They also discuss the ``projectable'' version of (e) Hořava's proposal and argue that this version can be understood as (=) a certain limit of the ghost condensate model. The theory is still problematic since (e) the additional field generically forms (eb) caustics and, again, has a very low strong coupling scale. Authors clarify some subtleties that arise in (eb) the application of the Stückelberg formalism to Hořava's model due to its non-relativistic nature. On the extra mode and inconsistency of Hořava gravity D. Blasa, O. Pujolàsb and S. Sibiryakova,c Published 12 October 2009 • Journal of High Energy Physics, Volume 2009, JHEP10(2009) STOP Petr Hořava presents a candidate quantum field theory of gravity with (e&eb) dynamical critical exponent equal to z=3 in the UV. (As in condensed-matter systems, z measures (eb) the degree of anisotropy between space and time.) This theory, which at short distances describes (eb) interacting nonrelativistic gravitons, is (=) power-counting renormalizable in (eb) 3+1 dimension. When restricted to satisfy the condition of detailed balance,(eb) this theory is intimately related to (e&eb) topologically massive gravity in three dimensions, and the geometry of (e) the Cotton tensor. At long distances, this theory flows (e&eb) naturally to the relativistic value z=1, and could therefore serve as (=) a possible candidate for (e) a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from (e) relevant deformations of the deeply nonrelativistic z=3 theory at short distances. Quantum gravity at a Lifshitz point Petr Hořava Phys. Rev. D 79, 084008 – Published 6 April 2009 STOP Petr Hořava1 proposes a quantum theory of membranes designed such that (e) the ground-state wavefunction of the membrane with (e&eb) compact spatial topology Σh reproduces (eb) the partition function of the bosonic string on (e&eb) worldsheet Σh. The construction involves (e&eb) worldvolume matter at (eb) quantum criticality, described in the simplest case by (e) Lifshitz scalars with (e&eb) dynamical critical exponent z = 2.This matter system must be coupled to (e&eb) a novel theory of worldvolume gravity, also exhibiting (eb) quantum criticality with z = 2. Authors first construct such a nonrelativistic ``gravity at a Lifshitz point'' with z = 2 in D+1 spacetime dimensions, and then specialize to (e&eb) the critical case of D = 2 suitable for the membrane worldvolume. They also show that (eb) in the second-quantized framework, the string partition function is reproduced if (e) the spacetime ground state takes the form of (e) a Bose-Einstein condensate of membranes in (eb) their first-quantized**

**ground states, correlated across all general. Membranes at quantum criticality Petr Hořava1, 2 Published 4 March 2009 • Journal of High Energy Physics, Volume 2009, JHEP03 (2009) STOP Petr Hořavaa, , Edward Wittenb propose that the ten-dimensional E8 × E8 heterotic string is related to an eleven-dimensional theory on the orbifold Full-size image (<1 K) in the same way that the Type IIA string in ten dimensions is related to Full-size image (<1 K). This in particular determines the strong coupling behavior of the ten-dimensional E8 × E8 theory. It also leads to a plausible scenario whereby duality between SO (32) heterotic and Type I superstrings follows from the classical symmetries of the eleven-dimensional world, just as the Full-size image (<1 K) duality of the ten-dimensional Type IIB theory follows from eleven-dimensional diffeomorphism invariance. Heterotic and Type I string dynamics from eleven dimensions Petr Hořavaa, , Edward Wittenb, Nuclear Physics B Volume 460, Issue 3, 12 February 1996, Pages 506–524 STOP Hořava-Lifshitz theory of gravity with detailed balance is plagued by (e) the presence of a negative bare (or geometrical) cosmological constant which makes (eb) its cosmology clash with (e&eb) observations. Authors argue that adding the effects of (e) the large vacuum energy of quantum matter fields, this bare cosmological constant can be approximately compensated to account for (e) the small observed (total) cosmological constant ΛOBS, thus resulting in (eb) a self-contained model of gravity and particle physics. Even though authors donot cannot address the fine-tuning problem in this way, they establish a relation between the smallness of ΛOBS and (e&eb) the scale ℓUV at which dimension 4 corrections to (e) the Einstein gravity become (eb) significant for cosmology. This scale turns out to be (eb) ℓUV sime 5 ℓP for ΛOBS sime 0 and authors therefore argue that the smallness of ΛOBS guarantees that (eb) Lorentz invariance is broken only at (eb) very small scales. They provide a first rough estimation for the values of (e) the parameters of the theory μ and ΛW.Four-dimensional Quantum Einstein Gravity (QEG) is likely to be (=) an asymptotically safe theory which is applicable at (e&eb) arbitrarily small distance scales. On sub-planckian distances it predicts (eb) that spacetime is a fractal with (e&eb) an effective dimensionality of 2. The original argument leading to (eb) this result was based upon (e) the anomalous dimension of Newton's constant. In the present paper authors demonstrate that also the spectral dimension equals 2 microscopically, while it is equal to 4 on macroscopic scales. This result is an exact consequence of asymptotic safety and does not (e) rely on any truncation. Contact is made with recent Monte Carlo simulations. Authors calculate the renormalized vacuum energy of (e) a massless scalar field confined between (e&eb) two nearby parallel plates formed by (e) ideal uncharged conductors, placed tangentially to (e) the surface of a sphere with (e&eb) mass M and radius R. This study will take into account the static and spherically symmetric solution of (e) Hořava-Lifshitz gravity found by (e) Kehagias-Sfetsos (KS), in (eb) both weak field and infrared limits. A slight amplification of (e) the Casimir force between (e&eb) the conducting plates is found. Thermal corrections to (e&eb) the Casimir energy are analyzed. Based on (e) current Casimir effect measurements, a constraint on (e) the ω parameter of KS metric is (=) also obtained. Received 1 October 2013 DOI: http://dx.doi.org/10.1103/PhysRevD.88.104035 Hořava-Lifshitz gravity effects on Casimir energy in weak field approximation and infrared regime C. R. Muniz, V. B. Bezerra, and M. S. Cunha Phys. Rev. D 88, 104035 – Published 26 November 2013 STOP Recently Hořava proposed a renormalizable gravity theory in (eb) four dimensions which reduces to (e&eb) Einstein gravity with (e&eb) a non-vanishing cosmological constant in (eb) IR but with improved UV behaviors. Here, I study an IR modification which breaks (e) ``softly'' the detailed balance condition in (eb) Hořava model and allows (eb) the asymptotically flat limit as well. Author obtains the black hole and cosmological solutions for (e) ``arbitrary'' cosmological constant that represent (eb) the analogs of the standard Schwarzschild-(A) dS solutions which can be (=) asymptotically (A)dS as well as flat and he discusses their thermodynamical properties Author also obtains solutions for FRW metric with (e&eb) an arbitrary cosmological constant. He studies its implication to (e&eb) the dark energy and finds (eb) that it seems to be consistent with current observational data. The black hole and cosmological solutions in IR modified Hořava gravity Mu-In Park Published 29 September 2009 Journal of High Energy Physics, Volume 2009, JHEP09 (2009) STOP Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Given a deformation (X,E) of the pair (X,E) over a small polydisk B centered at the origin, authors study**

**the jumping phenomenon of the cohomology groups dimCHq(Xt,Et) near t=0. Authors show that there are precisely two cohomological obstructions to the stability of dimCHq (Xt, Et), which can be expressed explicitly in terms of the Maurer-Cartan element associated to the deformation. This generalizes the results of X. Ye. Subjects: Differential Geometry (math.DG); Algebraic Geometry (math.AG); Complex Variables (math.CV) Cite as: arXiv: 1601.06472 [math.DG] (or arXiv: 1601.06472v1 [math.DG] for this version) This article gives an exposition of the deformation theory for (e) pairs (X, E), where (e) X is a compact complex manifold and E is (=) a holomorphic vector bundle over X, adapting (e&eb) an analytic viewpoint \`{a} la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer--Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed Subjects: Differential Geometry (math.DG); Complex Variables (math.CV) Journal reference: Complex Manifolds 3 (2016), 16-40 DOI:10.1515/coma-2016-0002 Cite as: arXiv:1406.6753 [math.DG] (or arXiv:1406.6753v4 [math.DG] for this version) A differential-geometric approach to deformations of pairs (X, E) Kwokwai Chan, Yat-Hin Suen and On the jumping phenomenon of dimCHq (Xt, Et) Kwokwai Chan, Yat-Hin Suen STOP Authors study the dynamic service migration problem in mobile edge-clouds that host cloud-based services at the network edge. This offers the benefits of reduction in network overhead and latency but requires service migrations as user locations change over time. It is challenging to make these decisions in an optimal manner because of the uncertainty in node mobility as well as possible non-linearity of the migration and transmission costs. In this paper, we formulate a sequential decision making problem for service migration using the framework of Markov Decision Process (MDP). Our formulation captures general cost models and provides a mathematical framework to design optimal service migration policies. In order to overcome (e) the complexity associated with (e&eb) computing the optimal policy, we approximate the underlying state space by the distance between the user and service locations. We show that the resulting MDP is exact for uniform one-dimensional mobility while it provides a close approximation for uniform two-dimensional mobility with a constant additive error term. We also propose a new algorithm and a numerical technique for computing the optimal solution which is significantly faster in computation than traditional methods based on value or policy iteration. We illustrate the effectiveness of our approach by simulation using real-world mobility traces of taxis in San Francisco. Comments: in Proc. of IFIP Networking 2015Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1506.05261 [cs.DC] (or arXiv: 1506.05261v1 [cs.DC] for this version) Dynamic Service Migration in Mobile Edge-Clouds Shiqiang Wang, Rahul Urgaonkar, Murtaza Zafer, Ting He, Kevin Chan, Kin K. Leung STOP Let kE denote the group algebra of an elementary abelian p-group of rank r over an algebraically closed field of characteristic p. We investigate the functors Fi from kE-modules of constant Jordan type to vector bundles on Pr−1(k), constructed by Benson and Pevtsova. For a kE-module M of constant Jordan type, we show that restricting the sheaf Fi(M) to a dimension s−1 linear subvariety of Pr−1(k) is equivalent to restricting M along a corresponding rank s shifted subgroup of kE and then applying Fi. In the case r=2, we examine the generic kernel filtration of M in order to show that Fi(M) may be computed on certain sub quotients of M whose Loewy lengths are bounded in terms of i. More precise information is obtained by applying similar techniques to the nth power generic kernel filtration of M. The latter approach also allows us to generalise our results to higher ranks r. Subjects: Representation Theory (math.RT) MSC classes: 16G10, 20C20 Cite as: arXiv: 1504.01994 [math.RT] (or arXiv: 1504.01994v1 [math.RT] for this version) Modules of constant Jordan type, pullbacks of bundles and generic kernel filtrations Shawn Baland, Kenneth Chan STOP Mobile micro-cloud is an emerging technology in distributed computing, which is aimed at providing seamless computing/data access to the edge of the network when a centralized service may suffer from poor**

**connectivity and long latency. Different from the traditional cloud, a mobile micro-cloud is smaller and deployed closer to users, typically attached to a cellular basestation or wireless network access point. Due to (e) the relatively small coverage area of each basestation or access point, when a user moves across areas covered by different base stations or access points which are attached to (e&eb) different micro-clouds, issues of service performance and service migration become (=) important. In this paper, authors consider such migration issues. They model the general problem as (=) a Markov decision process (MDP), and show that, in the special case where (e) the mobile user follows(eb) a one-dimensional asymmetric random walk mobility model, the optimal policy for (e) service migration is (=) a threshold policy. Authors obtain the analytical solution for (e) the cost resulting from arbitrary thresholds, and then propose an algorithm for finding the optimal thresholds. The proposed algorithm is more efficient than standard mechanisms for solving MDPs. Comments: in Proc. of IEEE MILCOM 2014, Oct. 2014 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) DOI: 10.1109/MILCOM.2014.145 Cite as: arXiv: 1503.05141 [cs.DC] (or arXiv: 1503.05141v1 [cs.DC] for this version) Mobility-Induced Service Migration in Mobile Micro-Clouds Shiqiang Wang, Rahul Urgaonkar, Ting He, Murtaza Zafer, Kevin Chan, Kin K. Leung STOP Seamless computing and data access is enabled by the emerging technology of mobile micro-clouds (MMCs). Different from traditional centralized clouds, an MMC is typically connected directly to a wireless base-station and provides services to a small group of users, which allows users to have instantaneous access to cloud services. Due to the limited coverage area of base-stations and the dynamic nature of mobile users, network background traffic, etc., the question of where to place the services to cope with these dynamics arises. In this paper, we focus on dynamic service placement for MMCs. We consider the case where there is an underlying mechanism to predict the future costs of service hosting and migration, and the prediction error is assumed to be bounded. Our goal is to find the optimal service placement sequence which minimizes the average cost over a given time. To solve this problem, we first propose a method which solves for the optimal placement sequence for a specific look-ahead time-window, based on the predicted costs in this time-window. We show that this problem is equivalent to a shortest-path problem and propose an algorithm with polynomial time-complexity to find its solution. Then, we propose a method to find the optimal look-ahead window size, which minimizes an upper bound of the average cost. Finally, we evaluate the effectiveness of the proposed approach by simulations with real-world user-mobility traces. Comments: in Proc. of IEEE ICC 2015 Subjects: Distributed, Parallel, and Cluster Computing (cs.DC); Networking and Internet Architecture (cs.NI); Optimization and Control (math.OC) Cite as: arXiv: 1503.02735 [cs.DC] (or arXiv: 1503.02735v1 [cs.DC] for this version) Dynamic Service Placement for Mobile Micro-Clouds with Predicted Future Costs Shiqiang Wang, Rahul Urgaonkar, Kevin Chan, Ting He, Murtaza Zafer, Kin K. Leung STOP Authors give a mathematically rigorous construction of Rozansky-Witten's 3-dimensional σ-model as a perturbative quantum field theory (QFT) by applying Costello's approach using the Batalin-Vilkovisky (BV) formalism. The quantization of our model is obtained via the technique of configuration spaces. We also investigate the observable theory following the work of Costello-Gwilliam. In particular, we show that the cohomology of local quantum observables on a genus g handle body is given by H∗(X,(Λ∗TX)⊗g), where X is the target hyperk\"ahler manifold. We further give a mathematical definition of the partition function and prove that it coincides with the Rozansky-Witten invariants. Subjects: Quantum Algebra (math.QA); High Energy Physics - Theory (hep-th); Algebraic Geometry (math.AG); Differential Geometry (math.DG); Geometric Topology (math.GT) Cite as: arXiv:1502.03510 [math.QA] (or arXiv:1502.03510v1 [math.QA] for this version) A mathematical foundation of Rozansky-Witten theory Kwokwai Chan, Naichung Conan Leung, Qin Li STOP Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 study the perturbative dynamics of (e) noncommutative field theories on (eb) Script Rd, and find (eb) an intriguing mixing of the UV and the IR. High energies of virtual particles in loops produce (eb) non-analyticity at low momentum. Consequently, the low energy effective action is (=) singular at zero momentum even when (e) the original noncommutative field theory is (=) massive. Some of the nonplanar diagrams of (e) these theories are (=) divergent, but they**

**interpret these divergences as (=) IR divergences and deal with (e&eb) them accordingly. Authors explain how this UV/IR mixing arises from (e) the underlying noncommutativity. This phenomenon is reminiscent of (e) the channel duality of (e) the double twist diagram in (eb) open string theory. Noncommutative perturbative dynamics Shiraz Minwalla1, Mark Van Raamsdonk1 and Nathan Seiberg2 Published 15 March 2000 • Journal of High Energy Physics, Volume 2000, JHEP02 (2000 STOP Authors give a derivation of the gravitational Hamiltonian starting from (e) the Einstein - Hilbert action, keeping track of (e) all surface terms. This derivation can be applied to (e&eb) any spacetime that asymptotically approaches (eb) a static background solution. The surface term that arises in (eb) the Hamiltonian can be taken as (=) the definition of the `total energy', even for (e) spacetimes that are not (e) asymptotically flat. (In the asymptotically flat case, it agrees with (eb) the usual ADM energy.) They also discuss the relation between the Euclidean action and (e&eb) the Hamiltonian when (e) there are horizons of infinite area (e.g. acceleration horizons) as well as (e&eb) the usual finite area black hole horizons. Acceleration horizons seem to be (=) more analogous to (e) extreme than nonextreme black holes, since they find evidence that their horizon area is not (e) related to the total entropy The gravitational Hamiltonian, action, entropy and surface terms S W Hawking and Gary T Horowitz Classical and Quantum Gravity, Volume 13, Number 6 http://dx.doi.org/10.1088/0264-9381/13/6/017 STOP Recent observations of Type 1a supernovae indicating (eb) an accelerating universe have (e) once more drawn attention to the possible existence, at the present epoch, of (e) a small positive Λ-term (cosmological constant). In this paper authors review both observational and theoretical aspects of (e) a small cosmological Λ-term. Authors discuss the current observational situation focusing on cosmological tests of Λ including (e) the age of the universe, high redshift supernovae (e&eb) gravitational lensing (e&eb) galaxy clustering and (e&eb) the cosmic microwave background. They also review the theoretical debate surrounding Λ: the generation of Λ in models with (e&eb) spontaneous symmetry breaking and through(e&eb) quantum vacuum polarization effects — mechanisms which are known to give rise to (eb) a large value of Λ hence leading to (eb) the "cosmological constant problem." More recent attempts to generate (eb) a small cosmological constant at the present epoch using (e) either field theoretic techniques or by (e) modelling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided. ARUN SAHNI and ALEXEI STAROBINSKY, Int. J. Mod Phys. D 09, 373 (2000) DOI: 10.1142/S0218271800000542 THE CASE FOR A POSITIVE COSMOLOGICAL Λ-TERM STOP Authors explore the observational adequacy of (e) a class of unified dark energy and dark matter (UDE/M) models with (e&eb) a fast transition. Constraints are set using (e) a combination of geometric probes—some low redshift ones and (e&eb) some high redshift ones (CMB-related included). The transition is phenomenologically modeled by (e) two different transition functions corresponding to (e&eb) a fast and to an ultrafast transition, respectively. They find that the key parameters governing (e&eb) the transition can be (=) well constrained, and from (e) the statistical point of view it follows (eb) that the models cannot be discarded when compared to (e&eb) LCDM. Authors find the intriguing result that standard or input parameters such as Ωm and Ωb are (=) far better constrained than in (eb) LCDM, and this is the case for (e) the derived or output parameter measuring (eb) the deceleration value at present, q0. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043537 © 2016 American Physical Society Cosmological constraints on fast transition unified dark energy and dark matter models Ruth Lazkoz, Iker Leanizbarrutia, and Vincenzo Salzano Phys. Rev. D 93, 043537 – Published 22 February 2016 STOP Authors report on constraints to (e) the cosmological jerk parameter (j) and to possible variability of (e) the fine-structure constant (Δα/α) based on (e) stochastic quintessence models of (e) dark energy, discussed by Chongchitnan and Efstathiou [Phys. Rev. D 76, 043508 (2007)]. Authors confirm the results by (e) these authors in the sense that many viable solutions can be obtained, obeying (e&eb) current observational constraints in low redshifts. They add the observables j and Δα/α to (e&eb) this conclusion. However, they find peculiarities that may produce (eb) in the nearby universe, potential observational imprints in (eb) future cosmological data. Authors conclude, for redshifts z≲3, that (i) j(z) fluctuates due to (e&eb) the stochasticity of the models, reaching (eb) an amplitude of up to 5% relatively to (e&eb) the Λ cold**

**dark matter model value (jΛCDM=1); and (ii) by contrasting two distinct (“extreme”) types of solutions, variabilities in α(z), linked to (e&eb) a linear coupling (ζ) between the dark energy and (e&eb) electromagnetic sectors, are (eb) weakly dependent on redshift, for (e) couplings of the order |ζ|∼10−4, even for (e) large variations in the equation of state parameter at (eb) relatively low redshifts. Nonlinear couplings produce (eb) an earlier and steeper onset of (e) the evolution in Δα/α (z), but can still accommodate (e) the data for weak enough couplings. DOI: http://dx.doi.org/10.1103/PhysRevD.93.043509 © 2016 American Physical Society The Void Galaxy Survey: Star Formation Properties B. Beygu1,6,★, K. Kreckel2, J. M. van der Hulst1, T. H Jarrett3, R. Peletier1, R. van de Weygaert1, J. H. van Gorkom4 and M. A. Aragon-Calvo5 STOP Stochastic quintessence models: Jerk and fine-structure variability constraints Christine C. Dantas and André L. B. Ribeiro Phys. Rev. D 93, 043509 – Published 8 February 2016 STOP Authors study the star formation properties of 59 void galaxies as part of the Void Galaxy Survey (VGS). Current star formation rates are derived from (e) $\rm {H\alpha} $ and recent star formation rates from (e) near-UV imaging. In addition, infrared 3.4 $\rm {\mu m}$, 4.6 $\rm {\mu m}$, 12 $\rm {\mu m}$ and 22 $\rm {\mu m}$ WISE emission is used as (=) star formation and mass indicator. Infrared and optical colours show that (eb) the VGS sample displays (eb) a wide range of dust and metallicity properties. They combine these measurements with (e&eb) stellar and $\rm {H\small {I}} $ masses to measure (eb) the specific SFRs ($\rm {SFR/M_ {*}} $) and (e&eb) star formation efficiencies ($\rm {SFR/M_ {H\small {I}}}$). Authors compare the star formation properties of their sample with (e&eb) galaxies in (eb) the more moderate density regions of (e) the cosmic web, ‘the field’. They find that specific SFRs of (e) the VGS galaxies as (=) a function of stellar and $\rm {H\small {I}}$ mass are similar to (e&eb, =) those of the galaxies in these field regions. Their $\rm {SFR\alpha} $ is slightly elevated than (e) the galaxies in the field for (e) a given total $\rm {H\small {I}} $ mass. In the global star formation picture presented by (e) Kennicutt-Schmidt, VGS galaxies fall into (eb)the regime of low average star formation and correspondingly (e&eb) low $\rm {H\small {I}} $ surface density. Their mean $\rm {SFR\alpha /M_{H\small {I}}}$ and (e&eb) $\rm {SFR\alpha /M_{*}}$ are of (e) the order of $\rm {10^{-9.9}}$ $\rm {yr^{-1}}$. Authors conclude that while the large scale underdense environment must play (e&eb) some role in galaxy formation and growth through accretion, they find that even with respect to other galaxies in the more mildly underdense regions, the increase in star formation rate is (=) only marginal. Key words galaxies: star formation galaxies: formation —galaxies: structure large-scale structure of universe© 2016 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Authors study the formation and evolution of (e) filamentary configurations of dark matter haloes in (eb) voids. Extant investigation uses (e) the high-resolution Λ cold dark matter simulation Cosmo Grid to (e) look for (e) void systems resembling the VGS_31 elongated system of (e) three interacting galaxies that was recently discovered by (e) the Void Galaxy Survey inside (eb) a large void in the Sloan Digital Sky Survey galaxy redshift survey. H I data revealed these galaxies to be embedded in (eb) a common elongated envelope, possibly embedded in (eb) intravoid filament. In the Cosmo Grid simulation authors look for systems similar to (e) VGS_31 in mass, size and environment. They find (eb) a total of eight such systems. For these systems, authors study the distribution of (e) neighbour haloes, the assembly and evolution of (e) the main haloes and (e&eb) the dynamical evolution of the haloes, as well as the evolution of (e) the large-scale structure in which the systems are (=) embedded. The spatial distribution of the haloes follows that of (e&eb) the dark matter environment. Authors find that VGS_31-like systems have (e) a large variation in formation time, having (e) formed between (e&eb) 10 Gyr ago and the present epoch. However, the environments in which the systems are embedded evolved to (e) resemble each other substantially. Each of the VGS_31-like systems is embedded in (eb) an intravoid wall, that no later than z = 0.5 became the only prominent feature in its environment. While parts of the void walls retain (eb) a rather featureless character, authors find that around half of them are marked by (e) a pronounced and rapidly evolving substructure. Five haloes find themselves in a tenuous filament of a few h−1 Mpc long inside (eb) the intravoid wall. Finally, authors compare the results to (e&eb) observed data from VGS_31. Study implies that (eb) the VGS_31 galaxies formed in (eb) the same (proto) filament, and did not (e) meet just recently. The diversity amongst the simulated halo systems indicates (eb)**

**that VGS_31 may not (e) be typical for (e) groups of galaxies in voids. Key words galaxies: formation galaxies: interactions cosmology (e&eb) theory dark matter large-scale structure of Universe © 2013 The Authors Published by Oxford University Press on behalf of the Royal Astronomical Society Assembly of filamentary void galaxy configurations Steven Rieder1, 2, Rien van de Weygaert3, Marius Cautun3, Burcu Beygu3 and Simon Portegies Zwart1First published online August 20, 2013 STOP Authors measure the star formation properties of (e) two large samples of galaxies from (e) the SDSS in large-scale cosmic voids on (e&eb) time scales of 10 Myr and 100 Myr, using (e) Hα emission line strengths and GALEX FUV fluxes, respectively. The first sample consists of 109,818 optically selected galaxies. They find that void galaxies in this sample have (e) higher specific star formation rates (SSFRs; star formation rates per unit stellar mass) than (e&eb) similar stellar mass galaxies in (eb) denser regions. The second sample is (=) a subset of (e) the optically selected sample containing (e) 8070 galaxies with (e&eb) reliable HI detections from ALFALFA. For the full HI detected sample, SSFRs do not (e) vary systematically with (e&eb) large-scale environment. However, investigating only the HI detected dwarf galaxies reveals (eb) a trend towards higher SSFRs in (eb) voids. Furthermore, authors estimate the star formation rate per unit HI mass (known as the star formation efficiency; SFE) of a galaxy, as (=) a function of environment. For the overall HI detected population, authors notice no (e) environmental dependence. (AC=DC=0)Limiting the sample to dwarf galaxies again reveals (eb) a trend towards higher SFEs in (eb) voids. These results suggest (eb) that void environments provide (eb) a nurturing environment for (e) dwarf galaxy evolution allowing (eb) for higher specific star formation rates and efficiencies. Comments: submitted to ApJ 12-2015 Subjects: Astrophysics of Galaxies (astro-ph.GA) Cite as: arXiv: 1601.04092 [astro-ph.GA]. Can Grover's algorithm speed up (eb+) search of a physical region—for example a 2-D grid of size √ n × √ n? The problem is that √n time seems to be needed for (e) each query, just to move amplitude across the grid. Here authors show that this problem can be surmounted (e) refuting a claim to (e) the contrary by Benioff. In particular, authors show how to search a d-dimensional hypercube in (eb) time O (√n) for d ≥ 3, or O (√nlog5/2n) for d = 2. More generally, authors introduce a model of (e) quantum query complexity on graphs, motivated by (e) fundamental physical limits on (e&eb) information storage, particularly the holographic principle from (e) black hole thermodynamics. Results in this model include (e) almost-tight upper and lower bounds for (e) many search tasks; a generalized algorithm that works for (e) any graph with good expansion properties, not just hypercubes; and relationships among several notions of 'locality' for (e) unitary matrices acting on graphs. As an application of results, authors give an O ( √ n)-qubit communication protocol for (e) the disjointness problem, which improves (eb+) an upper bound of Høyer and de Wolf and matches (e&eb) a lower bound of Razborov. ACM Classification: F.1.2, F.1.3 AMS Classification: 81P68, 68Q10 Key words and phrases: Quantum computing, (e&eb) Grover search, amplitude amplification, (e&eb) quantum communication complexity, disjointness, lower bounds ⁎This work was mostly done while the author was a PhD student at UC Berkeley, supported by an NSF Graduate Fellowship and by ARO grant DAAD19-03-1-0082.†Supported by an IQC University Professorship and by CIAR. STOP Grover's quantum search algorithm provides (eb) a way to speed up (eb) combinatorial search, but is not (e) directly applicable to searching (e&eb) a physical database. Nevertheless, Aaronson and Ambainis showed (eb) that a database of N items laid out in (eb) d spatial dimensions can be searched in time of (e) order N−−√ for d>2, and in time of order N−−√poly (logN) for d=2. Authors consider an alternative search algorithm based on (e) a continuous-time quantum walk on (e&eb) a graph. The case of the complete graph gives (eb) the continuous-time search algorithm of (e) Farhi and Gutmann, and other previously known results can be used to show that (eb) N−−√ speedup can also be achieved on (e&eb) the hypercube. Authors show that full N−−√ speedup can be achieved on (eb) a d-dimensional periodic lattice for d>4. In d=4, the quantum walk search algorithm takes time of (e) order N−−√poly (logN), and (e&eb) in d<4, the algorithm does not (e) provide substantial speedup. Received 16 June 2003 DOI:http://dx.doi.org/10.1103/PhysRevA.70.022314 Spatial search by quantum walk Andrew M. Childs and Jeffrey Goldstone Phys. Rev. A 70, 022314 – Published 23 August 2004 STOP Quantum walks are considered in (eb) a one-dimensional random medium characterized by (e) static or dynamic disorder. Quantum**

**interference for (e) static disorder can lead to (eb) Anderson localization which completely hinders (e) the quantum walk and it is contrasted with (e&eb) the decoherence effect of (e) dynamic disorder having (e) strength W, where (e) a quantum to (e&eb) classical crossover at time tc∝W−2 transforms (e&eb) the quantum walk into (e&eb) an ordinary random walk with (e&eb) diffusive spreading. Authors demonstrate this localization and decoherence phenomena in (eb) quantum carpets of the observed time evolution, authors relate results to (e&eb) previously studied models of (e) decoherence for quantum walks, and examine (e&eb) in detail a dimer lattice which corresponds to(e&eb) a single qubit subject to randomness. Received 11 August 2007 DOI:http://dx.doi.org/10.1103/PhysRevA.77.022302 Quantum walks on a random environment Yue Yin, D. E. Katsanos, and S. N. Evangelou Phys. Rev. A 77, 022302 – Published 5 February 2008 STOP Feedback quantum control, where (e) the sample `teaches' a computer-controlled arbitrary light form generator to find (e&eb) the optimal light field, is experimentally demonstrated for (e) a molecular system. Femtosecond pulses tailored by a computer-controlled acousto-optic pulse shaper excite (eb) fluorescence from laser dye molecules in (eb) solution. Fluorescence and laser power are (=) monitored, and the computer uses (e) the experimental data and a genetic algorithm to (e) optimize population transfer from ground to (e&eb) first excited state. Both efficiency (the ratio of excited state population to (e&eb) laser energy) and (e&eb) effectiveness (total excited state population) are optimized. Potential use as an `automated theory tester' is discussed. Copyright © 1997 Elsevier Science B.V. All rights reserved. Chemical Physics Letters Volume 280, Issues 1–2, 28 November 1997, Pages 151–158 Feedback quantum control of molecular electronic population transfer Christopher J Bardeena, Vladislav V Yakovleva, Kent R Wilsona, Scott D Carpenterb, Peter M Weberc, Warren S Warrend STOP Quantum robots are described as (=) mobile quantum computers and ancillary systems that move in and interact with (e&eb) arbitrary environments. Their dynamics is given as (=) tasks which consist of (e) sequences of (e) alternating computation and (e&eb) action phases. A task example is considered in which a quantum robot searches (e&eb) a space region to find the location of a system. The possibility that the search can be (=) more efficient than a classical search is examined by (e) considering use of (e) Grover's Algorithm to process the search results. For reversible searches this is (=) problematic for two reasons. One is (=) the removal of entanglements generated by (e) the search process. The other is that even if the entanglement can be avoided, (eb) the search process in 2-D space regions is (=) no more efficient than a classical search. However, quantum searches of space regions with (e&eb) 3 or more dimensions are (=) more efficient than classical searches. Reasons why quantum robots are interesting independent of these results are briefly summarized. Subjects: Quantum Physics (quant-ph) Journal reference: AMS Contemporary Math Series, Vol 305, (2002) Cite as: arXiv: quant-ph/0003006 (or arXiv: quant-ph/0003006v2 for this version) Space Searches with a Quantum Robot Paul Benioff (Physics Division, Argonne National Lab, Argonne, IL) (Submitted on 1 Mar 2000 (v1), last revised 26 Jun 2001 (this version, v2))STOP The relationship between the foundations of mathematics and (e&eb) physics is a topic of of much interest. This paper continues this exploration by (e) examination of the effect of space and time dependent number scaling on (e&eb) theoretical descriptions of (e) some physical and geometric quantities. Fiber bundles provide (eb) a good framework to introduce (eb) a space and time or space time dependent number scaling field. The effect of the scaling field on (e&eb) a few nonlocal physical and geometric quantities are described. The effect on gauge theories is to introduce (eb) a new complex scalar field into (e&eb) the derivatives appearing in Lagrangians. U (1) invariance of Lagrangian terms does not (e) affect the real part of the scaling field. For this field, any mass is possible. The scaling field is also shown to affect (e&eb) quantum wave packets and (e&eb) path lengths, and (e&eb) geodesic equations even on (e&eb) flat space. Scalar fields described so far in physics, are (=) possible candidates for (e) the scaling field. The lack of direct evidence for (e) the field in physics restricts the scaling field in that the gradient of (e) the field must be close to (e) zero in (eb) a local region of (e) cosmological space and time. There are no restrictions outside (e) the region. It is also seen that the scaling field does not (e) affect comparisons of computation or measurements outputs with (e&eb) one another. However it does affect (e&eb) the assignment of numerical values to the outputs of computations or measurements. These are needed because (e) theory predictions are**

**in terms of (e&eb) numerical values. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum Information Processing, 15(3), 1081-1102, (2016) DOI: 10.1007/s11128-015-1084-z Cite as: arXiv: 1508.01732 [math-ph] (or arXiv: 1508.01732v1 [math-ph] for this version) Space and time dependent scaling of numbers in mathematical structures: Effects on physical and geometric quantities Paul Benioff STOP The purpose of this paper is to put the description of number scaling and its effects on (e&eb) physics and geometry on (e&eb) a firmer foundation, and to make it more understandable. A main point is that two different concepts, number and (e&eb) number value are combined in the usual representations of (e) number structures. This is valid as (e) long as just one structure of each number type is being considered. It is not valid when (e) different structures of each number type are (=) being considered. Elements of base sets of (e) number structures, considered by (e) themselves, have no meaning. They acquire meaning or value as (e) elements of a number structure. Fiber bundles over (e&eb) a space or space time manifold, M, are described. The fiber consists of (e) a collection of many real or complex number structures and (e&eb) vector space structures. The structures are parameterized by (e) a real or complex scaling factor, s. A vector space at a fiber level, s has (e) as scalars, real or complex number structures at (eb) the same level. Connections are described that relate scalar and (e&eb) vector space structures at both neighbor M locations and (e&eb) at neighbor scaling levels. Scalar and vector structure valued fields are (=) described and covariant derivatives of these fields are (=) obtained. Two complex vector fields, each with (e&eb) one real and one imaginary field, appear, with one complex field associated with (e&eb) positions in M and the other with (e&eb) position dependent scaling factors. A derivation of (e) the covariant derivative for (e) scalar and vector valued fields gives (eb) the same vector fields. The derivation shows (eb) that the complex vector field associated with (e&eb) scaling fiber levels is (=) the gradient of a complex scalar field. Use of these results in gauge theory shows (eb) that the imaginary part of the vector field associated with (e&eb) M positions acts (e&eb) like the electromagnetic field. The physical relevance of the other three fields, if any, is (=) not known. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Proceedings SPIE STA conference, Vol. 9500, 9500 0X, 2015 DOI: 10.1117/12.2176080 Cite as: arXiv: 1503.05600 [math-ph] (or arXiv: 1503.05600v1 [math-ph] for this version) Principal fiber bundle description of number scaling for scalars and vectors: Application to gauge theory Paul Benioff STOP This work uses (e) fiber bundles as a framework to describe some effects of number scaling on (e&eb) gauge theory and some geometric quantities. A description of number scaling and fiber bundles over (e&eb) a flat space time manifold, M, is followed by (e&eb) a description of gauge theory. A fiber at point x of M contains (e) a pair of scaled complex number and (e&eb) vector space structures, Ccx×Vcx for each c in GL(1,C). A space time dependent scalar field, g, determines, (eb) for each x, the scaling value of (e) the vector space structures that contain (e) the values of a vector valued matter field at (eb) x. The vertical components of (e) connections between neighboring fibers are taken to be (=) the gradient field A(x)+iB(x), of g. Abelian gauge theory for (e) these fields gives (eb) the result that B is (=) massless and no mass restrictions for (e) A. Addition of an electromagnetic field dies not (e) change these results. In the Mexican hat Higgs mechanism B combines with (e&eb) a Goldstone boson to create massive vector bosons, the photon field, and the Higgs field. For geometric quantities the fiber bundle is (=) a tangent bundle with (e&eb) a fiber at point x containing (e) scaled pairs, Rrx×Trx of real number and (e&eb) tangent space structures for (e) each x and and nonnegative real r. B is zero everywhere. The A field affects (e&eb) path lengths and the proper times of (e) clocks along paths. It also appears in (eb) the geodesic equation. The lack of physical evidence for (e) the gradient field, A(x)+iB(x) means (eb) that it either couples very weakly to (e&eb) matter fields or that it is close to zero for all x in (eb) a local region of cosmological space and (e&eb) time. It says nothing about (e&eb) the values outside the local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th); Quantum Physics (quant-ph) Journal reference: Quantum Studies: Mathematics and Foundations, 2, 289-313, 2015 DOI: 10.1007/s40509-015-**

**0049-3 Cite as: arXiv: 1412.1493 [math-ph] (or arXiv: 1412.1493v2 [math-ph] for this version) Fiber bundle description of number scaling in gauge theory and geometry Paul Benioff STOP Stronger foundations for (e) earlier work on the effects of (e) number scaling, and local mathematics is described. Emphasis is placed on the effects of scaling on (e&eb) coordinate systems. Effects of scaling are represented by (e) a scalar field, θ, which appears (eb) in gauge theories as (=) a spin zero boson. Gauge theory considerations led to (eb) the concept of local mathematics, as expressed through (e&eb) the use of universes, as collections of (e) local mathematical systems at (eb) each point, x, of a space time manifold, M. Both local and global coordinate charts are (=) described. These map M into (e&eb) either local or global coordinate systems within (eb) a universe or between (e&eb) universes, respectively. The lifting of global expressions of (e) nonlocal physical quantities, expressed by (e) space and or time integrals or derivatives on M, to (e&eb) integrals or derivatives on coordinate systems, is (=) described. The assumption of local mathematics and (e&eb) universes makes integrals and derivatives, on (e&eb) M or on global charts, meaningless. They acquire (e) meaning only when mapped into (e&eb) a local universe. The effect of scaling, by including (e) the effect of θ into (e&eb) the local maps, is described. The lack of experimental evidence for (e) θ so far shows that (eb) the coupling constant of θ to (e&eb) matter fields must be very small compared to the fine structure constant. Also the gradient of θ must be (=) very small in the local region of (e) cosmological space and time occupied by (e) us as observers. So far, there are no known restrictions on (e&eb) θ or its gradient in regions of space and/or time that are far away from (e) our local region. Subjects: Mathematical Physics (math-ph); General Relativity and Quantum Cosmology (gr-qc); Quantum Physics (quant-ph) Journal reference: Quantum information and computation XII, proceedings of SPIE conference, 8-9 May, 2014, Vol. 9123, 91230Q-1 DOI: 10.1117/12.2049705 Cite as:arXiv: 1405.3217 [math-ph] (or arXiv: 1405.3217v1 [math-ph] for this version) Effects of mathematical locality and number scaling on coordinate chart use Paul Benioff STOP Effects of local availability of (e) mathematics (LAM) and (e&eb) space time dependent number scaling on (e&eb) physics and, especially, geometry are described. LAM assumes (eb) separate mathematical systems as (=) structures at each space time point. Extension of gauge theories to include freedom of (e) choice of scaling for number structures, and other structures based on (e) numbers, results (eb) in a space time dependent scaling factor based on (e) a scalar boson field. Scaling has no effect on (e&eb) comparison of experimental results with (e&eb) one another or with theory computations. With LAM all theory expressions are elements of (e) mathematics at some reference point. Changing the reference point introduces (eb) (external) scaling. Theory expressions with integrals or derivatives over (e&eb) space or time include scaling factors (internal scaling) that cannot be removed by (e) reference point change. Line elements and path lengths, as integrals over (e&eb) space and/or time show (eb) the effect of scaling on geometry. In one example, the scaling factor goes to 0 as the (e&eb) time goes to 0, the big bang time. All path lengths, and values of physical quantities, are crushed to (eb) 0 as t goes to 0. Other examples have (e) spherically symmetric scaling factors about some point, x. In one type, a black scaling hole, the scaling factor goes to (e&eb) infinity as the distance, d, between any point y and x goes to 0. For scaling white holes, the scaling factor goes to (e&eb) 0 as d (e&eb) goes to 0. For black scaling holes, path lengths from (e) a reference point, z, to y become (=) infinite as (e) y approaches x. For white holes, path lengths approach (e&eb) a value much less than the unscaled distance from z to (e) x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.2014360 Cite as: arXiv: 1306.4613 [quant-ph] (or arXiv: 1306.4613v1 [quant-ph] for this version) Effects of gauge theory based number scaling on geometry Paul Benioff STOP In gauge theories, separate vector spaces, Vx, are assigned to (e) each space time point x with (e&eb) unitary operators as maps between basis vectors in neighboring Vx. Here gauge theories are extended by replacing (e&eb) the single underlying set of complex scalars, C, with (e&eb) separate sets, Cx, at each x, and including (e) scaling between the Cx. In gauge theory Lagrangians, number scaling shows (eb) as a scalar boson field, B, with (e&eb) small coupling to (e&eb) matter fields. Freedom of number scaling is extended to (e&eb) a model with (e&eb) separate number structures assigned to (e) each point x. Separate collections, Ux, of all mathematical systems based on (e) numbers, are assigned to (e) each x.**

**Mathematics available to (e) an observer, Ox, at x is (=) that in Ux. The B field induces (eb) scaling between structures in the different Ux. Effects of B scaling on (e&eb) some aspects of (e) physics and geometry are described. The lack of experimentally observed scaling means (eb) that B (z) is essentially constant for all points, z, in a region, Z, that can be occupied by (e) us as observers. This restriction does not (e) apply outside Z. The effects of B scaling on (e&eb) line elements, curve lengths, and distances between points are examined. Oz's description, using (e) Uz in Z, of elements at points, x, outside Z, includes (e) scaling from x to (e&eb) z. Integrals over curves include (e) scaling factors inside the integrals. Two examples are discussed. One shows that B (t) can be such that mathematical, physical, and geometric quantities approach (eb) zero as the time t approaches zero. This mimics the big bang in that distances (e) approach zero. Examples of black and white scaling holes are described in (eb) which B(x) is plus or minus infinity at a point x. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); High Energy Physics - Theory (hep-th) Cite as: arXiv: 1211.3381 [quant-ph] (or arXiv: 1211.3381v2 [quant-ph] for this version) Gauge theory extension to include number scaling by boson field: Effects on some aspects of physics and geometry Paul Benioff STOP Local availability of (e) mathematics and number scaling provide (eb) (an approach to (e) a coherent theory of physics and mathematics. Local availability of mathematics assigns (e&eb) separate mathematical universes, U_{x}, to (e) each space time point, x. The mathematics available to (e) an observer, O_{x}, at x is contained in (eb) U_{x}. Number scaling is based on (e) extending the choice freedom of vector space bases in gauge theories to (e) choice freedom of underlying (e&eb) number systems. Scaling arises in (eb) the description, in U_{x}, of (e) mathematical systems in U_{y}. If a_{y} or \psi_{y} is (=) a number or a quantum state in U_{y}, then (eb)the corresponding number or state in (eb) U_{x} is r_{y,x}a_{x} or r_{y,x}\psi_{x}. Here a_{x} and \psi_{x} are (=) the same number and state in U_{x} as (=) a_{y} and \psi_{y} are in (eb) U_{y}. If y=x+\hat{\mu}dx is a neighbor point of x, then (eb) the scaling factor is r_{y,x}=\exp(\vec{A}(x)\cdot\hat{\mu}dx) where \vec{A} is a vector field, assumed here to be (=) the gradient of a scalar field. The effects of scaling and local availability of mathematics on (e&eb) quantum theory show that scaling has (e) two components, external and internal. External scaling is shown above for (e) a_{y} and \psi_{y}. Internal scaling occurs in expressions with integrals or derivatives over (e&eb) space or space time. An example is (=) the replacement of the position expectation value, \int\psi^{*}(y)y\psi(y)dy, by \int_{x}r_{y,x}\psi^{*}_{x}(y_{x})y_{x}\psi_{x}(y_{x})dy_{x}. This is an integral in U_{x}. The good agreement between quantum theory and (=) experiment shows (eb) that scaling is negligible in (eb) a space region, L, in which experiments and calculations can be done, and results compared. L includes the solar system, but the speed of light limits the size of L to a few light years. Outside of L, at cosmological distances, the limits on (e&eb) scaling are not present. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) DOI: 10.1117/12.917731 Cite as: arXiv: 1205.0200 [quant-ph] (or arXiv: 1205.0200v1 [quant-ph] for this version) Local Availability of mathematics and number scaling: Effects on quantum physics Paul Benioff STOP The work is based on two premises: local availability of mathematics to an observer at any space time location, and (e&eb) the observation that number systems, as (=) structures satisfying axioms for (e) the number type being considered, can be scaled by arbitrary, positive real numbers. Local availability leads to (eb) the assignment of mathematical universes, Vx, to each point, x, of space time. Vx contains (e) all the mathematics that an observer, Ox, at x, can know. Each Vx contains (e) many types of mathematical systems. These include (e) the different types of numbers (natural numbers, integers, rationals, and real and complex numbers), Hilbert spaces, algebras, and many other types of systems. Space time dependent scaling of number systems is used to (e) define representations, in Vx, of real and complex number systems in (eb) Vy. The representations are scaled by (e) a factor ry, x relative to (e&eb) the systems in Vx. For y a neighbor point of x, ry, x is (=) the exponential of the scalar product of (e) a gauge field, A→(x), and the vector from x to (e&eb) y. For y distant from x, ry, x is a path integral from x to (e&eb) y. Some consequences of the two premises will be examined. Number scaling has no effect on (e&eb) general comparisons of numbers obtained as (=) computations or as experimental outputs. The effect is limited to (e) mathematical expressions that include (e) space or space**

**time integrals or derivatives. The effect of A→ on (e&eb) wave packets and canonical momenta in quantum theory, and some properties of A→ in gauge theories, is described. Subjects: Quantum Physics (quant-ph); Information Theory (cs.IT); General Relativity and Quantum Cosmology (gr-qc); Mathematical Physics (math-ph) Journal reference: Chapter 2, in Quantum Theory, I. Cotaescu, Editor, Intech open access publisher, 2012 Cite as:arXiv:1110.1388 [quant-ph] (or arXiv:1110.1388v2 [quant-ph] for this version) Effects on quantum physics of the local availability of mathematics and space time dependent scaling factors for number systems Paul Benioff STOP Gauge theories can be described by (e) assigning a vector space V(x) to (e&eb) each space time point x. A common set of complex numbers, C, is usually assumed to be (=) the set of scalars for all the V{x}. This is expanded here to assign (e&eb) a separate set of scalars, C{x}, to V{x} for each x. The freedom of choice of bases, expressed by (e) the action of a gauge group operator on (e&eb) the V{x}, is expanded here to include (e) the freedom of choice of (e) complex scale factors, c_{y, x}, as (=) elements of GL (1, C) that relate C{y} to (e&eb) C{x}. A gauge field representation of c_{y, x} gives (eb) two gauge fields, A (x) and iB(x). Inclusion of these fields in (eb) the covariant derivatives of Lagrangians results in (eb) A(x) appearing as (=) a gauge boson for which mass is (=) optional and B(x) as (=) a massless gauge boson. B(x) appears to be (=) the photon field. The nature of A(x) is not known at present. One does know that the coupling constant of A(x) to (e) matter fields is very small compared to (e&eb) the fine structure constant. Subjects: Mathematical Physics (math-ph); Quantum Physics (quant-ph) Journal reference: Proc.SPIE Int.Soc.Opt.Eng. 8057:80570X, 2011 DOI: 10.1117/12.895454 Cite as: arXiv: 1105.0633 [math-ph] (or arXiv: 1105.0633v1 [math-ph] for this version) New Gauge Fields from Extension of Parallel Transport of Vector Spaces to Underlying Scalar Fields Paul Benioff STOP This work is based on (e) the field of reference frames based on (e) quantum representations of (e) real and complex numbers described in other work. Here frame domains are expanded to include (e) space and time lattices. Strings of qukits are described as (=) hybrid systems as they are both mathematical and (e&eb) physical systems. As mathematical systems they represent numbers. As physical systems in each frame the strings have (e) a discrete Schrodinger dynamics on the lattices. The frame field has an iterative structure such that the contents of (e) a stage j frame have images in (eb) a stage j-1 (parent) frame. A discussion of parent frame images includes (e) the proposal that points of stage j frame lattices have (e) images as (=) hybrid systems in parent frames. The resulting association of energy with (e&eb) images of lattice point locations, as (=) hybrid systems states, is discussed. Representations and images of other physical systems in (eb) the different frames are also described. Subjects: Quantum Physics (quant-ph); General Relativity and Quantum Cosmology (gr-qc) Journal reference: Advances in Mathematical Physics, Vol. 2009, ID 452738 DOI: 10.1155/2009/452738 Cite as: arXiv: 0809.4499 [quant-ph] (or arXiv: 0809.4499v2 [quant-ph] for this version) A Possible Approach to Inclusion of Space and Time in Frame Fields of Quantum Representations of Real and Complex Numbers Paul Benioff STOP A quantum theory representations of real (R) and complex (C) numbers are given that is based on (e) states of single, finite strings of (e) qukits for (e) any base k > 1. Both unary representations and (e&eb) the possibility that qukits with (e&eb) k a prime number are (=) elementary and the rest composite are discussed. Cauchy sequences of (e) qukit string states are defined from (e) the arithmetic properties. The representations of (e) R and C, as equivalence classes of these sequences, differ from (e&eb) classical kit string state representations in two ways: the freedom of choice of basis states, and the fact that each quantum theory representation is (=) part of a mathematical structure that is itself based on (e) the real and complex numbers. These aspects enable (eb) the description of 3 dimensional frame fields labeled by (e) different k values, different basis or gauge choices, and different iteration stages. The reference frames in the field are based on (e) each R and C representation where (e) each frame contains (e) representations of *(e) all physical theories as mathematical structures based on (e) the R and C representation. Approaches to integrating this with physics are described. It is observed that R and C values of physical quantities, matrix elements, etc. which are viewed in (eb) a frame as elementary and featureless, are seen in a parent frame as equivalence classes of (e) Cauchy sequences of qukit string states. Subjects:Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); General Mathematics**

**(math.GM) Journal reference:Advances in Quantum Computation, K. Mahdavi and D. Koslover, Eds., Contemporary Mathematics Vol. 482, pp125-163, (2009) Cite as: arXiv:0709.2664 [quant-ph] (or arXiv:0709.2664v2 [quant-ph] for this version) Reference Frame Fields based on Quantum Theory Representations of Real and Complex Numbers Paul Benioff STOP This paper extends earlier work on quantum theory representations of natural numbers N, integers I, and rational numbers Ra to describe (eb) a space of these representations and transformations on (e&eb) the space. The space is parameterized by (e) 4-tuple points in a parameter set. Each point, (k,m,h,g), labels (e&eb) a specific representation of X = N, I, Ra as a Fock space F^{X}_{k,m,h} of (e) states of (e) finite length strings of qukits q and a string state basis B^{X}_{k,m,h,g}. The pair (m,h) locates the q string in (eb) a square integer lattice I \times I, k is the q base, and the function g fixes the gauge or basis states for (e) each q. Maps on the parameter set induce (eb) transformations on on the representation space. There are two shifts, a base change operator W_{k',k}, and (e&eb) a basis or gauge transformation function U_{k}. The invariance of (e) the axioms and theorems for N, I, and Ra under (e&eb) any transformation is discussed along with (e&eb) the dependence of the properties of W_{k',k} on (e&eb) the prime factors of k' and k. This suggests (eb) that one consider prime number q's, q_ {2}, q_{3}, q_{5}, etc. as (=) elementary and the base k q's as composites of (e) the prime number q's. Subjects: Quantum Physics (quant-ph); High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph) Cite as: arXiv: 0704.3574 [quant-ph] (or arXiv: 0704.3574v1 [quant-ph] for this version) Space of Quantum Theory Representations of Natural Numbers, Integers, and Rational Numbers Paul Benioff STOP In this paper fields of quantum reference frames based on (e) gauge transformations of (e) rational string states are described in a way that, hopefully, makes (eb) them (eb) more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are (=) a large number of different quantum theory representations of (e) natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give (eb) a representation of the real (and complex) numbers. A reference frame is associated to (e&eb) each representation. (3) Each frame contains (e) a representation of (e) all mathematical and physical theories that have (e) the representations of the real and complex numbers as (=) a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to (e&eb) physics are discussed. Comments: 24 pages, 5 figures; submitted to proceedings, 3rd Feynman Festival, University of Maryland, 2006 Journal reference: Journal of Physics: Conference Series 70 Fields of quantum reference frames based on different representations of rational numbers as states of qubit strings Paul Benioff STOP System studied is as follows: System and attendant properties thereof is studied. Gravity Kills Schrödinger's Cat , Warped Space-Time Prevents Quantum Superpositions Of Large-Scale Objects, Error Correcting Codes In Quantum Theory, Scheme For Reducing Decoherence In Quantum Computer Memory, Correcting Quantum Computer Errors, Qubits Based On Quantum Spins, Connection Between The Observers Will Typically Be Subject To Disturbances That Can Disrupt The Entanglement, Relationship Between Communication Complexity Problems Bell Nonlocal Correlations, Bell Nonlocal Correlations, Communication Complexity And The Violation Of A Bell Inequality, Exchange Of The Information After Simultaneous Teleportation To Reveal The Path Of The Teleported System In A Three-Round Protocol, Efficient Teleportation Protocol**

**104.MODEL FORMULATION**

Let $p_7(t, a_7)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_7}{dt} = -b_7 Q_7 + g_7(g_7)Q_7$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_7(0) = 0, \quad g_7(P_7) > 0, \quad P_7 \; 1, \qquad \lim_{P_7 \to \infty} g_7(P_7) = (K_7)_7 \; \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_7 + (\rho_a)_7 + D_7(a_7, P_7, Q_7)\rho_7 + 0, \qquad 0 < a_7 \; \infty,$$

$$\rho_7(t,0) = \int_0^\infty n_7(a_7, P_7, Q_7)\rho_7(t_7, a_7)da_7$$

Where $D_7$the mortality or debit type function is function and $n_7$ is the fertility function.

Put $_7$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_7}{dt}\right]_7 = x_7 \left[\frac{dQ_7}{dt}\right]_7 - x_7 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_7}{dt}\right]_7 = x_3 \left[\frac{dQ_3}{dt}\right]_7 - x_4 \left[\frac{dQ_7}{dt}\right]_7$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5 \left[\frac{dQ_7}{dt}\right]_7 - x_6 \left[\frac{dQ_3}{dt}\right]_7$$

Where $x_7, x_7, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_7$ and $g_7(P_7)Q_7$

Now when you add the term $(b_7Q_7)$ with $g_7(P_7)(Q_7)$ we are infect see is that $(b_7Q_7)$ is credited incrementally part of $g_7(P_7)(Q_7)$ or instant nearly, when subtracted from $\frac{dQ_7}{dt}$ , it means that $g_7(P_7)(Q_7)$ is slowly eaten by $\frac{dQ_7}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

Following equation throws light on important issues in the following cases

**Simulation suggests non-water based life could exist on Saturn's moon Titan July 5, 2016 by Bob Yirka report Titan True-color image of layers of haze in Titan's atmosphere. Credit: NASA**

(6046) (Phys.org)—A team of researchers at Cornell University has built and run a simulation that showed (eb) prebiotic reactions could possibly occur on (eb) the surface of one of Saturn's moons, Titan, suggesting (eb) the possibility of life evolving in a place where (e) it is too cold for water to be a factor.

(6047) In their paper published in Proceedings of the National Academy of Sciences, the team describes the simulation they created in response to (e&eb) the discovery (by the Huygens probe) that polymers such as polyimine might have already developed on (eb) the moon's surface.

(6048) As scientists continue to search for life on other planets, more and more they are beginning to concede that if it does exist, (eb) it may not necessarily exist in the Goldilocks, or habitual zone.

(6049) Such planets are all the "right" distance from (e) their star, and hopefully, also have water.

(6050) But recent evidence has suggested that some chemical reactions could lead to (eb) types of life forms where there is no water, which further suggests that it may exist outside of (e) what is now considered the habitual zone.

(6051) For life to come about in such places, researchers reason, there would likely need to be (=) some sort of action going on—and that is why there has been (=) so much focus on Titan; it is the only object in our solar system, besides Earth, that has both rainfall and erosion due to (e) liquid movement.

(6052) But the water it has is locked far underground and the moon is (=) too cold to support an impact by (e&eb) water anyway.

(6053) But, as the researchers with this new effort discovered after poring over data sent back by Huygens, the surface does have (e) hydrogen cyanide in its sediment, brought down from (e) the atmosphere by methane and ethane rain.

(6054) It was those molecules that drove (eb) the design of the simulations—the team wanted to see if they could form the basis of reactions that could lead to (eb) the creation of polymers such as polyimine,

which the team notes are (=) conducive to the formation of prebiotic reactions leading to (eb) a form of life.

(6055) Their simulations showed (eb) that such reactions are possible and that the structures that came about were also able to absorb (e) sunlight in the wavelengths present on (eb) the Titan surface.

(6056) The researchers suggest their work, and that done by others indicates (eb) that a return to the planet by (e) a new probe might be in order, one able to test for (e) a different form of life, or at least its precursors. More information: Polymorphism and electronic structure of polyimine and its potential significance for prebiotic chemistry on Titan, PNAS, www.pnas.org/cgi/doi/10.1073/pnas.1606634113

**Abstract**

(6057) The chemistry of hydrogen cyanide (HCN) is believed to be central to (e) the origin of life question.

(6058) Contradictions between Cassini–Huygens mission measurements of (e) the atmosphere and the surface of (e) Saturn's moon Titan suggest (eb) that HCN-based polymers may have formed on the surface from (e) products of atmospheric chemistry.

(6059) This makes Titan a valuable "natural laboratory" for exploring potential nonterrestrial forms of (e) prebiotic chemistry.

(6060) Authors have used theoretical calculations to investigate the chain conformations of polyimine (pI), a polymer identified as (=) one major component of polymerized HCN in laboratory experiments.

(6061) Thanks to its flexible backbone, the polymer can exist in (eb) several different polymorphs, which are (=) relatively close in energy.

(6062) The electronic and (e&eb) structural variability among them is extraordinary.

(6063) The band gap changes over (e&eb) a 3-eV range when moving from a planar sheet-like structure to (e&eb) increasingly coiled conformations.

(6064) The primary photon absorption is predicted to occur in (eb) a window of relative transparency in Titan's atmosphere, indicating (eb) that pI could be photochemically active and (e&eb) drive chemistry on the surface.

(6065) The thermodynamics for adding and removing HCN from pI under Titan conditions suggests (eb) that such dynamics is plausible, provided that (e) catalysis or photochemistry is available to sufficiently lower reaction barriers.

(6066) Authors speculate that the directionality of pI's intermolecular and (e&eb) intramolecular =N–H…N hydrogen bonds may drive (eb) the formation of partially ordered structures, some of which may synergize with (e&eb)photon absorption and act catalytically. Future detailed studies on proposed mechanisms and the solubility and density of the polymers will aid in the design of future missions to Titan. Journal reference: Proceedings of the National Academy of Sciences search and more info website © 2016 Phys.org Read more at: http://phys.org/news/2016-07-simulation-non-water-based-life-saturn.html#jCp

**Water waves travel back in time to retrace their ripples By Jacob Aron (New Scientist)**

(6067) Nudge your bath tub just right and (eb) you might end up reversing time.

(6068) Don’t worry – you won’t end up wet, naked and without a towel in the Cretaceous period. Emmanuel Fort at the Langevin Institute in Paris, France, and his colleagues have developed what they call a “time mirror”, which can make water waves rewind into (e&eb) their past. Humans and rubber duckies are (=) affected.

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| (6069) | To understand how this works, imagine dropping a stone in a pond and watching waves ripple out from a central point in an ever-expanding circle. The equations governing the behaviour of waves say that it should also be possible for a ring of ripples to refocus back into a single point, like a reversed movie of the dropped stone. But we never actually see this happening, because it is vanishingly unlikely that an outside-in ring would form spontaneously. Put it in reverse |
| (6070) | Fort and others have previously shown that it is possible to engineer the situation so the waves turn back on themselves by kitting out the edge of your pond with sensors and wave generators, effectively recording and then playing back the waves in reverse. The process has been done with water, sound and even microwaves, and is used to produce clearer ultrasound images for medical uses. |
| (6071) | But now the team has discovered that simply dropping the tub downwards is enough to send water waves scurrying in reverse, with no need for complex sensors. That's because the velocity of water waves depends on acceleration due to gravity. "If you do a sudden jolt, you change the effective gravity, and the velocity of the wave changes," says Fort. |
| (6072) | The disruption essentially resets the starting point of the wave. Once normal gravity is restored, it starts to travel both backwards and forwards. "Suddenly, you freeze the wave. When you release it again, it doesn't remember that it was going one way," says Fort. |
| (6073) | Ripples in the water returned exactly to their start point - the shape of the Eiffel Tower |
| (6074) | Ripples in the water returned exactly to their start point – the shape of the Eiffel Tower |

**Déjà vu**

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| (6075) | To show that the backward-travelling waves converge exactly on their starting point, just as if you were rewinding time, the team conducted a series of experiments. In some, they simply created a circular ripple of waves by dipping a probe in the water, similar to dropping a stone in the pond. In others, they used wave emitters in the shape of France, the Eiffel Tower and a smiley face to create complex wave-fronts. In each case, the reversed waves converged to recreate their original shape. |
| (6076) | The experiments aren't truly reversing time, because the waves continue to travel out again once they have converged at the central point, points out Fort. "A real time-reversal would have a sink that absorbs all the waves," he says. |
| (6077) | Now the team hopes to conduct similar experiments with other kinds of waves, like light or sound, although it may prove more difficult to find an analogue of their time-mirror jolt in other wave mediums, he says. |
| (6078) | "The work is spectacular," says Geoffrey Edelmann of the US Naval Research Laboratory in Washington, D.C. "They essentially change the rules that govern how the wave propagates and, in doing so, create an entirely new way to focus waves free of spatial constraints." He also thinks the technique could be extended to other kinds of waves, and perhaps have medical applications such as lithotripsy, which uses ultrasound waves to break up kidney stones. Journal reference: Nature Physics, DOI: 10.1038/nphys3810 |

**Time reversal and holography with spacetime transformations Vincent Bacot, Matthieu Labousse, Antonin Eddi, Mathias Fink & Emmanuel Fort Nature Physics (2016) doi: 10.1038/nphys3810**

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| (6079) | Wave control is usually performed by (e) spatially engineering the properties of a medium. |
| (6080) | Because time and space play similar roles (e&eb) in wave propagation, manipulating time boundaries provides (eb) a complementary approach. |

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| (6081) Here, authors experimentally demonstrate the relevance of this concept by (e) introducing instantaneous time mirrors. |
| (6082) Authors show with water waves that a sudden change of (e) the effective gravity generates (eb) time-reversed waves that refocus at the source. |
| (6083) They generalize this concept for (e) all kinds of waves, introducing (eb) a universal framework which explains (eb) the effect of any time disruption on (e&eb) wave propagation. |
| (6084) Authors show that sudden changes of the medium properties generate (eb) instant wave sources that emerge (eb) instantaneously from the entire space at (eb) the time disruption. |
| (6085) The time-reversed waves originate from (e) these 'Cauchy sources', which are (+) the counterpart of Huygens virtual sources on (eb) a time boundary. It allows us to revisit the holographic method and introduce a new approach for wave control. |

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| **Viewpoint: A Bird's Eye View of Circuit Photons Sebastian Schmidt, Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland June 30, 2016• Physics 9, 74 A scanning probe detects the quantum states of photons in a microwave circuit, providing the information needed for quantum simulations. Figure caption expand figure A. Houck/Princeton (deleted)** |

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| (6086) Figure 1: Scanning defect microscopy provides a map of photons in a resonator lattice. Houck and colleagues demonstrated the technique using (e) 49 resonators (grey lines) that were coupled together to (e&eb) form a kagome lattice. |
| (6087) Quantum mechanics rules the dynamics of (e) light and matter. |
| (6088) Yet performing a quantum-mechanical simulation of a material from first principles is (=) practically impossible on a classical computer because (e) the complexity of the simulation increases (eb) exponentially with the number of particles involved. |
| (6089) The solution, according to Richard Feynman, was to build a machine out of (e) quantum building blocks that could directly emulate (e&eb) the material itself [1]. |
| (6090) Prototypes of such quantum simulators that are based on (e) ultracold atoms, ions, photons, and superconducting microwave circuits are now available [2], with (e&eb) the latter, in particular, having attracted (e) Silicon Valley's interest. |
| (6091) The challenge with these circuit-based simulators, however, is that they are (=) 2D, which complicates the readout of their constituent elements. Andrew Houck from Princeton University, New Jersey, and colleagues have now delivered an attractive solution by developing a technique [3], called scanning defect microscopy (Fig. 1), that determines (eb) the number of photons occupying (e) each mode of a 2D microwave circuit. |
| (6092) It is this information that would serve as (=) the fundamental input and output for (e) certain quantum simulations. |
| (6093) Superconducting microwave circuits combine electronic and (e&eb) photonic degrees of freedom [4, 5]. |
| (6094) The main element of the circuit is (=) a transmission line, which is made up of (e) a central superconducting wire separated by a gap from (e) two grounded plates. |
| (6095) All of these structures are on a single plane, as if one had taken a 2D slice through (e&eb) a coaxial cable. |

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| (6096) When truncated, the transmission line becomes (=) a resonator, which can host (e&eb) discrete photon modes within (eb) its gaps |
| (6097) Large lattices of resonators can be engineered in various 1D or 2D geometries by coupling (e&eb) two, three, or more resonators together via (e&eb) a capacitive interface. In many ways, photons in such devices behave similarly to electrons in a solid. |
| (6098) To make microwave circuits that can simulate (e&eb) quantum phenomena faster than a classical computer, however, resonator lattices have to be integrated with (e&eb) superconducting qubits |
| (6099) . Such qubits are controlled with (e&eb) electrical currents in a Josephson tunnel junction, and in many respects, they behave (e&eb) like artificial atoms, which couple to (e&eb) the photons in the circuit. |
| (6100) As a result, superconducting microwave circuits can be used to (e) explore the coupling between the quantum states of light and (e&eb) matter, the regime of (e) circuit quantum electrodynamics (cQED). |
| (6101) Photons in these devices often exhibit (eb) striking matter-like behavior [6, 7], providing (eb) the basis for the simulation of complex materials. |
| (6102) Such circuits can be fabricated on a substrate using (e) standard lithographic techniques, with (e&eb) qubits and resonators that are (=) hundreds of micrometers or even millimeters in size. |
| (6103) Small-scale realizations of cQED lattices have already been used for (e) proof-of-principle quantum simulations [8, 9]. |
| (6104) However, future large-scale simulations will have to be initialized and measured (eb) efficiently—steps known as (=) drive and readout. |
| (6105) This requires connecting the lattices to (e) external circuitry, such as voltage sources, circulators, and amplifiers, which typically reside outside (e) the chip. |
| (6106) So far, this connectivity has been realized through (e&eb) external transmission lines, but these elements can only couple to (e&eb) the resonators at the edges of (e) the lattice because of (e) its 2D nature. |
| (6107) Such a side view provides (eb) only limited information about (e&eb) the quantum state and spatial correlations of photons inside (eb) the bulk resonators, seriously limiting (e) the applicability of circuit QED devices for (e) quantum simulations. |
| (6108) The probe developed by Houck and colleagues, scanning defect microscopy (SDM), solves (e) this problem by (e) moving outside the plane of the chip to provide (eb) a fuller image of photons within (eb) the chip. |
| (6109) In SDM, a small sapphire tip is positioned a few tens of (e) micrometers above the surface of one of the resonators (Fig. 1). |
| (6110) The presence of this dielectric modifies (e&eb) the evanescent electric field from the circuit photons, which extends (eb) just outside the plane of the chip. |
| (6111) Specifically, the tip increases (eb) the local capacitance of the transmission line, downshifting (e) the local resonator frequency. |
| (6112) This in turn causes (eb) a drop in the transmission of photons through (e&eb) the lattice when (e) it is driven at its nominal resonance frequency. |
| (6113) Houck's team showed that if the sapphire tip has only a weak (linear) effect on the resonator frequency, then (eb) a simple analytic relationship exists between the induced frequency shift, (e&eb) the change in transmission, and (e&eb) the local photonic occupation of each resonator. |
| (6114) This relationship provides (eb) a recipe for detecting the local photon occupancy inside (eb) the lattice. |

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| (6115) First, calibrate the probe by determining (eb) the frequency shift for (e) a single isolated resonator as (=) a function of the tip's position. |
| (6116) Second, measure the transmission through (e&eb) the whole lattice with and without (e&eb) the probe. |
| (6117) Finally, use the measured transmission drop with (e&eb) the probe present to find (eb) the photonic occupation of (e) an individual resonator in (eb) the lattice. |
| (6118) A map of the photon-mode occupancies can be obtained by (e) scanning the probe across the whole lattice. |
| (6119) Houck's team used (e) SDM to image the occupancies of (e) the photon modes of (e) 49 coupled resonators arranged in (eb) a special geometry called a kagome lattice (see details in Fig. 1). |
| (6120) (To simplify the measurement, no superconducting qubits were (=) incorporated with (e&eb) the resonators.) |
| (6121) The occupancies they measured were in excellent agreement with (=) theoretical predictions, corresponding to (e&eb) a fidelity of 0.99. |
| (6122) The high fidelity is even more impressive considering the weak signal-to-noise ratio of (e) their technique (which is limited by (e) the intentionally small size of the frequency-shifted region) and (e&eb) the technical challenge of (e) positioning the probe inside (eb) a cryostat for (e) in situ measurements. |
| (6123) While SDM shares similarities with (e&eb) so-called bead-pull experiments for accelerators and scanning gate microscopy for (e) nanostructures, it is truly a (=) new tool for quantum electronics. |
| (6124) First application of SDM is likely to be (=) the simulation of 2D topological phenomena and (e&eb) geometrically frustrated lattices (of which the kagome lattice is (=) one example.) |
| (6125) Both are associated with (e&eb) macroscopically degenerate systems, which are (=) notoriously difficult to simulate on (e&eb) a classical computer. |
| (6126) Such circuits are (=) genuinely open systems (photons can be added or lost). |
| (6127) This combined with (e&eb) their inherent strong light-matter coupling, makes (eb) such quantum simulators ideal candidates for (e) the study of many-body, out-of-equilibrium phenomena, such as (=) thermalization or nonequilibrium phase transitions. |
| (6128) SDM could also lead to (eb) the discovery of new exotic phenomena, which are (=) unknown in traditional solid-state systems. |
| (6129) In a nutshell, SDM constitutes a new and indispensible tool for (e) future experiments on quantum simulations with microwave photons. This research is published in Physical Review X. |

**Imaging Photon Lattice States by Scanning Defect Microscopy D. L. Underwood, W. E. Shanks, Andy C. Y. Li, Lamia Ateshian, Jens Koch, and A. A. Houck Phys. Rev. X 6, 021044 – Published 30 June 2016**

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| (6130) Microwave photons inside (eb) lattices of coupled resonators and superconducting qubits can exhibit (eb) surprising matter like behavior. |
| (6131) Realizing such open-system quantum simulators presents (eb) an experimental challenge and requires (e) new tools and measurement techniques. |
| (6132) Here, authors introduce scanning defect microscopy as one such tool and illustrate its use in mapping (e&eb) the normal-mode structure of microwave photons inside (eb) a 49-site kagome lattice of coplanar waveguide resonators |
| (6133) Scanning is accomplished by (e) moving a probe equipped with a sapphire tip across (e&eb) the lattice. |

(6134) This locally perturbs (e&eb) resonator frequencies and induces (eb) shifts of the lattice resonance frequencies, which authors determine by (e) measuring the transmission spectrum.

(6135) From the magnitude of mode shifts, author can reconstruct (eb) photon field amplitudes at (eb) each lattice site and thus create (eb) spatial images of the photon-lattice normal modes. Received 4 December 2015 DOI:http://dx.doi.org/10.1103/PhysRevX.6.021044 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

We assume that

$$D_7 = \mu_7(a_7) + (f_7)_7(P_7)_7 + (f_7)_7(Q)_7 \quad (7)$$

$$n_7 = \beta_7(a_7)h_7(Q_7), \quad (3)$$

Where $\mu_7$the age- dependent debit rate whit is $\mu_7(a_7)$ $\forall a > 0$; $(f_7)_7$ is the density- dependent death rate with $f_7(0) = 0$, $(f_7^7)_7(P_7)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_7)_7(P_7) = (k_7)_7$ ; $f_7$ is the functional response of creditors to prey with $[f_7]_7(0) = 0, (f_7^7)_7(Q) > 0, \forall Q$ , and $\lim_{Q\to\infty}(f_7)_7(Q_7) = [k_7]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^7(Q)$ $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ .

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_7 + (\rho_a)_7 = -\mu_7(a_7) + (f_7)_7P_7) + \big(f_7)_7(Q_7)\big)(\rho_7)_7 \quad (4)$$

$$\rho_7(t_7, 0) = h_7(Q_7)\int_0^\infty \beta_7(a_7)\rho_7(t, a_7)da, \quad (5)$$

$$\frac{dQ_7}{dt}$$

$$= -b_7Q_7 + g_7(P_7)Q_7 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_7$ and $Q_7$ of (4-6) satisfy

$$0 \leq \rho_7(t, a_7) \leq \rho_7^*(t, a_7),$$

$$0 \leq Q_7(t) \leq Q_7^*(t),$$

Where $\rho_7^*(t, a_7)$ and $Q_7^*(t)$, are solutions of

$$(\rho_t^*)_7 + (\rho_a^*)_7 = -(\mu_7(a_7) + (f_7)_7(P_7^*) + (f_7)_7 Q_7^*))\, \rho_7^*, \quad (7)$$

$$\rho_7^*(t, 0) = \int_0^\infty \beta_7(a_7)\, \rho_7^*(t, a_7) da_7 \quad (8)$$

$$\frac{dQ_7^*}{dt} = -b_7\, Q_7^* + g_7(P_7^*) Q_7^*, \quad (9)$$

With $P_7^*(t) = \int_0^\infty \rho_7^*(t, a_7) da_7$

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From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_7^*(t)}{dt} = P_7^*(t)\big(r_7 - (f_7)(P_7^*) - (f_7)_7(Q^*)\big), \quad (10)$$

$$\frac{dQ_7^*(t)}{dt} = Q_7^*(t)(-b_7 + g_7(P_7^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_7(a) e^{-\int_0^a \mu_7(\tau)dt} e^{-ra}\, da_7 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_7^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(t)dt}\, da_7 > 1, \qquad (H1)$$

Which guarantees $r_7 > 0$.

*Remark.* If $\beta_7(a) = [\beta_0]_7 a_7 e^{-\alpha_7 a_7}, \mu_7(a_7) \equiv \mu_7$ then the solution of (12) is

$$r_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_7]_7(P_7) = 0, [f_7](Q_7) = \lambda_7 Q_7\ h_7(Q_7) = 1, \qquad and\ \ g_7(P_7) = c_7 P_7$$

$$[\rho_7]_7 + [\rho_a]_7 = -(\mu_7 + \lambda_7 Q_7)\rho_7$$

$$\rho_7(t,0) = \int_0^\infty [\beta_0]_7 a_7 e^{-\alpha_7 a_7} \rho_7(t, a_7)\, da_7$$

$$\frac{dQ_7}{dt} = -b_7 Q_7 + c_7 P_7 Q_7$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_7}{dt} = \omega_7 P_7 - \lambda_7 P_7 Q_7$$

$$\frac{dQ_7}{dt} = b_7Q_7 - c_7P_7Q_7$$

Where $\omega_7 = \sqrt{\beta_7} - \alpha_7 - \mu_7$

## 105.THE EQUILIBRIUM $(\rho_0)_7(a_7), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_7(a_7), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_7 + (\rho_a)_7 = -\big(\mu_7(a_7) + (f_7)_7(P_7)\big)\rho,$$

$$\rho_7(t,0) = \int_0^\infty \beta_7(a_7)\rho_7(t,a_7)da_7 \qquad (13)$$

The solution of (13) has the form of

$$\rho_7(t,a_7) = Ce_7^{r(t-a_7)} - \int_0^a (\tau)d\tau - \int_0^7 (f_7)_7\big(P_7(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_7(t) = C_7e^{r_7t - \int_0^t (f_7)_7(P_7(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_7^{-7}(r)$ if $k_7 > r$. This is equivalent to

$$\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau)d\tau} e^{(k_7)_7a_7} da_7 < 1.$$

In fact, if we define

$$R_7(s_7) = \int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau)d\tau}\, e^{-s_7 a_7}\, da_7$$

$R(s)$ is a decreasing function. $R_7(r_7) = 1$ And $R_7(k_7)_7 < 1$ imply $r_7 < (k_7)_7$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_7(a_7), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{k_7 a_7} da_7$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_7 \equiv \int_0^\infty (\rho_0)_{77}(a_7)da_7$ *satisfies*

$$\int_0^\infty \beta_7(a_7) e^{-\int_0^a \mu_7(\tau)d\tau} e^{-(f_7)_7(P_0)a_7} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_7(a_7), 0)$

Let $u_7(t, a_7) = \rho(t, a_7) - \rho_0(a_7)$ Then

$$N_7(t) \equiv \int_0^\infty u_7\,(t, a_7)da_7 = \int_0^\infty \rho_7(t, a_7)da_7 - \int_0^\infty (\rho_0)_7(a_7)da = P_7(t) - P_0)_7$$

And

$$(u_7)_7 + (u_a)_7 = (\rho_t)_7 + (\rho_a)_7 - (\rho_0^7)_7(a_7)$$

$$= -\mu_7(a_7) + (f_7)_7(P_7) + (f_7)_7(Q_7))\,(u_7 + (\rho_0)_7) + \mu_7(a_7)(\rho_0)_7 + (f_7)_7(P_0)_7(\rho_0)_7$$

$$= -\mu_7(a_7)u_7 - (f_7)_7(P_7) + (f_7)_7(Q_7))\,u_7 - (\rho_0)_7\,(f_7)_7(P_7) - (f_7)_7(P_0)_7$$

$$+ (f_7)_7(Q_7)), \qquad (15)$$

Where $(\rho_0^7)_7(a_7)$ denotes $d(\rho_0)_7(a_7)/da_7$

Expansion $(f_7)_7(P_7)$ and $(f_7)_7(Q_7)$ at $(P_0)_7$ and 0 yield respectively

$$(f_7)_7(P_7) = (f_7)_7((P_0)_7 + N_7)$$

$$= (f_7)_7(P_0)_7 + (f_7^7)_7(P_0)_7 N_7 + o_7(N_7), \quad (16)$$

$$(f_7)_7(Q_7)$$

$$= (f_7^7)_7(0)Q + o_7(Q_7) \quad (17)$$

Write $f_7(P_0) = f_{70}, f_7'(P_0) = f_{70}'$, and $f_7'(0) = f_{70}'$ and substitute (16) and (17) into (15) to obtain

$$(u_7)_7 + (u_a)_7 \approx -(\mu_7(a_7) + (f_{70})_7)u - (f_{70}^7)_7(\rho_0)_7(N_7) - (f_{70}^7)_7(\rho_0)_7(Q_7)$$

$$u_7(t,0) = \rho_7(t,0)(\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a) + (\rho_0)_7(a_7)da_7 - (\rho_0)_7(0)$$

$$= h_7(Q_7)\int_0^\infty \beta_7(a_7)(u_7(t,a_7)da_7 + (h_7(Q_7) - 1)(\rho_0)_7(0) \quad (18)$$

Expanding

$$h_7(Q_7) = 1 + (h^7)_7(0)Q_7 + 0(Q_7) \equiv 1 + h_0'Q_7 + 0Q_7$$

And substituting it into (18), we arrive at

$$u_7(t,0) \approx \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + (h_{a_7}')(\rho_0)_7 Q_7$$





The linearization is the following :

$$[u_7] + [u_7] = -(\mu_7(a_7) + [t_{70}]_7 u_7 - [\alpha_7]_7(a_7)N_7 - [\alpha_7]_7(a_7)Q_7$$

$$u_7(t,0) = \int_0^\infty \beta_7(a_7)u_7(t,a_7)da_7 + [\xi_7]Q_7 \qquad (19)$$

$$\frac{dQ_7}{dt} = -b_7 Q_7$$

where

$$[\alpha_7]_7(a_7) = [t_7^7 a]_7[p,o]_7(a_7)d2(a_7) = [f_{70}^7]_7(\rho\, o\,)_7(a_7)\xi = [\mathrm{h\ o}^7]_7[\rho\mathrm{o}]_7(\mathrm{o}) \text{ and } \mathrm{b}_7 = \mathrm{b}_7 - g_7(P_7)$$

If $g_7(P_0) > b$, clearly, $(\rho_0)_7(a_7), 0)$ is unstable and the predator population is persistent. Assume $g_7(P_0)_7 < b$.

Then, $Q_7(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_7 + u_a = -(\mu_7(a_7) + (f_{70})_7)u_7 - (\alpha_7)_7(a_7)N_7$$

$$= -(\mu_7(a_7) + f_{70})u_7 - \alpha_7(a_7)\int_0^\infty u_7(t,a_7)da_7$$

$$u_7(t,0) = \int_0^\infty \beta_7\,(a_7)u_7(t,a_7)da_7$$

Using the method developed by Cushing [2, 3], we suppose

$$u_7(t,a_7) = \tilde{\mathrm{u}}_7(a_7)^{e^{\gamma(t-a_7)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_7}\, da_7$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_7)$ gives,

$$\tilde{\mathrm{u}}_7(a_7) = e^{-\int_0^a \mu_7(\tau_7)d\tau - (f_{70})_7 a}\left(\tilde{\mathrm{u}}_7(0) - w\int_0^a e^{\int_0^\tau \mu_7(s_7)ds + (f_{70})}\tau_7\,(\alpha_7)_7(\tau_7)e^{\gamma_7\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\right)\tilde{u}_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a(\alpha_7)_7(\tau_7)e^{-\gamma_7\tau_7}e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds}d\tau_7\,da_7$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}da\right)(\tilde{u}_7(0))_7$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(\tau_7)d\tau}\int_0^a e^{(f_{70})_7\tau_7+\int_0^\tau \mu_7(s_7)ds+y\tau}\alpha_7(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_7 a_7-(f_{70})_7 a_7-\int_0^a \mu_7(t_7)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_7\,(a_7)Y_7(\gamma_7,a_7)da_7\right)u_7(0)$$

$$+w\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)\int_0^a(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7=0,$$

(24)

$$\left(\int_0^{\infty}Y_7(\gamma_7,a_7)da_7\right)\tilde{u}_7(0)$$

$$+\left(-1-\int_0^{\infty} Y_7(\gamma_7,a_7)\int_0^{a}(\alpha_7)_7(\tau)_7(Y^{-7})_7(\gamma_7,\tau_7)d\tau_7\,da_7\right)w=0.$$

Equation (20) has a solution of the form of $u_7(t,a_7)=u_7(a_7)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_7(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_7(\gamma_7,a_7)=\int_0^a(\rho_0)_7(\tau_7)Y^{-7}(\gamma_7,a_7)d\tau_7$. We obtain the characteristic equation

$$f_{70}^{l}{}_7\left(\int_0^{\infty} Y_7(\gamma_7,a_7)da_7\right)\left(\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$+\left(1-\int_0^{\infty}\beta_7(a_7)Y_7(\gamma_7,a_7)da\right)\left(1+(f_{70}^{l})_7\int_0^{\infty}Y_7(\gamma_7,a_7)I_7(\gamma_7,a_7)da_7\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_7(a_7)=(\rho_0)_7(0)e^{-\int_0^a\mu_7(\tau_7)d\tau_7-(f_{70})_7a_7}$$

$$I_7(\gamma_7,a_7)=\int_0^a(\rho_0)_7(\tau_7)e^{\gamma_7\tau_7+(f_{70})_7\tau+\int_0^{\tau}\mu_7(s_7)ds_7}d\tau$$

$$=\frac{(\rho_0)_7(0)}{\gamma_7}(e^{\gamma_7a_7}-1)$$

Let

$$E_7(a_7)=e^{-(f_{70})_7a_7-\int_0^a\mu_7(\tau_7)d\tau_7}$$

Then $[Y_7]_7[\gamma_7]_7, a_7) = E_7(a_7)e^{-\gamma_7 a_7}$. by a simple calculation,

$$\int_0^\infty \beta_7(a_7)Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma}\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}\ da\right);$$

$$\int_0^\infty Y_7(\gamma_7, a_7)I(\gamma_7, a_7)da = \frac{[\rho_0]_7(0)}{\gamma_7}\left(\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}\ da_7\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_7(0)}{\gamma_7}[f_{70}^7]\int_0^\infty E_7(a_7)e^{-\gamma_7 a_7}da_7\ (\int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7)$$

$$+\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right).$$

$$\left(1 + \frac{(\rho_0)_7(0)}{\gamma_7}(f_{70}^{'})_7\left(\int_0^\infty E_7(\,a_7)da_7 - \int_0^\infty E_7(\,a_7)e^{-\gamma_7 a_7}da_7\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7\right)(1 + \frac{(f_{70}^{'})_7}{\gamma_7}(\rho_0)_7(0)\int_0^\infty E_7(\,a_7)da_7) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (24), to

$$1 - \int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)da_7 = 1 - \int_0^\infty \beta_7(a_7)E_7(a_7)e^{-\gamma_7 a_7}da_7 \neq 0,$$

Since $\int_0^\infty \beta_7(a_7)Y_7(\gamma_7 a_7)\ \int_0^a \alpha_7\ (\tau_7)Y^{-7}(\gamma_7, \tau_7)d\tau_7 da_7 > 0$. Therefore, the characteristic solution is

$$\gamma_7 = -(f_{70}^{'})_7(\rho_0)_7 \int_0^{\infty} E_7(a_7)da_7.$$

Since $f_7^{'}(P_0)_7 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_7), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE OF DEBIT AND CREDIT TYPES

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_7), 0)$ which is related to the extinction of predators and to The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_7 + (\mu_7(a_7) + [f_7]_7(P_7^*) + [f_7]_7(Q_7^*))P_7^* = 0, \quad (76)$$

$$p_7^*(0) = h_7(Q_7^*)\int_0^{\infty} \beta_7(a_7)p_7^*(a_J)p_7^*(a_7)da_7 \quad (77)$$

$$g_7(P_7^*) - b_7 = 0, \quad (78)$$

Where $P_7^* = \int_0^{\infty} p_7^*(a_7)\, da_7$

Equation (28) yields $P^* = g^{-7}(b)$,

From (26),

$$p_7^*(a) = p_7^*(0)e^{\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}$$

Substituting into (27), we need to solve

$$\left(1 - h_7(Q_7^*)\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(Q_7^*)a_7}da_7\right)p_7^*(0) = 0$$

For $Q_7^*$

Fix $p_7^*(0)$ and $P_7^*$ Let

$$H_7(z_7) = 1 - h_7(z_7)\int_0^\infty \beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(P_7^*)a_7 - [f_7]_7(z_7)a_7}da_{,7}$$

And assume that

$$[k_3]_7\int_0^\infty \beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - [f_7]_7(g_7^{-7}(b_7))a_7 - [k_7]_7a_7}\, da_7 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_7(z_7)] > 0.$$

Clearly,

$$\frac{dH_7}{dz_7} = h_7(z)([f_7']_7(z_7))\int_0^\infty a_7\beta_7(a_7)\, e^{-\int_0^a \mu_7(\tau_7)d\tau_7 - f_7(P_7^*)a_7 - f_7(z_7)a_7}da_{7}$$

$-h'(z_7)\int_0^{\infty}\beta_7(a_7)\,e^{\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7-[f_7]_7(z_7)a_7}\quad da_7$

(See author's It is a debit-credit world)

By the assumptions on $\left[f_7^7\right]_7 and\ [h^7]^7 dH_7/dz > 0$. Hence, if

$$H_7(0) = 1-\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7(P_7^*)a_7}da_7 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_7(0) < 0$ follows if $g_7^-(b_7) < [f_7^{-7}]_7(r_7)$, where $r_7$ is the solution of (12).

Indeed, since

$$R_7(s_7) = \int_0^{\infty}\beta_7(a_7)\,e^{-\int_0^a \mu_7(\tau_7)d\tau_7 e^{-y\frac{a}{7}}}da_7$$

Is a decreasing function of $s_7$ when $P_7^* = [g^{-7}]_7(b_7) < [f_7^{-7}]_7(r_7)$, that is $[f_7]_7(P_7^*) < r$, $R\left(f_7(P^*)\right) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

**THEOREM 3**

Assume that the hypothesis (H3) holds and $R(f_7(g^{-7}(b))) > 1, i.e.,$

$$\int_0^{\infty}\beta_7(a_7)e^{-\int_0^a \mu_7(\tau_7)d\tau_7-[f_7]_7\left(g_7^{-7}(b)\right)a_7} > 1. \qquad (H4)$$

Then the system (4)-(6) has a unique positive equilibrium $(P_7^*(a_7), Q_7^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_7^*(a_7), Q_7^*)$ is more complicated. We again consider perturbations $m_7(t_7 a_7) = p_7(t_7, a_7) - p_7^*(a_7)$ and $W_7(t) = Q_7(t) - Q^*$.Then

$$M_7(t) \equiv \int_0^\infty m_7(t, a_7) da_7 = P_7(t) - P_7^*,$$

$$[m_7]_7 + [m_a]_7 = [p_7]_7 + [p_a]_7 - [p^*]_7(a_7) \approx -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7$$

$$-[f_{7*}^7]_7 p_7^*(a_7)M - [f_{7*}^7]_7 p_7^*(\mathrm{a})W_7,$$

$$m_7(t,0)p_7(t,0) - p_7^*(0) \approx h_* \int_0^\infty \beta_7(a_7) m_7(t, a_7)\, da + \frac{[h_*^7]_7}{[h_*]_7}\, p_7^*(0) W_7$$

$$\frac{dW_7}{dt} = \frac{dQ_7}{dt} \approx [g_*^7]_7 Q_7^* M,_7$$

Where $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}]_7$ denoting $[f_7(P^*)]_7$, $[f_{7*}']_7$ denoting $[f'_7(P^*)]_7[f_{7*}']_7$ denoting $[f_7]_7$ $(Q_7^*$, $h_{*7}$ denoting $h_*'$ denoting $h_7'$ $(Q_7^*)$, and $[g_*']_7$

Denoting $[g']_7(P_7^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_7]_7 + [m_a]_7 = -(\mu_7(a_7) + [f_{7*}]_7 + [f_{7*}]_7)m_7 - [f'_{7*}]_7[p_7^*](a_7)M_7 - [f_{7*}']_7 p_7^*(a_7)W_7$$

$$m_7(t,0) = [h_*]_7 \int_0^\infty \beta_7(a_7) m_7(t, a_7) da_7 + \frac{[h_*^7]_7}{[h^*]_7}[p^*]_7(0)\, W,_7 \qquad (30)$$

$$\frac{dW_7}{dt} = [g_*^7]_7 [Q_7^*] M_7$$

Integrating $[m_7]_7(t, a_7)$ with respect to a from 0 to ∞, we have

$$\frac{dM_7(t)}{dt} = \int_0^\infty [mt]_7(t, a_7)da_7$$

$$= -\int_0^\infty [m_a]_7\,(t, a_7)da_7 - \int_0^\infty \mu(a_7)m(t, a_7)da_7$$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f^{'}{}_{7*}]_7 \int_0^\infty p_7^*(a_7)da_7\right)M_7 - [f_{7*}^7]_7 \int_0^\infty p_7^*(a_7)da_7 W_7$$

$= m_7(t, 0) - \int_0^\infty \mu_7(a_7)\, m_7(t, a_7)da_7$

$$-\left([f_{7*}]_7 + [f_{7*}]_7 + [f_{7*}^{\cdot}]_7 P_7^*\right)M_7 - [f_{7*}^{\cdot}]_7 P_7^* W_{\cdot 7}$$

From (30), we arrive at the equations

$$\frac{dW_7(t)}{dt} = \int_0^\infty ([h_*]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7$$

$$-\left([f_{7*}] + [f_{7*}]_7 + [f_{7*}^{'}]_7 P_7^*\right)M_7 + \left(\frac{[h_*^{'}]_7}{[h_*]^7}\, P_7^*(0) - f_{7*}^{'} P_7^*\right)W_7,$$

(31)

$$\frac{dW_7(t)}{dt}[g^{'}{}_*]_7\, Q_7^*\, M_{\cdot 7}$$

By the mean value theorem,

$$\int_0^\infty ([h_{n*}]_7\beta_7(a_7) - \mu_7(a_7))m_7(t, a_7)da_7 = ([h_*]_7\beta_7(\theta_7) - \mu_7(\theta_7))M_{,7}$$

Where $0< \theta_7 = \theta_7(t) < \infty$.

And if a $\frac{7}{7}$ [a $\frac{7}{7}$]2<4 [a $_7$]2<4 a $_7$e,

Re $\lambda_{\,7}\left[\lambda_{\,7}\right]_{\,7} = \mathrm{Re}\lambda_{\,7}\left[\lambda_{\,7}\right]_{\,7} = -\frac{7}{7}a_7[\mathrm{a2}\ ]_{\,7} < 0.$

Let ⊿=max$_{7<u<\infty}$ $\left|\mathrm{h}_{\,*}\,[\mathrm{h}_{\,*}]_{\,7}\beta_{\,7}(a_7) - \mu_7(a_7)\right|$. Then$|n_7(\mathrm{t})|$ ⊿∀t ⓪. Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $\frac{7}{7}$ <4 a $_7$ e and [a $_7$] $> 1\Delta$,or [a $\frac{7}{7}$] ⓪[a $_7$]e and (a $_7 - \sqrt{(\mathrm{a}\frac{7}{7})_{\,7} - 4(\mathrm{a}_{\,7})_{\,7}\,\mathrm{e}} > 1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{\mathrm{a}}{\mathrm{a}+7}\right)$ and $\beta_{\,7}(a_7) = \mathrm{ae}^{\text{-ma}}$ Then ⊿=$[\mu_0] + 2$ Simple calculation leads to

$$\int_0^{\infty} \beta_{\,7}(a_7)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_7(\tau_{\,7}) + d\tau_7 \mathrm{da}} = \frac{1}{(\mathrm{m} + \mu_0 + 1)^{\,7}} + \frac{1}{(\mathrm{m} + \mu_0 + 1)^{\,3}}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_7 + [\mu_0]$ ⓪.5. Assume$[f_7](x_7)= K_7x_7$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_7(x_7)$= $[0x_7]/(2 + x_7)$; the functional responses $[[f_7](x) = [0]\, x_7/(1 + x_7)$and $h_7(x_7)$=(1+$\omega_7x_7$)/$(1 + x_7)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_7 + [\mu_0] + [\mathrm{K}][P_0])}{[m_7]_{\,7} + (\mu_0)_{\,7} + [\mathrm{K}][P_0])\,\frac{3}{7}\ [\mathrm{K}]_{\,7}[P_0]\,\frac{3}{7}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_7$ i e $[P_0] < b/(\delta_7 - b_7)$then the equilibrium$(P_0)$ $[P_0]_7(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_7$

$[f_{7*}] = K_7\ \mathrm{P}^*\ [f_{7*}] = \frac{\sigma \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*} \quad [h_*] = \frac{7+\omega_7 \mathrm{Q}_7^*}{7+\mathrm{Q}_7^*}$

$[f_{7*}] = K_7\ [\mathrm{f}_{7*}^{7}] = \frac{\sigma_7}{(7+\mathrm{Q}_7^*)^7}[\ \mathrm{g}_*^7] = \frac{\sigma_7}{(7+\mathrm{P}_7^*)^7} \quad [\mathrm{h}_*^7] = \frac{\omega_7 - 7}{(7+\mathrm{Q}_7^*)^7}$

In order to have (H3) and (H4) hold,$m, \mu_0,$ K, b, δ, and $\omega$ need to satisfy

$$\frac{\omega_7}{(m+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma)^7} + \frac{1\omega_7}{([m_7]_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7+b_7}+\sigma_7)^3} < 1$$

And

$$\frac{1}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^7} + \frac{1\omega_7}{(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^*$ which can be solved from the

following equation:

$(m_7+[\mu_0]_7+1+\frac{K_7 b_7}{\delta_7-b_7}+(m_7+[\mu_0]_7+\frac{K_7 b_7}{\delta_7-b_7}+\sigma_7)\ \mathrm{Q}_7^*)^3$

$=m_7+[\mu_0]_7+3+\frac{K_7 b_7}{\delta_7-b_7}(1+\mathrm{Q}^*)(1+\omega_7 \mathrm{Q}_7^*$

Now

$$\left[\mathrm{a}\begin{matrix}1\\1\end{matrix}\right]-4\left[\mathrm{a}_{7}\right]\mathrm{e}=\left(\frac{1K_7b_7}{\delta_7-b_7}\right)^{7}+\frac{4K_7b_7\sigma_7\,\mathrm{Q}_{7}^{*}}{(\delta_7-b_7)(7+\mathrm{Q}_{7}^{*})}+\frac{\sigma_{7}^{7}\mathrm{Q}_{7}^{*7}}{\left(1+\mathrm{Q}_{7}^{*}\right)^{7}}$$

$$\frac{4\,b_7\sigma_7\,(\delta_{7}-\mathrm{b})\mathrm{Q}^{*}}{\delta_{7}(1+\mathrm{Q}_{7}^{*})^{7}}-\frac{4\,(\delta_{7}-b_7)^{7}(1-\omega_7)\mathrm{Q}_{7}^{*}}{\delta_{7}\left(1+\omega_7\mathrm{Q}_{7}^{*}\right)(1+\mathrm{Q}_{7}^{*})}\mathrm{P}^{*}(0)$$

And

$$\Delta=\lim_{0<a<\infty}|[h_*]_{7}\,ae_{7}^{mu}\;\;[\mu_0]_{7}-\frac{a_7}{a_7+7}$$

Hence if $[\mathrm{a}_{7}^{7}>4(a_7)_{7}\mathrm{e}$ and

$$[a_7]=\frac{1K_7b_7}{\delta_7-b_7}+\frac{\sigma_7\,\mathrm{Q}_{7}^{*}}{1+\mathrm{Q}_{7}^{*}}>1\Delta$$

The nontrivial equilibrium is stable.

Some of the examples in which stability analysis is of seminal importance are: These are the system to which the model becomes applicable

**Observation of a Dissipation-Induced Classical to Quantum Transition J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck Phys. Rev. X 4, 031043 – Published 8 September 2014**

(1035) Here, authors report the experimental observation of (e) a dynamical quantum phase transition in (eb) a strongly interacting open photonic system.

(1036) The system studied, comprising a Jaynes-Cummings dimer realized on (eb) a superconducting circuit platform, exhibits (eb) a dissipation-driven localization transition.

(1037) Signatures of the transition in the homodyne signal and photon number reveal (eb) this transition to be from (e) a regime of classical oscillations into (e&eb) a macroscopically self-trapped state manifesting revivals, (=) a fundamentally quantum phenomenon.

(1038) This experiment also demonstrates (eb) a small-scale realization of a new class of quantum simulator, whose well-controlled (e&eb) coherent and dissipative dynamics is suited to (e) the study of quantum many-body phenomena out of (e) equilibrium. Received 10 December 2013 DOI:http://dx.doi.org/10.1103/PhysRevX.4.031043 This article is available under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Published by the American Physical Society

**Viewpoint: Fermionic Vortices Find their Dual John McGreevy, Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319, USA June 27, 2016• Physics 9, 72Theoretical work reveals a surprising relationship between the physics of fermionic vortices and quantum electrodynamics. Figure caption expand figure APS/Alan Stonebraker Figure 1: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).**

(1039) Dirac fermions in 2+1 dimensions (two spatial dimensions plus time) have been found in (eb) graphene and on (e&eb) the surfaces of topological insulators (TIs).

(1040) A trio of papers now shows that these particles can also be found in (eb) a dual description of quantum electrodynamics (QED) [1–3].

(1041) This deceptively simple statement joins (e&eb) together several disparate lines of progress in theoretical physics: interacting topological insulators, (e&eb) anomalies in quantum field theory, (e&eb) spin liquids, and (e&eb) compressible Quantum Hall States.

(1042) Humans are not very good at understanding interacting many-body systems.

(1043) We understand the behavior of free particles and can sometimes use (e) perturbation theory to (e) incorporate weak interactions between (e&eb) them.

(1044) But what happens when we couple (e&eb) the particles strongly?

(1045) To reveal the interesting collective phenomena that can result, we've had to rely largely on (e) experiments.

(1046) Examples include (e) superconductivity (e&eb) superfluidity (e&eb) the integer and (e&eb) fractional quantum Hall effects (e&eb) the fractionalization of charge and spin, and (e&eb) the wetness of water.

(1047) The fractional quantum Hall effect (FQHE) is a physical phenomenon in which the Hall conductance of 2D electrons shows (eb) precisely quantised plateaus at (eb) fractional values of (e) {\displaystyle e^{2}/h} e^{2}/h.

(1048) It is a property of (e) a collective state in which electrons bind (e&eb) magnetic flux lines to make (eb) new quasiparticles, and excitations have (e) a fractional elementary charge and possibly also fractional statistics. The 1998 Nobel Prize in Physics was awarded to Robert Laughlin, Horst Störmer, and Daniel Tsui for the discovery and explanation of the fractional Hall effect.[1]

(1049) However, Laughlin's explanation was (=) a phenomenological guess and only applies to (e&eb) fillings {\displaystyle \nu =1/m} \nu =1/m where (e) {\displaystyle m} m is an odd integer. The

| |
|---|
| microscopic origin of the FQHE remains unknown and is a major research topic in condensed matter physics. (Wikipedia) |
| (1050) The quantum Hall effect (or integer quantum Hall effect) is a quantum-mechanical version of (e) the Hall effect, observed in (eb) two-dimensional electron systems subjected to (e&eb) low temperatures and strong magnetic fields, in which the Hall conductance σ undergoes (e&eb) certain quantum Hall transitions to take on (e&eb) the quantized values {\displaystyle \sigma ={\frac {I_{\text{channel}}}{V_{\text{Hall}}}}=\nu \;{\frac {e^{2}}{h}},} \sigma = \frac{I_\text{channel}}{V_\text{Hall}} = \nu \; \frac{e^2}{h}, where {\displaystyle I_{\text{channel}}} I_\text{channel} is the channel current, {\displaystyle V_{\text{Hall}}} V_\text{Hall} is the Hall voltage, e is the elementary charge and h is Planck's constant. The prefactor ν is known as the "filling factor", and can take on either integer (ν = 1, 2, 3, ...) or fractional (ν = 1/3, 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5, ...) values. |
| (1051) The quantum Hall Effect is referred to as (=) the integer or fractional quantum Hall effect depending on whether ν is an integer or fraction, respectively. |
| (1052) The striking feature of the integer quantum Hall Effect is (=) the persistence of the quantization (i.e. the Hall plateau) as (e) the electron density is varied. |
| (1053) Since the latter remains constant when (e) the Fermi level is in a clean spectral gap, it must be that the Fermi level is immersed in (eb) essential spectrum (i.e. of finite density of states), though Anderson localized. |
| (1054) The quantization and stability of the Hall conductance in such conditions was achieved in 1994 by (e) Bellissard, van Elst and Schulz-Baldes using the methods of (e) Non-Commutative Geometry.[1] |
| (1055) The fractional quantum Hall effect is more complicated, as its existence relies fundamentally on (e) electron–electron interactions. |
| (1056) Although the microscopic origins of the fractional quantum Hall effect are unknown, there are several phenomenological approaches that provide (eb) accurate approximations. |
| (1057) For example the effect can be thought of as (=) an integer quantum Hall effect, not of electrons but of (e) charge-flux composites known as (=) composite fermions. |
| (1058) In 1988, it was proposed that there was quantum Hall effect without Landau levels. This quantum Hall Effect is referred to as (=) the quantum anomalous Hall (QAH) effect. |
| (1059) There is also a new concept of the quantum spin Hall effect which is an analogue of the quantum Hall effect, where (e) spin currents flow instead of (e&eb) charge currents.[2] |
| (1060) In two dimensions, when classical electrons are subjected to (e&eb) a magnetic field they follow (eb) circular cyclotron orbits. |
| (1061) When the system is treated quantum mechanically (eb) these orbits are quantized. |
| (1062) The energy levels of these quantized orbitals take on (e) discrete values |
| (1063) :{\displaystyle \textstyle E_{n}=\hbar \omega _{c}(n+1/2),} \textstyle E_{n}=\hbar \omega _{c}(n+1/2), where ωc = eB/m is the cyclotron frequency |
| (1064) These orbitals are known as Landau levels, and at weak magnetic fields, their existence gives rise to (eb) many interesting "quantum oscillations" such as the Shubnikov–de Haas oscillations and the de Haas–van Alphen effect (which is often used to map the Fermi surface of metals). |
| (1065) For strong magnetic fields, each Landau level is (=) highly degenerate (i.e. there are many single particle states which have the same energy En). |
| (1066) Specifically, for a sample of area A, in magnetic field B, the degeneracy of each Landau level is (=) {\displaystyle \textstyle N=g_{s} BA/\phi _ {0},} \textstyle N=g_{s} BA/\phi _ {0}, where gs represents (eb) a factor of 2 for spin degeneracy, and ϕ0 = 2·10−15 Wb is the magnetic flux quantum. |

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| (1067) | For sufficiently strong B-fields, each Landau level may have (e) so many states that all of the free electrons in the system sit in only a few Landau levels; it is in this regime where (e) one observes the quantum Hall Effect. |
| (1068) | The integers that appear in the Hall Effect are (=) examples of **topological quantum numbers**. |
| (1069) | They are known in mathematics as the first **Chern numbers** and are closely related to (e&eb) **Berry's phase.** |
| (1070) | A striking model of much interest in this context is the Azbel-Harper-Hofstadter model whose quantum phase diagram is (=) the Hofstadter butterfly shown in the figure. (deleted due to spatial constraints) |
| (1071) | The vertical axis is (=) the strength of the magnetic field and the horizontal axis is the chemical potential, which fixes the electron density. The colors represent the integer Hall conductance. Warm colors represent positive integers and cold colors negative integers. The phase diagram is fractal and has structure on all scales. In the figure there is an obvious self-similarity. |
| (1072) | Concerning physical mechanisms, impurities and/or particular states (e.g., edge currents) are important for (e) both the 'integer' and 'fractional' effects. |
| (1073) | In addition, Coulomb interaction is also essential in (eb) the fractional quantum Hall effect. |
| (1074) | The observed strong similarity between integer and (=) fractional quantum Hall effects is explained by (e) the tendency of electrons to form bound states with (e&eb) an even number of magnetic flux quanta, called composite fermions. |
| (1075) | Duality is the circumstance in which cranking up interactions results in (eb) another system we understand, usually one that can be described in terms of (e&eb) another set of weakly coupled particles. |
| (1076) | When such a dual representation is found, (eb) it gives a solution to the strong coupling problem. |
| (1077) | A further reason that theoretical physicists get excited about dualities is that they undermine (e) the very notion of an “elementary particle.” |
| (1078) | In terms of the original variables, the dual, weakly coupled particles are (=) generally solitons—large, heavy, classical objects made up of (e) many quanta. |
| (1079) | The remarkable thing is that solitons can become (=) “elementary,” in the sense that the model that describes the system can be reformulated with the solitons as (=) the basic constituents. |
| (1080) | An example of a duality, which we will use below, arises in (eb) Maxwell’s theory of electromagnetism, where (e) the strength of the interactions is determined by (e) the electric charge e. |
| (1081) | The duality exchanges electric and (e&eb) magnetic point charges, and the interaction strength in the dual description is 1⁄e, which is small when (e) e is large. |
| (1082) | This idea has been generalized to (e) many other field theories, thanks, in particular, to tools of (e) supersymmetry [4]. |
| (1083) | Another classic example of duality involves (e&eb) vortices. |
| (1084) | A vortex is a localized object in the plane around which a phase field winds (e&eb). (A phase field takes values on the circle.) |
| (1085) | Dualities relating particles and vortices have (e) a rich history in particle physics and condensed-matter physics. |
| (1086) | The simplest version relates the particles in the normal state of a superfluid to (e&eb) the vortices in a dual superconductor [5]. |
| (1087) | This charge-vortex duality has been (=) a workhorse in the study of two-dimensional condensed-matter systems, such as (=) fractional Quantum Hall States, superconducting thin films, and beyond-Landau critical phenomena [6]. |

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| (1088) | Vortices can be (=) fermions (as they frequently are, for instance, in type-II superconductors). |
| (1089) | A dual description of a system with (e&eb) fermionic vortices thus requires (e) a new ingredient, since (e) it must include (e) such a fermionic particle. |
| (1090) | The new work [1–3] builds on (e&eb) earlier theoretical work [7] to make (eb) an appealing proposal for (e) the required duality: the fermionic vortex of QED in (eb) 2+1 dimensions is (=) a free Dirac fermion. The argument for this proposal builds on our understanding of TIs. |
| (1091) | TIs are distinguished only by (e) their edge states. |
| (1092) | The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host boundary states whose band structure is defined by (e) a single Dirac cone. |
| (1093) | A single time-reversal-symmetric Dirac cone is not possible without (e) the bulk TI, so its appearance in experiments provides (eb) a signature of the bulk phase. |
| (1094) | But the Dirac cone is not (e) the only possible signature. |
| (1095) | Deformations of the boundary conditions can dramatically change (e&eb) the spectrum of the edge states, yet preserve some essential "TI-ness." This essential property is (=) called an anomaly in high-energy theory. |
| (1096) | Max Metlitski of the University of California, Santa Barbara, and Ashvin Vishwanath of the University of California, Berkeley [1], and independently, Chong Wang and Todadri Senthil at the Massachusetts Institute of Technology, Cambridge [2], use this ambiguity in the edge theory and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to (e&eb) the TI to argue for the duality. |
| (1097) | When I said that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, one could have (e) complained that the electron is (=) a fermion, while (e) a magnetic monopole may not be. |
| (1098) | However, the authors observe that in a TI, a charge-two monopole is (=) also a fermion. |
| (1099) | By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and (e&eb) the charge-two monopole and, moreover, exchanges a TI with (e&eb) time-reversal symmetry with (e&eb) a "chiral TI," which instead has (e) particle-hole symmetry |
| (1100) | . But this is just what is required of (e) a symmetry that exchanges a fermionic particle with (e&eb) a fermionic vortex: time-reversal symmetry maps the electron to itself but maps the vortex to (e&eb) another vortex whose phase winds (e&eb) the other way (see Fig. 1). |
| (1101) | The exchange of particles and holes, in turn, preserves (eb) the vortex. |
| (1102) | The bulk duality implies (eb) a duality between edge theories. |
| (1103) | The freedom to deform the edge of the TI implies (eb) a weak form of equivalence: although the edge theories are (=) topologically the same (in terms of symmetry assignments and (e&eb) anomalies), they need not have (e) identical low-energy physics. |
| (1104) | One side of the duality is (=) a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is (=) a long-standing problem, which is important in the study of spin liquids. |
| (1105) | Many gauge theories in two dimensions develop (eb) a gap in their energy spectrum because of (e) the proliferation of monopoles. |
| (1106) | In the realization proposed by Metlitski and Vishwanath [1] and by Wang and Senthil [2], this cannot happen because (e) the monopoles carry electric charge, and indeed the stronger notion of duality as (=) an equivalence of low-energy physics may hold. |

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| (1107) Evidence for the stronger relation comes from (e) David Mross and colleagues at the California Institute of Technology, Pasadena [3], who have derived a lattice version of the duality using (e) a clever but explicit change of variables. |
| (1108) Their construction is particularly interesting because (e) it uses (e) a lattice of one-dimensional models which themselves can only occur at the edge of a TI. |
| (1109) Authors explicitly derive the duality between a free electronic Dirac cone and (e&eb) quantum electrodynamics in (eb) (2+1) dimensions (QED3) with (e&eb) N=1 fermion flavors. |
| (1110) The duality proceeds via (e&eb) an exact, nonlocal mapping from electrons to (e&eb) dual fermions with (e&eb) long-range interactions encoded by (e) an emergent gauge field. |
| (1111) This mapping allows (eb) us to construct (eb) parent Hamiltonians for exotic topological-insulator surface phases, derive (eb) the particle-hole-symmetric field theory of (e) a half-filled Landau level, and nontrivially constrain (e) QED3 scaling dimensions. |
| (1112) They similarly establish duality between bosonic topological insulator surfaces and (e&eb) N=2 QED3. |

**Black hole thermodynamics with generalized uncertainty principle Li Xiang, X. Q. Wen (Submitted on 6 Jan 2009 (v1), last revised 18 Mar 2009 (this version, v2))**

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| (1113) In the standard viewpoint, the temperature of a stationary black hole is proportional to (=) its surface gravity, $T_H=\hbar\kappa/2\pi$. |
| (1114) This is a semiclassical result and the quantum gravity effects are not (e) taken into consideration. |
| (1115) This Letter explores a unified expression for (e) the black hole temperature in the sense of (e) a generalized uncertainty principle (GUP). |
| (1116) Discussion involves a heuristic analysis of (e) a particle which is absorbed by (e) the black hole. |
| (1117) Besides a class of static and spherically symmetric black holes, an axially symmetric Kerr-Newman black hole is considered. Different from the existing literature, authors suggest (eb) that the black hole's irreducible mass represent (eb) the characteristic size in (eb) the absorption process. |
| (1118) The information capacity of a remnant is also discussed by (e) Bousso's D-bound in de Sitter spacetime. Subjects: General Relativity and Quantum Cosmology (gr-qc) Journal reference: JHEP 0910:046,2009 DOI: 10.1088/1126-6708/2009/10/046 Cite as: arXiv:0901.0603 [gr-qc] (or arXiv:0901.0603v2 [gr-qc] for this version) |

IF we define

$$[a_7]_{\,7} = [f_{7*}]_{\,7} + [f_{7*}]_{\,7} + [f_{7*}]\,\mathrm{P}^{*}_{7}$$

$$[a_7]_{\,7} = \left[\mathrm{f}^{7}_{7*}\right]\mathrm{P}^{*}_{7} - \frac{\mathrm{h}^{7}_{*}}{\mathrm{h}_{*}}\mathrm{P}^{*}(0)$$

$$e=[\,\mathrm{g}^{7}_{*}]_{\,7}\,\mathrm{Q}^{*}_{7}$$

$$\eta_7(t) = \left[h_*\right]_7 \beta_7\left(\theta_7(t)\right) - \mu_7(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_7\right]M_7 - [a_7]_7 W_7 - [n(t)]M_7$$

$$w_7(t) = eM_7$$

Write

$$A=\begin{pmatrix} -[a_7]_7 & -[a_7]_7 \\ e & 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_7 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_7 \\ W_7 \end{pmatrix}$$

The solution of (32)is

$$X_7(t) = e^{A,t}[X]_7(0) + \int_0 e^{At}$$

**PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION**

**ATTRITION MODELS**

**MODULE EIGHT**

Jia li studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal.

## 106.MODEL FORMULATION

**Model is applied to systems given in the following:**

**Researchers blur the line between classical and quantum physics by connecting chaos and entanglement July 12, 2016 by Sonia Fernandez**

(6136) Experimental link between quantum entanglement (left) and (e&eb) classical chaos (right) found using (e) a small quantum computer. Credit: University of California - Santa Barbara Using a small quantum system consisting of three superconducting qubits, researchers at UC Santa Barbara and Google have uncovered a link between aspects of classical and (e&eb) quantum physics thought to be unrelated: classical chaos and (e&eb) quantum entanglement.

(6137) Their findings suggest (eb) that it would be possible to use (e) controllable quantum systems to investigate certain fundamental aspects of nature.

(6138) "It's kind of surprising because (e) chaos is this totally classical concept—there's (=) no idea of chaos in a quantum system," Charles Neill, a researcher in the UCSB Department of Physics and lead author of a paper that appears in Nature Physics.

(6139) "Similarly, there's no concept of entanglement within (eb) classical systems. And yet it turns out that chaos and (e&eb) entanglement are really very strongly and clearly related."

(6140) Initiated in the 15th century, classical physics generally examines and describes (eb) systems larger than atoms and molecules.

(6141) It consists of hundreds of years' worth of study including Newton's laws of motion, electrodynamics, relativity, thermodynamics as well as chaos theory—the field that studies the behavior of (e) highly sensitive and unpredictable systems.

(6142) One classic example of chaos theory is the weather, in which a relatively small change in one part of (e) the system is enough to foil predictions—and vacation plans—anywhere on the globe.

(6143) At smaller size and length scales in nature, however, such as those involving (e&eb) atoms and photons and their behaviors, classical physics falls short.

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| (6144) | In the early 20th century quantum physics emerged, with its seemingly counterintuitive and sometimes controversial science, including (e) the notions of superposition (the theory that a particle can be located in (eb) several places at once) and entanglement (particles that are deeply linked behave as (=) such despite physical distance from one another). |
| (6145) | And so began the continuing search for connections between the two fields. All systems are fundamentally quantum systems, according Neill, but the means of describing in a quantum sense the chaotic behavior of, (e) say, air molecules in an evacuated room, remains limited. |
| (6146) | Imagine taking a balloon full of air molecules, somehow tagging them so you could see them and then releasing them (eb) into a room with no air molecules, noted co-author and UCSB/Google researcher Pedram Roushan. |
| (6147) | One possible outcome is that the air molecules remain clumped together (e&eb) in a little cloud following the same trajectory around the room |
| (6148) | And yet, he continued, as we can probably intuit, the molecules will more likely take off in (eb) a variety of velocities and directions, bouncing off walls and interacting with (e&eb) each other, resting after the room is (=) sufficiently saturated with them. |
| (6149) | "The underlying physics is (=) chaos, essentially," he said. |
| (6150) | The molecules coming to rest—at least on (eb) the macroscopic level—is the result of (e) thermalization, or of reaching equilibrium after they have achieved (eb) uniform saturation within the system. |
| (6151) | But in the infinitesimal world of quantum physics, there is (=) still little to describe that behavior. |
| (6152) | The mathematics of quantum mechanics, Roushan said, do not allow for (e) the chaos described by Newtonian laws of motion. |
| (6153) | To investigate, the researchers devised an experiment using (e) three quantum bits, the basic computational units of (e) the quantum computer. |
| (6154) | Unlike classical computer bits, which utilize (e) a binary system of two possible states (e.g., zero/one), a qubit can also use (e) a superposition of both states (zero and one) as a single state. |
| (6155) | Additionally, multiple qubits can entangle, or link so closely that their measurements will (=) automatically correlate. |
| (6156) | By manipulating these qubits with electronic pulses, Neill caused them to interact, rotate and evolve (e&eb, e&eb, eb) in the quantum analog of (e) a highly sensitive classical system. |
| (6157) | The result is a map of entanglement entropy of (e) a qubit that, over time, comes to strongly resemble that of (e) classical dynamics—the regions of entanglement in (eb) the quantum map resemble the regions of (e) chaos on the classical map. |
| (6158) | The islands of low entanglement in (eb) the quantum map are located in the places of (e) low chaos on the classical map. |
| (6159) | "There's a very clear connection between entanglement and (e&eb) chaos in these two pictures," said Neill. |
| (6160) | "And, it turns out that thermalization is (=) the thing that connects chaos and (e&eb) entanglement. |
| (6161) | It turns out that they are actually the driving forces behind (eb) thermalization. |
| (6162) | "What we realize is that in almost any quantum system, including on quantum computers, if you just let it evolve and you start to study what happens as a function of time, (eb) it's going to thermalize," added Neill, referring to the quantum-level equilibration. "And this really ties together the intuition between classical thermalization and (e&eb) chaos and how it occurs (eb) in quantum systems that entangle." |

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| (6163) | The study's findings have fundamental implications for (e) quantum computing. |
| (6164) | At the level of three qubits, the computation is relatively simple, said Roushan, but as researchers push to build increasingly sophisticated and powerful quantum computers that incorporate (e) more qubits to study highly complex problems that are beyond the ability of classical computing—such as those in the realms of machine learning, artificial intelligence, fluid dynamics or chemistry—a quantum processor optimized for such calculations will be a very powerful tool. |
| (6165) | "It means we can study things that are completely impossible to study right now, once we get to bigger systems," said Neill.More information: Nature Physics, DOI: 10.1038/nphys3830 , http://www.nature.com/nphys/journal/vaop/ncurrent/full/nphys3830.html Journal reference: Nature Physics search and more info website Provided by: University of California - Santa Barbara |

**Title: coherent control of macroscopic quantum states in a single-Cooper-pair box Authors: Nakamura, Y.; Pashkin, Yu. A.; Tsai, J. S. Publication: Nature, Volume 398, Issue 6730, pp. 786-788 (1999). (Nature Homepage) Publication Date: 04/1999 Origin: NATURE DOI: 10.1038/19718 Bibliographic Code: 1999Natur.398...786N**

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| (6166) | A nanometre-scale superconducting electrode connected to (e&eb) a reservoir via (e&eb) a Josephson junction constitutes (e) an artificial two-level electronic system: a single-Cooper-pair box. |
| (6167) | The two levels consist of (e) charge states (differing by 2e, where (e) e is the electronic charge) that are coupled by (e) tunnelling of Cooper pairs through (e&eb) the junction. |
| (6168) | Although the two-level system is macroscopic, containing (e) a large number of electrons, the two charge states can be (=) coherently superposed. |
| (6169) | The Cooper-pair box has therefore been suggested as (=) a candidate for a quantum bit or `qubit'- the basic component of a quantum computer. |
| (6170) | Here authors report the observation of quantum oscillations in (eb) a single-Cooper-pair box. |
| (6171) | By applying a short voltage pulse via (e&eb) a gate electrode, we can control (e&eb) the coherent quantum state evolution: the pulse modifies (e&eb) the energies of (e) the two charge states non-adiabatically, bringing them into (e&eb) resonance. |
| (6172) | The resulting state-a superposition of (e) the two charge states-is detected by (e) a tunnelling current through (e&eb) a probe junction. |
| (6173) | Results demonstrate electrical coherent control of (e) a qubit in a solid-state electronic device. |

**Superconducting quantum computing From Wikipedia, the free encyclopedia**

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| (6174) | Superconducting quantum computing is a promising implementation of (e) quantum information technology that involves (e&eb) nanofabricated superconducting electrodes coupled through (e&eb) Josephson junctions. |
| (6175) | As in a superconducting electrode, the phase and (e&eb) the charge are conjugate variables. |
| (6176) | There exist three families of superconducting qubits, depending on (=) whether the charge, (e&eb) the phase, or neither of (e) the two is good quantum number. |
| (6177) | These are respectively termed charge qubits (e&eb) flux qubits, and (e&eb) hybrid qubits. |

**Theory**

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| (6178) | Unlike many other physical implementations of a qubit which involve (e&eb) exclusively two-level systems (such as nuclear spin and (e&eb) photon polarization), the integrated quantum circuit |

involved in (e&eb) a superconducting qubit is (=) a multi-level system, of which only the first two levels are used as (=) the computational basis.

(6179) A basic requirement for such an implementation is that the energy levels are (=) not uniformly spaced, so that (e) photons of a particular frequency which cause (eb) transition between the 0 and 1 levels do not(e) cause transitions from the first level to (e&eb) the higher levels as well.

(6180) For electronic signals to be carried from one part of the circuit to (e&eb) another without energy loss and (e&eb) hence decoherence, the system also needs to be (=) non-dissipative, i.e., the metallic parts involved should have (e) zero resistance.

(6181) These circuits currently need to be (=) operated at very low temperatures so that, 1) superconductivity is (=) realized, and 2) thermal fluctuations do not (e) cause transitions between (e&eb) energy levels.

**Quantum LC Circuit**

(6182) The simplest non-dissipative quantum circuit consists simply of (e) an inductor L {\displaystyle L} L and (e&eb) capacitor C {\displaystyle C} C, with (e&eb) the metallic wires connecting them being superconducting.

(6183) Here the flux Φ {\displaystyle \Phi } \Phi through (e&eb) the inductor and (e&eb) the charge Q {\displaystyle Q} Q in the capacitor are (=) canonically conjugate variables which obey (eb) the commutation relation [ Φ , Q ] = i ℏ {\displaystyle [\Phi ,Q]=i\hbar } [\Phi ,Q]=i\hbar

**The Hamiltonian associated with this system is**

(6184) H = Φ 2 2 L + Q 2 2 C {\displaystyle H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}} H={\frac {\Phi ^{2}}{2L}}+{\frac {Q^{2}}{2C}}

(6185) This gives rise to the energy levels E = ℏ ω 0 ( n + 1 2 ) {\displaystyle E=\hbar \omega _{0}(n+{\frac {1}{2}})} E=\hbar \omega _{0}(n+{\frac {1}{2}}) Here ω 0 = 1 L C {\displaystyle \omega _{0}={\frac {1}{\sqrt {LC}}}} \omega _{0}={\frac {1}{{\sqrt {LC}}}}

**Quantum gravity Carlo Rovelli (2008), Scholarpedia, 3(5):7117. doi:10.4249/scholarpedia.7117 revision #135539 [link to/cite this article] Post-publication activity Curator: Carlo Rovelli Dr. Carlo Rovelli, Center of Theoretical Physics, and Marseille, France**

(6186) Quantum Gravity is the name given to any theory that describes gravity in (eb) the regimes where quantum effects cannot be disregarded.

(6187) At present, there is no such a theory which is universally accepted and confirmed by experience. Therefore the term "Quantum Gravity" indicates (eb) more an open problem than a specific theory.

(6188) Several research lines, at different levels of development, offer tentative solutions to the problem. These tentative quantum-gravity theories are variously viewed as competing research directions, or as contributions to the common goal of finding the physically correct theory. The quest for the good quantum theory of gravity bears on a number of fundamental issues and it is sometimes presented as the most important open problem in fundamental physics: the "Holy Grail" of contemporary theoretical physics.

**The problem of quantum gravity**

(6189) The current theory of (e) gravity is general relativity, a theory developed by Albert Einstein at the beginning of the XX century, which has changed in depth our understanding of (e) what are space and time.

(6190) Similarly, quantum mechanics has changed in depth our understanding of matter, energy and causality. These two theories, today very well empirically supported, have opened (eb) a major conceptual revolution in physics.

(6191) However, they appear to be incompatible, at least at a first reading, because (e) each of the two is formulated on the basis of (e) principles that are explicitly contradicted by (e) the other theory.

(6192) Therefore in spite of the immense development of scientific knowledge it has delivered, the physics of the XX century has left us with (e&eb) a major confusion on what are the basic conceptual ingredients for (e) understanding the physical world.

(6193) In other words, the scientific revolution opened by general relativity and quantum mechanics at the beginning of (e) the XX century is not concluded yet, and a new synthesis is required. Quantum gravity, merging general relativity and quantum mechanics should be this new synthesis.

**Quantum space and quantum time**

(6194) Quantum gravity is expected to force us to further modifications of (e&eb) the concepts of space and time, along (e&eb) the direction opened by Einstein's general relativity, in order to make them fully compatible with quantum theory.

(6195) In general relativity, space and time lose (e) their properties of being a fixed framework in which the dynamical world is (=) immersed.

(6196) They are identified with (=) the gravitational field and acquire (e) dynamical properties: spacetime can "fold" and (e&eb) "stretch" like a rubber sheet.

(6197) When we take quantum mechanics into account, we realize that this "rubber sheet" is (=) in fact a quantum field, and therefore, like all quantum fields, it must have (e) a microscopic granular structure (like the photons forming the electromagnetic field) and a probabilistic dynamics.

(6198) Therefore quantum gravity is likely to be (=) the theory of granular and probabilistic "quantum space" and (e&eb) "quantum time". (Model is given in the next paper). Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

**The Planck scale**

(6199) Simple dimensional arguments show that (eb) the physical phenomena where (e) quantum gravitational effects becomes relevant as those characterized by (e) the length scale lPlanck=ħG/c3−−−−−√∼10−33cm, called the "Planck length".

(6200) Here $\hbar$ is the Planck constant that governs (e&eb) the scale of the quantum effects, G is (=) the Newton constant that governs (e&eb) the strength of the gravitational force, and c is the speed of light, which governs (e&eb) the scale of the relativistic effects.

(6201) The Planck length is extremely small. To have an idea, the Planck length is (=) as many times smaller than an atom, as an atom is (=) smaller than the solar system.

(6202) Current technology is not yet capable of observing physical effects at (eb) scales that are so small (although several recent suggestions of how it could be possible to do so have appeared.)

(6203) Because of this, we have no direct experimental guidance for building a quantum theory of gravity. This is not by itself a complete impediment, because general relativity and quantum mechanics are

themselves strong guidance for constructing the theory: several major advances in the history of physics have been obtained in (eb) the absence of new experiments, from (e) the effort of merging two empirically supported but apparently contradictory theories. (Examples are Newton's merge of Kepler's and Galileo's theories, Maxwell's merge of electric and (e&eb) magnetic theory, or Einstein's derivation of special relativity from (e) the apparent contradiction between electromagnetism and mechanics.) However, until genuine quantum gravitational phenomena are directly or indirectly observed, (eb) we cannot confirm or falsify any of the current tentative theories.

## Physical applications

Some phenomena where quantum gravity plays a major role are the following.

### The microscopic structure of spacetime

(6204) If we could measure the geometry of space and time at the Planck scale, (eb) we should be able to see quantum gravitational effects.

(6205) Many arguments indicate (eb) that the Planck length may appear as a limit to the infinite divisibility of space, that is, as (=) a minimal length

(6206) Intuitively, any attempt to measure smaller distances would result in (eb) the concentration of (e) too much energy in too small a space, with the result of forming (eb) a micro-black-hole, effectively subtracting (e) the region from observation.

(6207) A minimal length would complete the turn of fundamental scales in Nature, together with (e&eb) the speed of light, which is the maximal velocity of a body, and the Planck constant, which is (=) the minimal amount of action exchanged between two systems.

### Early cosmology

(6208) According to the currently standard cosmological model, the Universe was very dense and hot in the past. Extrapolating back the model, we encounter (e&eb) a singular point of infinite density, temperature and curvature conventionally denoted the Big Bang.

(6209) However, this final extrapolation is certainly incorrect because (e) quantum gravitational phenomena become (=) dominant when (e) the universe is very dense and hot, and these effects are not (e) included in the usual model.

(6210) A quantum theory of gravity is needed in order to take these effects into (e) account and study the early instants in the life of our universe.

(6211) Some current theories of gravity (in particular loop quantum gravity, see below), indicate (eb) that the singular Big-Bang point is never reached, and the current expansion of the universe might have been preceded by (e) a collapsing phase, and a "Big Bounce".

(6212) One of the major hopes of observing traces of quantum gravitational phenomena is (=) in this cosmological context, as traces of (e) early universe phenomena left in (eb) the cosmic background radiation currently under (e&eb) intense observation, or in the background gravitational wave radiation, which is expected to be observed in (eb) the next decade.

### Black holes

(6213) Quantum gravity should play a role in (e&eb) several aspects of black-hole physics.

(6214) First, it should give a complete understanding of (e) the thermal radiation that black holes are expected to produce (eb) first computed by Stephen Hawking.

(6215) Second, Hawking's analysis shows that black holes carry (e&eb) enormous entropy about 10 to the power 77 for a solar mass black hole.

(6216) What is the statistical mechanical origin of this number which is enormous even by the standards of thermodynamics? Third, quantum gravity is expected to replace the infinite singularity that general relativity predicts at the center of black holes with (e&eb) a more physically reasonable picture.

(6217) Finally, the theory should explain what happens at the end of the Hawking evaporation of (e) a black hole.

**Astrophysical effects**

(6218) Several astrophysical quantum-gravitational effects have been suggested. None has been observed so far, but different calculations suggest (eb) that they might observed be in the near future.

(6219) An example is a small dependence of the speed of light on the color of the light, caused by (e) the Planck-scale granularity of space.

(6220) The effect is very small because of (e) the smallness of the Planck scale, but it might become detectable if (e) it is cumulated over a very long intergalactic path traveled by (e) the light. Observations for testing this prediction are ongoing.

**Main approaches**

(6221) The following approaches to quantum gravity open fascinating perspectives on the structure of Nature, and might be correct. But for the moment, none of these offers a complete theory of quantum gravity and none of these has been corroborated by experience. Because we do not have experimental data with direct bearing on quantum gravity, of necessity each program singles out certain aspects as being fundamental and focuses on them in the hope that other problems can be (=) easily sorted out later.

(6222) Opinions vary widely on the relative merit of (e) these approaches and the polemic, as often in science, is sometimes fiery.

(6223) The list that follows is roughly ordered according to (e&eb) the number of scientists working in each line of research. One should however remember that history teaches us that often a large majority of scientists has been working on the wrong tentative theory, with only a few working on the theory that ended up to be the correct: scientific truth is judged by experimental corroboration, not by a democratic counting of votes! These theories must all be considered tentative, at present.

**String theory**

(6224) String theory is not just a theory of quantum gravity in the strict sense, because (e) its objective is wider: the theory aims at giving a unified description of (e) the physical world, where all physical entities are understood as manifestations of (e) the motion of a single object: a string.

(6225) Gravity emerges in the theory as (=) one of the aspects of (e) the dynamics of the string.

(6226) String theory can be defined in terms of (e&eb) a perturbation expansion around (e&eb) a fixed spacetime.

(6227) Remarkably, in this formulation certain infinities that plague (e&eb) perturbative quantum general relativity do not (e) appear and there is an ongoing program to establish that each term in the series is (=) in fact finite.

(6228) However, when summed the entire series appears to be (=) divergent.

(6229) In any case, a definition of the theory as (=) a perturbation expansion is not sufficient for describing genuine quantum gravitational phenomena, which appear in (eb) the nonperturbative regime.

(6230) Therefore much of the research in this area in recent years focuses on (e&eb) nonperturbative effects.

(6231) Numerous indications point out to the existence of (e) a fundamental nonperturbative definition of (e) string theory, to which the various perturbative formulations should (eb) converge, and various partial attempts to define (eb) this nonperturbative theory are ongoing.

(6232) In this sought-for nonperturbative formulation, the characteristic features of quantum gravity become (=) manifest: for instance, the lack of a fixed background space and time, and (e&eb) the resulting conceptual difficulties. Research is currently active to try to find this fundamental description of string theory.

**Loop quantum gravity and spin foams**

(6233) Loop quantum gravity is the most developed attempt to solve (e) the specific problem of finding a quantum version of general relativity, and thus a genuine merge of the conceptual novelties represented by (e) general relativity and quantum mechanics.

(6234) The theory is consistent with the other fundamental physical theories (such as the standard model of particle physics), but it does not unify gravity and (e&eb) these theories in the sense of presenting them as manifestation of the dynamics of a single physical entity.

(6235) Loop quantum gravity is based in (eb) the formulation of general relativity developed by (e) Abhay Ashtekar, and offers a precise mathematical description of quantum spacetime.

(6236) The granular properties of space can be (=) explicitly computed.

(6237) In particular, a result of the theory is that the area and the volume of (e) any physical surface of region turns out to be (=) "quantized" (like the energy of a hydrogen atom), and the corresponding (e&eb) discrete values that area and volume can take have been computed within the theory.

(6238) The quantum states of physical space are described in the theory by (e) labelled graphs called spin networks.

(6239) Each node of a spin network represents (eb) an elementary "quantum of space", and the links between these indicates (eb) who is next to whom, building (eb) the spacial structure.

(6240) The main incompleteness of the theory regards the relation between the Planck scale and (e&eb) macroscopic physics, and the consistency of (e) its classical limit.

(6241) Related to loop quantum gravity is (=) a covariant approach to (e) quantum that goes under (e&eb) the name of spin foam formalism.

(6242) This is sometimes presented as (=) the path-integral or, covariant (Lagrangian) version of (e) the canonical (Hamiltonian) loop formalism.

**Noncommutative geometry**

(6243) Einstein's discovery is that gravity, which is a dynamical field, is (=) the geometry of spacetime.

(6244) Quantum mechanics teaches us that dynamical quantities are (=) noncommutative --meaning that the outcome of the measurement of two of their properties depends on (eb) the order in which the measurement is performed.

(6245) It is therefore natural to suspect that the mathematics needed to describe quantum spacetime is (=) a noncommutative version of geometry.

(6246) A number of different formulations of such a noncommutative theory of geometry are under study. One line of research sees (eb) noncommutative spaces as (=) effective (approximate) descriptions of the deep quantum structure of spacetime and studies quantum field theory over (e&eb) such spaces.

(6247) Alain Connes has developed a beautiful mathematical framework for (e) describing noncommutative metric spaces.

(6248) This framework appears to shed surprising light on the physical geometry underlying (e&eb) the standard model of particle physics and its relation with (e&eb) general relativity, and opens an intriguing perspectives towards quantum gravity.

**Others**

(6249) Among these, Roger Penrose has introduced an approach to quantum gravity denoted "Twistor theory", in which the primary elements for (e) the description of spacetime are not (e) the spacetime points but the light rays

(6250) . Rafael Sorkin has introduced an approach denoted "Coset (or Causal Set) theory", where (e) quantum spacetime is described by (e) probabilistic superpositions of sets of points whose only structure is (=)their causal connection.

(6251) Finally, approaches that must be mentioned is (=) simply the possibility of taking general relativity and merge it with (e&eb) quantum mechanics following the conventional methods of quantum field theory, but circumventing (e) the traditional difficulties by (e) an intelligent retuning of (e&eb) the quantization method itself. Several authors have questioned the conventional assumption that this is manifestly impossible.

(6252) The common hope, today, is that the development of the theoretical research and, hopefully, some direct input from experiments and observations could soon bring to a convergence or a selection among these tentative theories, and therefore more clarity on the quantum nature of (e&eb) space and time.

Let $p_8(t, a_8)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_8}{dt} = -b_8 Q_8 + g_8(g_8)Q_8$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_8(0) = 0, \quad g_8(P_8) > 0, \quad P_8 \ 0, \qquad \lim_{P_8 \to \infty} g_8(P_8) = (K_8)_8 \ \infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_8 + (\rho_a)_8 + D_8(a_8, P_8, Q_8)\rho_8 + 0, \qquad 0 < a_8 \ \infty,$$

$$\rho_8(t, 0) = \int_0^{\infty} n_8(a_8, P_8, Q_8)\rho_8(t_8, a_8)da_8$$

Where $D_8$ is the mortality function and $n_8$ is the fertility function.

Put $_8$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_8}{dt}\right]_8 = x_8\left[\frac{dQ_8}{dt}\right]_8 - x_8\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_8}{dt}\right]_8 = x_3\left[\frac{dQ_3}{dt}\right]_8 - x_4\left[\frac{dQ_8}{dt}\right]_8$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_8}{dt}\right]_8 - x_6\left[\frac{dQ_3}{dt}\right]_8$$

Where $x_8, x_8, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_8$ and $g_8(P_8)Q_8$

Now when you add the term $(b_8Q_8)$ with $g_8(P_8)(Q_8)$ we are infect see is that $(b_8Q_8)$ is credited incrementally part of $g_8(P_8)(Q_8)$ or instant nearly, when subtracted from $\frac{dQ_8}{dt}$ , it means that $g_8(P_8)(Q_8)$ is slowly eaten by $\frac{dQ_8}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_8 = \mu_8(a_8) + (f_8)_8(P_8)_8 + (f_8)_8(Q)_8 \quad (2)$$

$$n_8 = \beta_8(a_8)h_8(Q_8), \quad (3)$$

Where $\mu_8$the age- dependent debit rate whit is $\mu_8(a_8)$ 0 $\forall a > 0$; $(f_8)_8$ is the density- dependent death rate with $f_8(0) = 0$, $(f_8^8)_8(P_8)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_8)_8(P_8) = (k_8)_8$ ; $f_8$ is the functional response of creditors to prey with $[f_8]_8(0) = 0, (f_8^8)_8(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_8)_8(Q_8) = [k_8]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ 0, $\forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^8(Q)$ 0, $\forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_8 + (\rho_a)_8 = -\mu_8(a_8) + (f_8)_8P_8) + \big(f_8)_8(Q_8)\big)(\rho_8)_8 \quad (4)$$

$$\rho_8(t_8, 0) = h_8(Q_8)\int_0^\infty \beta_8(a_8)\rho_8(t, a_8)da, \quad (5)$$

$$\frac{dQ_8}{dt} = -b_8Q_8 + g_8(P_8)Q_8 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_8$ and $Q_8$ of (4-6) satisfy

$$0 \ \rho_8(t, a_8) \ \rho_8^*(t, a_8),$$

$$0 \ Q_8(t) \ Q_8^*(t),$$

Where $\rho_8^*(t, a_8)$ and $Q_8^*(t)$, are solutions of

$$(\rho_t^*)_8 + (\rho_a^*)_8 = -(\mu_8(a_8) + (f_8)_8(P_8^*) + (f_8)_8Q_8^*))\,\rho_8^*, \quad (7)$$

$$\rho_8^*(t,0) = \int_0^\infty \beta_8(a_8)\,\rho_8^*(t,a_8)da_8 \quad (8)$$

$$\frac{dQ_8^*}{dt} = -b_8\,Q_8^* + g_8(P_8^*)Q_8^*, \quad (9)$$

With $P_8^*(t) = \int_0^\infty \rho_8^*(t,a_8)da_8$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_8^*(t)}{dt} = P_8^*(t)\big(r_8 - (f_8)(P_8^*) - (f_8)_8(Q^*)\big), \quad (10)$$

$$\frac{dQ_8^*(t)}{dt} = Q_8^*(t)(-b_8 + g_8(P_8^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_8(a)e^{-\int_0^a \mu_8(\tau)dt}e^{-ra}\,da_8 = 1. \quad (12)$$

If $r < 0$, the trivial solution $P_8^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(t)dt}\,da_8 > 1, \quad (H1)$$

Which guarantees $r_8 > 0$.

*Remark.* If $\beta_8(a) = [\beta_0]_8 a_8 e^{-\alpha_8 a_8}, \mu_8(a_8) \equiv \mu_8$ then the solution of (12) is

$$r_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_8]_8(P_8) = 0, [f_8](Q_8) = \lambda_8 Q_8\ h_8(Q_8) = 1, \qquad and\ \ g_8(P_8) = c_8 P_8$$

$$[\rho_8]_8 + [\rho_a]_8 = -(\mu_8 + \lambda_8 Q_8)\rho_8$$

$$\rho_8(t,0) = \int_0^\infty [\beta_0]_8 a_8 e^{-\alpha_8 a_8} \rho_8(t, a_8)\, da_8$$

$$\frac{dQ_8}{dt} = -b_8 Q_8 + c_8 P_8 Q_8$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_8}{dt} = \omega_8 P_8 - \lambda_8 P_8 Q_8$$

$$\frac{dQ_8}{dt} = b_8 Q_8 - c_8 P_8 Q_8$$

Where $\omega_8 = \sqrt{\beta_8} - \alpha_8 - \mu_8$

## 107.THE EQUILIBRIUM $(\rho_0)_8(a_8), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_8(a_8), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_8 + (\rho_a)_8 = -\big(\mu_8(a_8) + (f_8)_8(P_8)\big)\rho,$$

$$\rho_8(t,0) = \int_0^\infty \beta_8(a_8)\rho_8(t,a_8)da_8 \qquad (13)$$

The solution of (13) has the form of

$$\rho_8(t,a_8) = Ce_8^{r(t-a_8)} - \int_0^a (\tau)d\tau - \int_0^8 (f_8)_8\big(P_8(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_8(t) = C_8 e^{r_8 t - \int_0^t (f_8)_8(P_8(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_8^{-8}(r)$ if $k_8 > r$. This is equivalent to

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{(k_8)_8 a_8} da_8 < 1.$$

In fact, if we define

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau)d\tau} e^{-s_8 a_8}\, da_8$$

$R(s)$ is a decreasing function. $R_8(r_8) = 1$ And $R_8(k_8)_8 < 1$ imply $r_8 < (k_8)_8$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_8(a_8), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{k_8 a_8} da_8 < 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_8 \equiv \int_0^\infty (\rho_0)_{88}(a_8)da_8$ *satisfies*

$$\int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau)d\tau} e^{-(f_8)_8(P_0)a_8} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_8(a_8), 0)$

Let $u_8(t, a_8) = \rho(t, a_8) - \rho_0(a_8)$ Then

$$N_8(t) \equiv \int_0^{\infty} u_8(t, a_8) da_8 = \int_0^{\infty} \rho_8(t, a_8) da_8 - \int_0^{\infty} (\rho_0)_8(a_8) da = P_8(t) - P_0)_8$$

And

$$(u_8)_8 + (u_a)_8 = (\rho_t)_8 + (\rho_a)_8 - (\rho_0^8)_8(a_8)$$

$$= -\mu_8(a_8) + (f_8)_8(P_8) + (f_8)_8(Q_8))\,(u_8 + (\rho_0)_8) + \mu_8(a_8)(\rho_0)_8 + (f_8)_8(P_0)_8(\rho_0)_8$$

$$= -\mu_8(a_8)u_8 - (f_8)_8(P_8) + (f_8)_8(Q_8))\,u_8 - (\rho_0)_8\,(f_8)_8(P_8) - (f_8)_8(P_0)_8$$

$$+ (f_8)_8(Q_8)), \qquad (15)$$

Where $(\rho_0^8)_8(a_8)$ denotes $d(\rho_0)_8(a_8)/da_8$

Expansion $(f_8)_8(P_8)$ and $(f_8)_8(Q_8)$ at $(P_0)_8$ and 0 yield respectively

$$(f_8)_8(P_8) = (f_8)_8((P_0)_8 + N_8)$$

$$= (f_8)_8(P_0)_8 + (f_8^8)_8(P_0)_8 N_8 + o_8(N_8), \qquad (16)$$

$$(f_8)_8(Q_8)$$

$$= (f_8^8)_8(0)Q + o_8(Q_8) \qquad (17)$$

Write $f_8(P_0) = f_{80}, f_8'(P_0) = f_{80}'$, and $f_8'(0) = f_{80}'$ and substitute (16) and (17) into (15) to obtain

$$(u_8)_8 + (u_a)_8 \approx -(\mu_8(a_8) + (f_{80})_8)u - (f_{80}^8)_8(\rho_0)_8(N_8) - (f_{80}^8)_8(\rho_0)_8(Q_8)$$

$$u_8(t, 0) = \rho_8(t, 0)(\rho_0)_8(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a) + (\rho_0)_8(a_8)da_8 - (\rho_0)_8\,(0)$$

$$= h_8(Q_8)\int_0^{\infty} \beta_8(a_8)(u_8(t,a_8)da_8 + (h_8(Q_8) - 1)(\rho_0)_8\,(0) \qquad (18)$$

Expanding

$$h_8(Q_8) = 1 + \ (h^8)_8(0)Q_8 + 0(Q_8) \equiv 1 + h_0^{'}Q_8 + 0Q_8$$

And substituting it into (18), we arrive at

$$u_8(t,0) \approx \int_0^{\infty} \beta_8\,(a_8)u_8(t,a_8)da_8 + (h_{a_8}^{'})(\rho_0)_8Q_8$$

The linearization is the following :

$$[u_8] + [u_8] = -(\mu_8(a_8) + [t_{80}]_8u_8 - [\alpha_8]_8(a_8)N_8 - [\alpha_8]_8(a_8)Q_8$$

$$u_8(t,0) = \int_0^{\infty} \beta_8(a_8)u_8(t,a_8)da_8 + [\xi_8]Q_8 \qquad (19)$$

$$\frac{dQ_8}{dt} = -b_8Q_8$$

where

$$[\alpha_8]_8(a_8) = [t_8^8a]_8[p,o]_8(a_8)d2(a_8) = [f_{80}^8]_8(\rho\ o\ )_8(a_8)\xi = [\mathrm{h\ o}^8]_8[\rho\mathrm{o}]_8(\mathrm{o}) \text{ and } \mathrm{b}_8 = \mathrm{b}_8 - g_8(P_8)$$

If $g_8(P_0) > b$, clearly, $(\rho_0)_8(a_8), 0)$ is unstable and the predator population is persistent. Assume $g_8(P_0)_8 < b$.

Then, $Q_8(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_8 + u_a = -(\mu_8(a_8) + (f_{80})_8)u_8 - (\alpha_8)_8(a_8)N_8$$

$$= -(\mu_8(a_8) + f_{80})u_8 - \alpha_8(a_8)\int_0^\infty u_8(t, a_8)da_8$$

$$u_8(t, 0) = \int_0^\infty \beta_8\,(a_8)u_8(t, a_8)da_8$$

Using the method developed by Cushing [2, 3], we suppose

$$u_8(t, a_8) = \tilde{u}_8(a_8)^{e^{\gamma(t-a_8)}}$$

And define $w = \int_0^\infty \tilde{u}(a)e^{-\gamma a_8}\,da_8$. Then,

A solution of the first differential equation for $\tilde{u}(a_8)$ gives,

$$\tilde{u}_8(a_8) = e^{-\int_0^a \mu_8(\tau_8)d\tau - (f_{80})_8 a}\left(\tilde{u}_8(0) - w\int_0^a e^{\int_0^\tau \mu_8(s_8)ds + (f_{80})}\tau_8\,(\alpha_8)_8(\tau_8)e^{\gamma_8\tau}d\tau\right). \quad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1 - \int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\right)\tilde{u}_8(0)$$

$$+w\int_0^\infty \beta_8(a_8)e^{-\gamma_8 a_8 - (f_{80})_8 a_8 - \int_0^a \mu_8(\tau_8)d\tau}\int_0^a (\alpha_8)_8(\tau_8)e^{-\gamma_8\tau_8}e^{(f_{80})_8\tau_8 + \int_0^\tau \mu_8(s_8)ds}d\tau_8\,da_8$$

$$= 0 \quad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau} da\right)(\tilde{u}_8(0))_8$$

$$+\left(-1-\int_0^{\infty} e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(\tau_8)d\tau}\int_0^a e^{(f_{80})_8\tau_8+\int_0^\tau \mu_8(s_8)ds+y\tau}\alpha_8(\tau)d\tau\, da\right)w=0. \quad (23)$$

Define $Y(\gamma,a)=e^{-\gamma_8 a_8-(f_{80})_8 a_8-\int_0^a \mu_8(t_8)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da_8\right)u_8(0)$$

$$+w\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8=0, \quad (24)$$

$$\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\tilde{u}_8(0)$$

$$+\left(-1-\int_0^{\infty}Y_8(\gamma_8,a_8)\int_0^a(\alpha_8)_8(\tau)_8(Y^{-8})_8(\gamma_8,\tau_8)d\tau_8\, da_8\right)w=0.$$

Equation (20) has a solution of the form of $u_8(t,a_8)=u_8(a_8)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_8(0)\not\equiv 0$ and $\gamma,w$ in the complwx plane.

Let $I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)Y^{-8}(\gamma_8,a_8)d\tau_8$. We obtain the characteristic equation

$$f_{80_8}^{l}\left(\int_0^{\infty}Y_8(\gamma_8,a_8)da_8\right)\left(\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)da\right)\left(1+(f_{80}^{l})_8\int_0^{\infty}Y_8(\gamma_8,a_8)I_8(\gamma_8,a_8)da_8\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_8(a_8)=(\rho_0)_8(0)e^{-\int_0^a\mu_8(\tau_8)d\tau_8-(f_{80})_8a_8}$$

$$I_8(\gamma_8,a_8)=\int_0^a(\rho_0)_8(\tau_8)e^{\gamma_8\tau_8+(f_{80})_8\tau+\int_0^{\tau}\mu_8(s_8)ds_8}d\tau$$

$$=\frac{(\rho_0)_8(0)}{\gamma_8}(e^{\gamma_8a_8}-1)$$

Let

$$E_8(a_8)=e^{-(f_{80})_8a_8-\int_0^a\mu_8(\tau_8)d\tau_8}$$

Then $[Y_8]_8[\gamma_8]_8,a_8)=E_8(a_8)e^{-\gamma_8a_8}$. by a simple calculation,

$$\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma}\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}\ da\right);$$

$$\int_0^{\infty}Y_8(\gamma_8,a_8)I(\gamma_8,a_8)da=\frac{[\rho_0]_8(0)}{\gamma_8}\left(\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}\ da_8\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_8(0)}{\gamma_8}[f_{80}^{8}]\int_0^{\infty}E_8(a_8)e^{-\gamma_8a_8}da_8\left(\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8a_8}da_8\right)$$

$$+\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right).$$

$$\left(1+\frac{(\rho_0)_8(0)}{\gamma_8}(f_{80}^{'})_8\left(\int_0^{\infty}E_8(a_8)da_8-\int_0^{\infty}E_8(a_8)e^{-\gamma_8 a_8}da_8\right)\right)=0.$$

Hence the characteristic equation becomes

$$\left(1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\right)(1+\frac{(f_{80}^{'})_8}{\gamma_8}(\rho_0)_8(0)\int_0^{\infty}E_8(a_8)da_8)=0.$$

We need $\tilde{u}(a)\not\equiv 0$, namely $w\neq 0$. This necessarily leads, from (24), to

$$1-\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)da_8=1-\int_0^{\infty}\beta_8(a_8)E_8(a_8)e^{-\gamma_8 a_8}da_8\neq 0,$$

Since $\int_0^{\infty}\beta_8(a_8)Y_8(\gamma_8 a_8)\int_0^{a}\alpha_8(\tau_8)Y^{-8}(\gamma_8,\tau_8)d\tau_8 da_8>0$. Therefore, the characteristic solution is

$$\gamma_8=-(f_{80}^{'})_8(\rho_0)_8\int_0^{\infty}E_8(a_8)da_8.$$

Since $f_8^{'}(P_0)_8>0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_8),0)$ is asymptotically stable if $g(P_0)<b$. It is unstable if $g(P_0)>b$.

3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_8), 0)$ which is related to the extinction of predators and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_8 + (\mu_8(a_8) + [f_8]_8(P_8^*) + [f_8]_8(Q_8^*))P_8^* = 0, \quad (26)$$

$$p_8^*(0) = h_8(Q_8^*)\int_0^\infty \beta_8(a_8)p_8^*(a_J)p_8^*(a_8)da_8 \quad (27)$$

$$g_8(P_8^*) - b_8 = 0, \quad (28)$$

Where $P_8^* = \int_0^\infty p_8^*(a_8)\, da_8$

Equation (28) yields $P^* = g^{-8}(b)$,

From (26),

$$p_8^*(a) = p_8^*(0)e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}$$

Substituting into (27), we need to solve

$$\left(1 - h_8(Q_8^*)\int_0^\infty \beta_8(a_8)e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(Q_8^*)a_8}da_8\right)p_8^*(0) = 0$$

For $Q_8^*$

Fix $p_8^*(0)$ and $P_8^*$ Let

$$H_8(z_8) = 1 - h_8(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\, da_{,8}$$

And assume that

$$[k_3]_8 \int_0^{\infty} \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(g_8^{-8}(b_8))a_8 - [k_8]_8 a_8}\, da_8 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_8(z_8)] > 0.$$

Clearly,

$$\frac{dH_8}{dz_8} = h_8(z)(\,[f_8^{'}]_8(z_8)\,)\int_0^{\infty} a_8\beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - f_8(P_8^*)a_8 - f_8(z_8)a_8}\, da_{\square}$$

$$\text{-}h^{'}(z_8)\int_0^{\infty} \beta_8(a_8)\, e^{\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8 - [f_8]_8(z_8)a_8}\quad da_8$$

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By the assumptions on $\left[f_8^8\right]_8 and\ [h^8]^8 dH_8/dz > 0$. Hence, if

$$H_8(0) = 1 - \int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8(P_8^*)a_8} da_8 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_8(0) < 0$ follows if $g_8^-(b_8) < [f_8^{-8}]_8(r_8)$, where $r_8$ is the solution of (12).

Indeed, since

$$R_8(s_8) = \int_0^\infty \beta_8(a_8)\, e^{-\int_0^a \mu_8(\tau_8)d\tau_8 e^{-y_8^a} da_8}$$

Is a decreasing function of $s_8$ when $P_8^* = [g^{-8}]_8(b_8) < [f_8^{-8}]_8(r_8)$, that is $[f_8]_8(P_8^*) < r$, $R\big(f_8(P^*)\big) > 1$ which implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_8(g^{-8}(b))) > 1, i.e.,$

$$\int_0^\infty \beta_8(a_8) e^{-\int_0^a \mu_8(\tau_8)d\tau_8 - [f_8]_8\left(g_8^{-8}(b)\right)a_8} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_8^*(a_8), Q_8^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_8^*(a_8), Q_8^*)$ is more complicated. We again consider perturbations $m_8(t_8 a_8) = p_8(t_8, a_8) - p_8^*(a_8)$ and $W_8(t) = Q_8(t) - Q^*$.Then

$M_8(t) \equiv \int_0^\infty m_8(t, a_8)da_8 = P_8(t) - P_8^*,$

$$[m_8]_8 + [m_a]_8 = [p_8]_8 + [p_a]_8 - \left[p^*\right]_8(a_8) \approx -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8$$

$$-[f_{8*}^8]_8 p_8^*(a_8)M - [f_{8*}^8]_8 p_8^*(\mathrm{a})W_8,$$

$$m_8(t,0)p_8(t,0) - p_8^*(0) \approx h_* \int_0^\infty \beta_8(a_8)m_8(t,a_8)\,da + \frac{[h_*^8]_8}{[h_*]_8}\, p_8^*(0)W_8$$

$$\frac{dW_8}{dt} = \frac{dQ_8}{dt} \approx [g_*^8]_8 Q_8^* M,_8$$

Where $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $[f_{8*}]_8$ denoting $[f_8(P^*)]_8$, $\left[f_{8*}^{'}\right]_8$ denoting $\left[f^{'}{}_8(P^*)\right]_8 \left[f_{8*}^{'}\right]_8$ denoting $[f_8]_8$ $(Q_8^*,$ $h_{8}^{*}$ denoting $h_*^{'}$ denoting $h_8^{'}$ $(Q_8^*)$, and $\left[g_*^{'}\right]_8$

Denoting $[\dot{g}]_8(P_8^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_8]_8 + [m_a]_8 = -(\mu_8(a_8) + [f_{8*}]_8 + [f_{8*}]_8)m_8 - [\dot{f}_{8*}]_8[p_8^*](a_8)M_8 - [\dot{f}_{8*}]_8 p_8^*(a_8)W_8$$

$$m_8(t,0) = [h_*]_8 \int_0^\infty \beta_8(a_8)m_8(t,a_8)da_8 + \frac{[h_*^8]_8}{[h^*]_8}[p^*]_8(0)\, W_{,8} \qquad (30)$$

$$\frac{dW_8}{dt} = [g_*^8]_8[Q_8^*]M_8$$

Integrating $[m_8]_8(t,a_8)$ with respect to a from 0 to $\infty$, we have

$$\frac{dM_8(t)}{dt} = \int_0^\infty [mt]_8(t,a_8)da_8$$

$$= -\int_0^\infty [m_a]_8\,(t,a_8)da_8 - \int_0^\infty \mu(a_8)m(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 \int_0^\infty p_8^*(a_8)da_8\right)M_8 - [f_{8*}^8]_8 \int_0^\infty p_8^*(a_8)da_8 W_8$$

$$= m_8(t,0) - \int_0^\infty \mu_8(a_8)\, m_8(t,a_8)da_8$$

$$-\left([f_{8*}]_8 + [f_{8*}]_8 + [\dot{f}_{8*}]_8 P_8^*\right)M_8 - [\dot{f}_{8*}]_8 P_8^* W_{,8}$$

From (30), we arrive at the equations

$$\frac{dW_8(t)}{dt} = \int_0^\infty ([h_*]_8\beta_8(a_8) - \mu_8(a_8))m_8(t,a_8)da_8$$

$$-\left([f_{8*}]+[f_{8*}]_8+\left[f_{8*}^{'}\right]_8 P_8^*\right)M_8+\left(\frac{\left[h_*^{'}\right]_8}{[h_*]^8}\,P_8^*(0)-f_{8*}^{'}P_8^*\right)W_8,$$

(31)

$$\frac{dW_8(t)}{dt}\left[g^{'}{}_*\right]_8 Q_8^*\,M_{.8}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_8\beta_8(a_8)-\mu_8(a_8))m_8(t,a_8)da_8=([h_*]_8\beta_8(\theta_8)-\mu_8(\theta_8))M_{,8}$$

Where $0<\ \theta_8=\theta_8(t)<\infty$.

And if a $^8_8$ [a $^8_8$]2<4 [a $_8$]2<4 a $_8$e,

Re $\lambda_8\left[\lambda_8\right]_8=\mathrm{Re}\lambda_8\left[\lambda_8\right]_8=-\frac{8}{8}a_8[\mathrm{a2}\ ]_8<0.$

Let ⊿=max$_{8<u<\infty}\left|\mathrm{h}_*\,[\mathrm{h}_*]_8\beta_8(a_8)-\mu_8(a_8)\right|$. Then|$n_8$(t)| ⊿∀t . Hence we obtain the following result.

THEOREM 4. Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium(ρ $^*$ (a), $\boldsymbol{Q}^*$) is asymptotically stable if either a $^8_8$ <4 a $_8$ e and [a $_8$] > 1⊿,or [a $^8_8$] [a $_8$]e and (a $_8-\sqrt{(\mathrm{a}^8_8)_8-4(\mathrm{a}_8)_8\,\mathrm{e})}>1\Delta$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a)=\mu_0+\left(\frac{\mathrm{a}}{\mathrm{a}+8}\right)$ and β $_8(a_8)$ = ae-$^{\mathrm{ma}}$ Then ⊿=[$\mu_0$] + 2 Simple calculation leads to

$$\int_0^{\infty} \beta_8(a_8)\mathrm{e}^{-\int_0^{\mathrm{a}} \mu_8(\tau_8)+d\tau_8}\mathrm{da} = \frac{1}{(\mathrm{m}+\mu_0+1)^8}+\frac{1}{(\mathrm{m}+\mu_0+1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_8 + [\mu_0]$ 0.5. Assume$[f_8](x_8)= K_8x_8$ Then the hypothesis (H2) is satisfied We take the conversion rate function $g_8(x_8)= [0x_8]/(2 + x_8)$; the functional responses $[[f_8](x) = [0]\, x_8/(1 + x_8)$and $h_8(x_8)$=$(1+\omega_8x_8)/(1 + x_8)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_8 + [\mu_0] + [\mathrm{K}][P_0])}{[m_8]_8 + (\mu_0)_8 + [\mathrm{K}][P_0])\,{}_8^3\,[\mathrm{K}]_8[P_0]_8^3} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1 + P_0) < b_8$i e $[P_0] < b/(\delta_8 - b_8)$then the equilibrium$(P_0)$ $[P_0]_8(\mathrm{a}), 0)$ is stable This implies that the cedit confederation goes to extinction but that the cedit confederation persists $b_8$

$[f_{8*}] = K_8\ \mathrm{P}^*\ [f_{8*}] = \frac{\sigma \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$ $[h_*] = \frac{8+\omega_8 \mathrm{Q}_8^*}{8+\mathrm{Q}_8^*}$

$[f_{8*}] = K_8\ [\mathrm{f}_{8*}^{\,8}] = \frac{\sigma_8}{(8+\mathrm{Q}_8^*)^8}[\ \mathrm{g}_*^{\,8}] = \frac{\sigma_8}{(8+\mathrm{P}_8^*)^8}$ $[\mathrm{h}_*^{\,8}] = \frac{\omega_8-8}{(8+\mathrm{Q}_8^*)^8}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta$, and $\omega$ need to satisfy

$$\frac{\omega_8}{(m+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma)^8}+\frac{1\omega_8}{([m_8]_8+[\mu_0]_8+1+\frac{K_8b_8}{\delta_8+b_8}+\sigma_8)^3}<1$$

And

$$\frac{1}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^8} + \frac{1\omega_8}{(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8})^3} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $Q^*$ which can be solved from the following equation:

$(m_8 + [\mu_0]_8 + 1 + \frac{K_8 b_8}{\delta_8 - b_8} + (m_8 + [\mu_0]_8 + \frac{K_8 b_8}{\delta_8 - b_8} + \sigma_8)\, Q_8^*)^3$

$= m_8 + [\mu_0]_8 + 3 + \frac{K_8 b_8}{\delta_8 - b_8}(1 + Q^*)(1 + \omega_8 Q_8^*$

Now

$$\left[a_1^1\right] - 4\left[a_8\right] e = \left(\frac{1K_8 b_8}{\delta_8 - b_8}\right)^8 + \frac{4K_8 b_8 \sigma_8 Q_8^*}{(\delta_8 - b_8)(8 + Q_8^*)} + \frac{\sigma_8^8 Q_8^{*8}}{(1 + Q_8^*)^8}$$

$$\frac{4\, b_8 \sigma_8 (\delta_8 - b) Q^*}{\delta_8 (1 + Q_8^*)^8} - \frac{4\, (\delta_8 - b_8)^8 (1 - \omega_8) Q_8^*}{\delta_8 (1 + \omega_8 Q_8^*)(1 + Q_8^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_8\, ae_8^{mu}\ \ [\mu_0]_8 - \frac{a_8}{a_8 + 8}$$

Hence if $[a_8^8 > 4(a_8)_8 e$ and

$$[a_8] = \frac{1K_8 b_8}{\delta_8 - b_8} + \frac{\sigma_8 \, Q_8^*}{1 + Q_8^*} > 1\vartriangle$$

**The nontrivial equilibrium is stable.**

IF we define

$$[a_8]_{\,8} = [f_{8*}]_{\,8} + [f_{8*}]_{\,8} + [f_{8*}]\, P_8^*$$

$$[a_8]_{\,8} = \left[f_{8*}^{\,8}\right] P_8^* - \frac{h_*^8}{h_*} P^*(0)$$

$$e = [\, g_*^8]_{\,8}\, Q_8^*$$

$$\eta_8(t) = \left[h_*\right]_{\,8}\, \beta_8 \left(\theta_8(t)\right) - \mu_8\left(\theta(t)\right),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_8\right] M_8 - [a_8]_{\,8} W_8 - [n(t)] M_8$$

$$w_8(t) = eM_8$$

Write

$$A = \begin{pmatrix} -[a_8]_{\,8} & -[a_8]_{\,8} \\ e & 0 \end{pmatrix}$$

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| $B=(t)=\begin{pmatrix} [\eta(t)]_8 & 0 \\ 0 & 0 \end{pmatrix}$ |
| And |
| $X(t)=\begin{pmatrix} M_8 \\ W_8 \end{pmatrix}$ |
| The solution of (32)is |
| $X_8(t) = e^{A,t}[X]_8(0) + \int_0 e^{At}$ |
| This equation notwithstanding its importance in all the models stand out for its importance for the following models<br><br>to which it is applied: |
| **Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions David F. Mross, Jason Alicea, and Olexei I. Motrunich Phys. Rev. Lett. 117, 016802 – Published 27 June 2016** |

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| (694)These developments solve outstanding theoretical puzzles about (e&eb) interacting TIs and (e&eb) their symmetric, gapped, topologically ordered surface states [1, 2], and they have inspired furious activity in the field. |
| (695)The "auxiliary" bulk gauge theory formulated in Refs. [1, 2] actually predicts (eb) a novel (3+1)-dimensional spin liquid with (e&eb) time-reversal symmetry [8], which is (=) a candidate description of certain pyrochlore magnets. |
| (696) Perhaps most interesting is the connection to the manifestly particle-hole symmetric theory of (e) a half-filled Landau level [7], the best-understood non-Fermi liquid state. |
| (697) Briefly, the successful theory of this state makes a distinction between half-filled and (e&eb) half-empty electronic bands. |
| (698)The associated particle-hole symmetry is (=) emergent and an exact local realization can only occur at (eb) the edge of a TI |
| (699) The main outcome is a new understanding of (e) the Dirac-like nature of the composite fermion excitation of (e) this system, which provides (eb) a new point of view on the phenomenology [7, 9]. |
| (700)This circle of ideas has also led to (eb) new insights [10] on two-dimensional superfluid-insulator transitions with (e&eb) nearby metallic phases, where (e) the fermions come alive. |
| (701)More recently, consequences of (e) the duality have been checked mathematically and numerically and the fermionic vortex duality has been fit into (e&eb) the wider web of field-theory dualities [11]. This research is published in Physical Review Letters, Physical Review B, and Physical Review X. |

**Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators Max A. Metlitski and Ashvin Vishwanath Phys. Rev. B 93, 245151 – Published 27 June 2016**

(702)Particle-vortex duality is (=) a powerful theoretical tool that has been used to (e) study bosonic systems.

(703)Here, authors propose an analogous duality for (e) Dirac fermions in 2+1 dimensions.

(704) The physics of a single Dirac cone is proposed to be described by (e) a dual theory, QED3, with again a single Dirac fermion but coupled to (e&eb) a gauge field.

(705)This duality is established by considering two alternate descriptions of (e) the three-dimensional topological insulator (TI) surface.

(706)The first description is the usual Dirac fermion surface state. The dual description is accessed via (e&eb) an electric-magnetic duality of (e) the bulk TI coupled to (e&eb) a gauge field, which maps it to (e&eb) a gauged chiral topological insulator.

(707)This alternate description ultimately leads to (eb) a new surface theory, QED3, which provides (eb) a simple description of otherwise intractable interacting electronic states.

(708) For example, an explicit derivation of (e) the T-Pfaffian state, a proposed surface topological order of (e) the TI, is obtained by (e) simply pair condensing the dual fermions.

(709)The roles of time-reversal and (e&eb) particle-hole symmetries are exchanged by (e&eb) the duality, which connects some of results to (e&eb) a recent conjecture by Son on particle-hole symmetric Quantum Hall States. Received 15 April 2016 DOI:http://dx.doi.org/10.1103/PhysRevB.93.245151 ©2016 American Physical Society

**Viewpoint: Many-Body Localization Needs a Bath David A. Huse, Physics Department, Princeton University, Princeton, NJ 08544, USA and Institute for Advanced Study, Princeton, NJ 08540, USA July 5, 2016• Physics 9, 76 Theoretical work proves that interacting quantum systems can enter a many-body localized phase in which they cannot reach thermal equilibrium without an external bath. Figure caption expand figure Chris Laumann/University of Washington (deleted due to spatial restrictions). For references please see the original article**

(710)Figure 1: The work of Imbrie [2] provides (eb) the proof that quantum systems can enter a (e&eb) many-body localized (MBL) phase, in which they cannot thermally equilibrate under (e&eb) their own internal dynamics.

(711)We learn in our statistical mechanics classes that a system of many interacting degrees of (e) freedom isolated from (e) the external environment will, under (e&eb) its own internal dynamics, reach (eb) thermal equilibrium after (e) a sufficiently long time.

(712)When this is true—as is often the case—the system serves as (=) a "bath" for its own subsystems: Any small part of the full system sees (e) the rest of the system as (=) a reservoir with which it can exchange (e&eb) energy, particles, and quantum entanglement. This process is called **thermalization.**

(713)However, there are some systems of interacting quantum particles or spins that are believed not to be able to serve as (=) a bath for themselves.

(714)Such many-body localized (MBL) [1] systems remain localized in states that are close to (e) their initial conditions (see Fig. 1).

(715)Bringing them to thermal equilibrium requires (e) an external bath.

(716) But so far, there has been no rigorous proof of (e) the existence of **such many-body localization**. Now, theoretical work by John Imbrie at the University of Virginia, Charlottesville [2], provides such a proof—although relying on (e) one very reasonable assumption.

(717)The concept of **localization** was introduced by Philip Anderson in 1958 [3], when (e) he showed that interacting spins in (eb) lightly doped semiconductors may not be able to (e) thermally equilibrate without (e) any help from phonons (the material's lattice vibrations).

(718)The relationship between Anderson localization and (e&eb) thermalization received little attention for almost half a century.

(719)But more recently, this topic was turned into a trending research field by great progress seen in (eb) many-body atomic, molecular, and optical physics, and quantum information.

(720)Such developments have allowed researchers to engineer and investigate (e&eb) experimentally a wide variety of systems—from trapped cold atoms to (e&eb) spin ensembles—containing (e) many interacting quantum degrees of (e) freedom that are (=) well isolated from (e) their environment.

(721)For quantum particles or waves in (eb) a disordered medium that do not (e) interact with each other, Anderson localization is well understood.

(722) It has been experimentally observed for phonons [4], photons [5], and ultracold atoms [6]. In 2006, Denis Basko, Igor Aleiner, and Boris Altshuler investigated whether an Anderson-localized system at (eb) nonzero temperature would remain localized when (e) small interactions between nearby particles are (=) introduced [7].

(723) Studying such systems with perturbation theory, they found (eb) that this is indeed true to (e) all orders of approximation, thus establishing (eb) the concept of the MBL phase as (=) a state of matter at (eb) nonzero temperature.

(724) One striking feature of such MBL systems is that they have (e) strictly zero thermal and electrical conductivity: they are perfect insulators.

(725)However, even if "true to all orders" in perturbation theory, the conclusion by Basko, Aleiner, and Altshuler does not (e) rule out another possibility: many-body localization might only be (=) metastable, with (e&eb) the system appearing localized but actually thermalizing on (e&eb) long time scales because of (e) subtle effects that are not captured by (e) perturbation theory. One such scenario has been published earlier this year [8].

(726)Imbrie's new work [2] makes important progress towards a conclusive proof of a MBL phase by essentially ruling out (e) all such nonperturbative effects for certain systems.

(727) Treating a particular model of interacting spins, he shows (eb) that all eigenstates of its Hamiltonian are (=) MBL.

(728)As a consequence, the system—evolving under (e&eb) its own Hamiltonian—cannot (e) reach a non-MBL state, and thermalization is (=) blocked.

(729)His conclusion holds for a broad class of 1D system with (e&eb) short-range interactions between particles or spins on a lattice.

(730) An example that could be realized experimentally with ultracold atoms is (=) a Bose-Hubbard system: Bosonic atoms hop on (eb) a 1D lattice in which the potential energy varies randomly from site to (e) site, and the atoms interact when (e) they are on the same site.

(731)This result gives us confidence that some 1D systems undergo (e&eb) a true many-body localization phase transition between a delocalized or thermal phase (in which the system is (=) a bath for itself) and (e&eb) the MBL phase (in which it fails to be (=) a bath for itself).

(732)This novel quantum phase transition is a topic of active research, but it remains poorly understood [1]. Recent experiments—one with ions trapped in (e&eb) 1D (but with longer-range interactions than (e) those considered by Imbrie) [9] and others with (e&eb) ultracold atoms with (e&eb) short-range interactions (but trapped in (e&eb) more than one dimension) [10]—explored such a possible phase transition.

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| (733)However, in these cases, which are not covered by (e) Imbrie's proof, it remains possible that many-body localization is (=) not a true phase but (=) only a metastable regime: |
| (734) These systems might be (=) almost localized but still become (=) a bath for themselves on extremely long time scales (too long to be detected in these experiments) because of (e) nonperturbative effects. |
| (735)It is important to note that Imbrie's proof is valid under (e&eb) an assumption that is actually very reasonable, which he calls (=) "limited level attraction": the hypothesis that the eigenenergies of (e) the system's Hamiltonian do not (e) accumulate too strongly near (eb) degenerate points (states with almost the same energies). |
| (736) This is a realistic scenario for (e) all known systems. |
| (737)On the one hand, in systems that thermalize, near-degenerate eigenstates show (eb) energy-level repulsion due to (e) quantum-mechanical effects. |
| (738)On the other hand, in MBL systems, energy levels neither attract nor repel e*(e&eb) each other. |
| (739)No scenario with strong level attraction is known that would violate (e) Imbrie's assumption, but Imbrie could not mathematically prove that this seemingly obvious assumption is true. |
| (740) His proof thus remains, in principle, incomplete until this assumption is (=) proven or a general proof that does not (e) rely on this assumption is found. |
| (741)Many intriguing questions remain open. First, what is the nature of the quantum phase transition between quantum thermalization and (e&eb) many-body localization? |
| (742)Second, what are the possible systems in which there is (=) a sharply distinct MBL phase that is stable against (e&eb) all nonperturbative effects? |
| (743)Imbrie's proof partially answers the second question, saying that the MBL phase is indeed stable in (=) certain 1D systems with (e&eb) short-range interactions. |
| (744)So in those systems, the quantum phase transition does indeed exist. |
| (745)But Imbrie's proof does not work for (e) longer-range interactions or in more than one dimension. |
| (746)Does this limitation reflect (e&eb) the impossibility of a stable MBL phase outside of (e) the cases his approach can tackle, or is this (=) just a limitation of the mathematical approach he uses, and we may thus find stable many-body localization in (eb) a much wider class of systems? This research is published in Physical Review Letters. |

## PRESCIENTIAL DYNAMICS AND CONTENTIAL STATICITY OF ACCENTUATION ATTRITION MODELS

## MODULE NINE

**Abstract: Jiali studied predator – prey model in which the prey population is assumed to have age structure and is governed by the McKendrick- von Forester partial differential equation and the predator population is described by the classical Volterra- Lotka ordinary differential equation. Quite general hypotheses are assumed for the mortality function, the fertility function, and the functional responses of predation. Existence and stability of three biologically meaningful equilibria, corresponding to extinction of both species, persistence of one species prey, and coexistence of two species, are studied. A particular example**

**and some numerical result are given. (Vide journal of mathematical analysis and applications 152, 399-415) 1990. We demonstrate its applicability cosmology/ quantum computation economics latal. Model is applied to following formulations Current bounds from (e) the polarization of the CMB predict (eb) the scale-invariant gravitational wave (GW) background of (e) inflation to be out of reach for (e) upcoming GW interferometers. This prospect dramatically changes if the inflaton is (=) a pseudoscalar, in which case its generic coupling to (e&eb) any abelian gauge field provides (eb) a new source of GWs, directly related to (e&eb) the dynamics of inflation. This opens up (eb) new ways of probing (e&eb) the scalar potential responsible for (e) cosmic inflation. Dividing inflation models into universality classes, authors analyze (e&eb) the possible observational signatures. One of the most promising scenarios is (=) Starobinsky inflation, which may lead to (eb) observational signatures both in (eb) direct GW detection as well as in (eb) upcoming CMB detectors. In this case, the complementarity between the CMB and (e&eb) direct GW detection, as well as the possibility of a multi-frequency analysis with (e&eb) upcoming ground and (e&eb) space based GW interferometers, may provide (eb) a first clue to the microphysics of (e) inflation. Subjects: Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Phenomenology (hep-ph) Cite as: arXiv: 1603.01287 [astro-ph.CO] (or arXiv: 1603.01287v1 [astro-ph.CO] for this version Primordial gravitational waves for universality classes of pseudoscalar inflation Valerie Domcke, Mauro Pieroni, Pierre Binétruy (Submitted on 3 Mar 2016)STOP An analysis of general scalar-tensor gravitation theory, containing (e) two arbitrary functions of (e) the scalar field, is presented. The weak-field limit is considered in detail, and predictions for (e) the classical tests of gravitation theory are derived. A definite relationship between the light propagation and (e&eb) perihelion shift effects is found to hold under (e&eb) very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and (e&eb) tensor components are indicated. Finally, the relationship between the properties of the source and (e&eb) its radiation is considered in the weak-source limit, and expressions for (e) the rate of energy loss are derived. It is shown that the existing observational data are consistent with (e&eb) the possibility that the scalar field represents (eb) a major component of gravitational radiation from (e) astronomical sources. Received 18 February 1970 DOI:http://dx.doi.org/10.1103/PhysRevD.1.3209 ©1970 American Physical Society Scalar-Tensor Theory and Gravitational Waves Robert V. Wagoner Phys. Rev. D 1, 3209 – Published 15 June 1970 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. We adopt several slicing conditions to (e) check whether a long time integration is (=) possible in those conditions. For the case in which the amplitude of (e) the gravitational waves is low, (eb)a long time integration is possible by using (e) the harmonic slice and (e&eb) the maximal slice, while (e) in the geodesic slice ($\alpha$=1) it is not possible. As in the axisymmetric case and also in the 3D case, gravitational waves with (e&eb) a sufficiently high amplitude collapse by (e) their self-gravity and their final fates seem to be (=) as black holes. In this case, the singularity avoidance property of (e) the harmonic slice seems weak, so that (e) it may be inappropriate for (e) the formation problems of (e) the black hole. By means of the gauge-invariant wave extraction technique authors compute the waveform of (e) the gravitational waves at (eb) an outer region. They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP Masaru Shibata and Takashi Nakamura perform numerical simulations of (e) a three-dimensional (3D) time evolution of (e) pure gravitational waves. Authors use a conformally flat and K=0 initial condition for (e) the evolution of the spacetime. 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They find that the nonlinearity of Einstein gravity induces (eb) the higher multipole modes even if only a quadrupole mode exists initially. Received 7 April 1995 DOI:http://dx.doi.org/10.1103/PhysRevD.52.5428 ©1995 American Physical Society Evolution of three-**

**dimensional gravitational waves: Harmonic slicing case Masaru Shibata and Takashi Nakamura Phys. Rev. D 52, 5428 – Published 15 November 1995 STOP This paper is divided into four parts. In part A, some general considerations about (e&eb) gravitational radiation are followed by (e) a treatment of the scalar wave equation in the manner later to be applied to (e&eb) Einstein's field equations. In part B, a co-ordinate system is specified which is suitable for (e) investigation of outgoing gravitational waves from (e) an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in (eb) negative powers of (e) a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by (e&eb) a single function of (e) two variables called the news function. Together with initial conditions specified on a light cone, this function fully defines (eb) the behaviour of the system. No constraints of (e) any kind are encountered. In part C, the transformations leaving (eb) the metric in the chosen form are determined. An investigation of the corresponding transformations in (eb) Minkowski space suggests (eb) that no generality is lost by (e) assuming that the transformations, like the metric, may be expanded in (eb) negative powers of r. Gravitational Waves in General Relativity VII Waves from Axi-Symmetric Isolated Systems Royal Society Proceedings H. Bondi, M. G. J. van der Burg, A. W. K. Metzner Published 21 August 1962.DOI: 10.1098/rspa.1962.0161 STOP There is a deep tension between the well-developed theory of gravitational waves from (e) isolated systems and the presence of (e) a positive cosmological constant Λ, however tiny. In particular a generalization of Einstein's 1918 quadrupole formula that would allow (eb) a positive Λ is not yet available. Authors first explain the principal difficulties and then show that it is possible to overcome (e) them in the weak field limit. These results also provide (eb) concrete hints for constructing the Λ>0 generalization of (e) the Bondi-Sachs framework for full, nonlinear general relativity. Received 18 October 2015 DOI:http://dx.doi.org/10.1103/PhysRevLett.116.051101 © 2016 American Physical Society Gravitational Waves from Isolated Systems: Surprising Consequences of a Positive Cosmological Constant Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan Phys. Rev. Lett. 116, 051101 – Published 1 February 2016 STOP Searching for gravitational waves (GWs) from (e) binary black holes (BBHs) with LIGO and Virgo involves matched-filtering data against (e&eb) a set of representative signal waveforms—a template bank—chosen to (e) cover the full signal space of interest with (e&eb) as few template waveforms as possible. Although the component black holes may have (e) significant angular momenta (spin), previous searches for BBHs have filtered LIGO and Virgo data using (e) only waveforms where (e) both component spins are zero. This leads to (eb) a loss of signal-to-noise ratio for (e) signals where (e) this is not the case. Combining the best available template placement techniques and (e&eb) waveform models, authors construct a template bank of (e) GW signals from BBHs with component spins $\chi_{1,2}\in[-0.99,0.99]$ aligned (e&eb) with the orbital angular momentum, component masses $m_{1,2}\in[2,48]M_\odot$, and (e&eb) total mass $M_{total}\leq 50\ M_\odot$. Using effective-one-body waveforms with (e&eb) spin effects, authors show that less than 3% of (e) the maximum signal-to-noise ratio (SNR) of (e) these signals is lost due to (e) the discreetness of the bank, using (e) the early Advanced LIGO noise curve. Authors use simulated Advanced LIGO noise to (e) compare the sensitivity of this bank to (e) a nonspinning bank covering (e&eb) the same parameter space. In doing so, we consider the competing effects between improved SNR and (e&eb) signal-based vetoes and the increase in the rate of false alarms of (e) the aligned-spin bank due to (e) covering a larger parameter space. Authors find that the aligned-spin bank can be (=) a factor of 1.3–5 more sensitive than a nonspinning bank to (e) BBHs with dimensionless spins >+0.6 and component masses $\gtrsim 20\,M_\odot$. Even larger gains are obtained for systems with (e&eb) equally high spins but smaller component masses. Implementing a search for gravitational waves from binary black holes with nonprecessing spin Collin Capano, Ian Harry, Stephen Privitera, and Alessandra Buonanno Phys. Rev. D 93, 124007 – Published 3 June 201**

## 108.MODEL FORMULATION

Let $p_9(t, a_9)$ denote the distribution function of debited types the number of debts with age from $\alpha$ to $\beta$ and $P(t) = \int_0^\infty \rho(t,a)da$ is the total confederation size of debit summation. Denote the total number of by $Q$. Assume that the credit population is governed by the classical Volterra-Lotka equation

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(g_9)Q_9$$

Where $b$ is the natural death rate in the absence of prey and $g(P)$ is the conversion rate function of debits into credit creations which is assumed to be differentiable and satisfies

$$g_9(0) = 0, \quad g_9(P_9) > 0, \quad P_9 \text{ 0}, \qquad \lim_{P_9 \to \infty} g_9(P_9) = (K_9)_9 \text{ }\infty. \qquad (1)$$

The debit confederation is assumed to be described by the McKendrick-von Forester equation

$$(\rho_t)_9 + (\rho_a)_9 + D_9(a_9, P_9, Q_9)\rho_9 + 0, \qquad 0 < a_9 \text{ }\infty,$$

$$\rho_9(t,0) = \int_0^\infty n_9(a_9, P_9, Q_9)\rho_9(t_9, a_9)da_9$$

Where $D_9$ is the mortality function and $n_9$ is the fertility function.

Put $_9$ subscript to all parameters.

We assume following equations held

$$\left[\frac{dQ_9}{dt}\right]_9 = x_9\left[\frac{dQ_9}{dt}\right]_9 - x_9\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_9}{dt}\right]_9 = x_3\left[\frac{dQ_3}{dt}\right]_9 - x_4\left[\frac{dQ_9}{dt}\right]_9$$

$$\left[\frac{dQ_3}{dt}\right]_3 = x_5\left[\frac{dQ_9}{dt}\right]_9 - x_6\left[\frac{dQ_3}{dt}\right]_9$$

Where $x_9, x_9, x_3, x_4, x_5, x_6$ are accentuation and attention (sufficient credit- debit summation, radiation rate debits raised) , credit summation (amount of credits raised), the pathway that are naturally extinct has trivial and nontrivial equilibrium debit – credit co existence is also studied.

On similar liras we can its $bQ_9$ and $g_9(P_9)Q_9$

Now when you add the term $(b_9Q_9)$ with $g_9(P_9)(Q_9)$ we are infect see is that $(b_9Q_9)$ is credited incrementally part of $g_9(P_9)(Q_9)$ or instant nearly, when subtracted from $\frac{dQ_9}{dt}$ , it means that $g_9(P_9)(Q_9)$ is slowly eaten by $\frac{dQ_9}{dt}$ or instant nearly these factors are to be borne as we proceed to obtain conditions for co existence. When there is no existence of age structure, all the three categories in the nodule can be taken to be equivalent.

We assume that

$$D_9 = \mu_9(a_9) + (f_9)_9(P_9)_9 + (f_9)_9(Q)_9 \quad (2)$$

$$n_9 = \beta_9(a_9)h_9(Q_9), \quad (3)$$

Where $\mu_9$the age- dependent debit rate whit is $\mu_9(a_9)$ $\forall a > 0$; $(f_9)_9$ is the density- dependent death rate with $f_9(0) = 0$, $(f_9^9)_9(P_9)0$, $\forall P$ 0, and $\lim_{p\to\infty}(f_9)_9(P_9) = (k_9)_9$ ; $f_9$ is the functional response of creditors to prey with $[f_9]_9(0) = 0, (f_9^9)_9(Q) > 0, \forall Q$ 0, and $\lim_{Q\to\infty}(f_9)_9(Q_9) = [k_9]$ ; $\beta$ is the age dependent birth rate with $\beta(a)$ $0, \forall a > 0$; and $h$ measures the effects of debit on the of credit with $h(0) = 1, h^9(Q)$ $0, \forall Q$ 0, and $\lim_{Q\to\infty} h(Q) = k_3$ 0.

Summarizing these assumptions, we arrive at the following equations:

$$(\rho_t)_9 + (\rho_a)_9 = -\mu_9(a_9) + (f_9)_9P_9) + \big(f_9)_9(Q_9)\big)(\rho_9)_9 \quad (4)$$

$$\rho_9(t_9, 0) = h_9(Q_9)\int_0^\infty \beta_9(a_9)\rho_9(t, a_9)da, \quad (5)$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + g_9(P_9)Q_9 \quad (6)$$

Using the comparison technique developed in Li [10], we can verify that the solution $\rho_9$ and $Q_9$ of (4-6) satisfy

$$0 \le \rho_9(t, a_9) \le \rho_9^*(t, a_9),$$

$$0 \le Q_9(t) \le Q_9^*(t),$$

Where $\rho_9^*(t, a_9)$ and $Q_9^*(t)$, are solutions of

$$(\rho_t^*)_9 + (\rho_a^*)_9 = -(\mu_9(a_9) + (f_9)_9(P_9^*) + (f_9)_9 Q_9^*))\, \rho_9^*, \quad (7)$$

$$\rho_9^*(t, 0) = \int_0^\infty \beta_9(a_9)\, \rho_9^*(t, a_9) da_9 \quad (8)$$

$$\frac{dQ_9^*}{dt} = -b_9 Q_9^* + g_9(P_9^*)Q_9^*, \quad (9)$$

With $P_9^*(t) = \int_0^\infty \rho_9^*(t, a_9) da_9$

From Eqs. (7)-(9), we can derive the following equation for *t* sufficiently large:

$$\frac{dP_9^*(t)}{dt} = P_9^*(t)\big(r_9 - (f_9)(P_9^*) - (f_9)_9(Q^*)\big), \quad (10)$$

$$\frac{dQ_9^*(t)}{dt} = Q_9^*(t)(-b_9 + g_9(P_9^*)), \quad (11)$$

Where *r* is the real solution of the equation

$$\int_0^\infty \beta_9(a)e^{-\int_0^a \mu_9(\tau)dt}e^{-ra}\,da_9 = 1. \qquad (12)$$

If $r < 0$, the trivial solution $P_9^* = 0$ and $Q^* = 0$ is stable and, hence, two species go to extinction. This is trivial in mathematics. Therefore, we subsequently, throughout this paper, assume that

$$\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(t)dt}\,da_9 > 1, \qquad (H1)$$

Which guarantees $r_9 > 0$.

*Remark.* If $\beta_9(a) = [\beta_0]_9 a_9 e^{-\alpha_9 a_9}, \mu_9(a_9) \equiv \mu_9$ then the solution of (12) is

$$r_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$$

Corresponding to the result in Gurtun and Levine [6]. In their model,

$$[f_9]_9(P_9) = 0, [f_9](Q_9) = \lambda_9 Q_9\ h_9(Q_9) = 1, \qquad and\ \ g_9(P_9) = c_9 P_9$$

$$[\rho_9]_9 + [\rho_a]_9 = -(\mu_9 + \lambda_9 Q_9)\rho_9$$

$$\rho_9(t,0) = \int_0^\infty [\beta_0]_9 a_9 e^{-\alpha_9 a_9}\rho_9(t,a_9)\,da_9$$

$$\frac{dQ_9}{dt} = -b_9 Q_9 + c_9 P_9 Q_9$$

Which, obviously, is a special case of the model in this paper. They arrived at a pair of ordinary differential equations

$$\frac{dP_9}{dt} = \omega_9 P_9 - \lambda_9 P_9 Q_9$$

$$\frac{dQ_9}{dt} = b_9 Q_9 - c_9 P_9 Q_9$$

Where $\omega_9 = \sqrt{\beta_9} - \alpha_9 - \mu_9$

THE EQUILIBRIUM $(\rho_0)_9(a_9), 0)$

To study the case where only the debit confederation congregation the behavior of the equilibrium $(\rho_0)_9(a_9), 0)$ of (4)-(6).

Obviously, the existence of such equilibrium is equivalent to the existence of the equilibrium of the equations

$$(\rho_t)_9 + (\rho_a)_9 = -\big(\mu_9(a_9) + (f_9)_9(P_9)\big)\rho,$$

$$\rho_9(t,0) = \int_0^\infty \beta_9(a_9)\rho_9(t,a_9)da_9 \qquad (13)$$

The solution of (13) has the form of

$$\rho_9(t,a_9) = Ce_9^{r(t-a_9)} - \int_0^a (\tau)d\tau - \int_0^9 (f_9)_9\big(P_9(\tau)\big)d\tau$$

For $t$ sufficiently large, where $r$ is the solution of (12). By integrating with respect to a from 0 to ∞, WE OBTAIN

$$P_9(t) = C_9 e^{r_9 t - \int_0^t (f_9)_9(P_9(\tau))d\tau} \int_0^\infty e^{-ra}\, e^{-\int_0^a \mu(\tau)d\tau} da.$$

Hence, there exists an equilibrium $P_0 = \int_9^{-9}(r)$ if $k_9 > r$. This is equivalent to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{(k_9)_9 a_9} da_9 < 1.$$

In fact, if we define

$$R_9(s_9) = \int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau)d\tau} e^{-s_9 a_9}\, da_9$$

$R(s)$ is a decreasing function. $R_9(r_9) = 1$ And $R_9(k_9)_9 < 1$ imply $r_9 < (k_9)_9$.

Thus, we can state an existence result of the equilibrium $(\rho_0)_9(a_9), 0)$ as follows.

THEOREM 1. *Assume that the hypotheses* (H1) *and*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{k_9 a_9} da_9$$

$$< 1 \qquad (H2)$$

*Hold. Then the system* (4)-(6) *has the equilibrium*$(\rho_0(a), 0)$ *where*$(P_0)_9 \equiv \int_0^\infty (\rho_0)_{99}(a_9)da_9$ *satisfies*

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau)d\tau} e^{-(f_9)_9(P_0)a_9} da$$

$$= 1 \qquad (14)$$

Next, we study the stability of $(\rho_0)_9(a_9), 0)$

Let $u_9(t, a_9) = \rho(t, a_9) - \rho_0(a_9)$ Then

$$N_9(t) \equiv \int_0^\infty u_9\,(t, a_9)da_9 = \int_0^\infty \rho_9(t, a_9)da_9 - \int_0^\infty (\rho_0)_9(a_9)da = P_9(t) - P_0)_9$$

And

$$(u_9)_9 + (u_a)_9 = (\rho_t)_9 + (\rho_a)_9 - (\rho_0^9)_9(a_9)$$

$$= -\mu_9(a_9) + (f_9)_9(P_9) + (f_9)_9(Q_9))\,(u_9 + (\rho_0)_9) + \mu_9(a_9)(\rho_0)_9 + (f_9)_9(P_0)_9(\rho_0)_9$$

$$= -\mu_9(a_9)u_9 - (f_9)_9(P_9) + (f_9)_9(Q_9))\,u_9 - (\rho_0)_9\,(f_9)_9(P_9) - (f_9)_9(P_0)_9$$

$$+ (f_9)_9(Q_9)), \qquad (15)$$

Where $(\rho_0^9)_9(a_9)$ denotes $d(\rho_0)_9(a_9)/da_9$

Expansion $(f_9)_9(P_9)$ and $(f_9)_9(Q_9)$ at $(P_0)_9$ and 0 yield respectively

$$(f_9)_9(P_9) = (f_9)_9((P_0)_9 + N_9)$$

$$= (f_9)_9(P_0)_9 + (f_9^9)_9(P_0)_9N_9 + o_9(N_9), \qquad (16)$$

$$(f_9)_9(Q_9)$$

$$= (f_9^9)_9(0)Q + o_9(Q_9) \qquad (17)$$

Write $f_9(P_0) = f_{90}, f_9'(P_0) = f_{90}'$, and $f_9'(0) = f_{90}'$ and substitute (16) and (17) into (15) to obtain

$$(u_9)_9 + (u_a)_9 \approx -(\mu_9(a_9) + (f_{90})_9)u - (f_{90}^9)_9(\rho_0)_9(N_9) - (f_{90}^9)_9(\rho_0)_9(Q_9)$$

$$u_9(t,0) = \rho_9(t,0)(\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a) + (\rho_0)_9(a_9)da_9 - (\rho_0)_9(0)$$

$$= h_9(Q_9)\int_0^\infty \beta_9(a_9)(u_9(t,a_9)da_9 + (h_9(Q_9) - 1)(\rho_0)_9(0) \qquad (18)$$

Expanding

$$h_9(Q_9) = 1 + (h^9)_9(0)Q_9 + 0(Q_9) \equiv 1 + h_0'Q_9 + 0Q_9$$

And substituting it into (18), we arrive at

$$u_9(t,0) \approx \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + (h_{a_9}')(\rho_0)_9Q_9$$





The linearization is the following :

$$[u_9] + [u_9] = -(\mu_9(a_9) + [t_{90}]_9u_9 - [\alpha_9]_9(a_9)N_9 - [\alpha_9]_9(a_9)Q_9$$

$$u_9(t,0) = \int_0^\infty \beta_9(a_9)u_9(t,a_9)da_9 + [\xi_9]Q_9 \qquad (19)$$

$$\frac{dQ_9}{dt} = -b_9Q_9$$

where

$$[\alpha_9]_9(a_9) = [t_9^9 a]_9[p,o]_9(a_9)d2(a_9) = [f_{90}^9]_9(\rho\, o\,)_9(a_9)\xi = [\mathrm{h\ o}^9]_9[\rho\mathrm{o}]_9(\mathrm{o}) \text{ and } \mathrm{b}_9 = \mathrm{b}_9 - g_9(P_9)$$

If $g_9(P_0) > b$, clearly, $(\rho_0)_9(a_9), 0)$ is unstable and the predator population is persistent. Assume $g_9(P_0)_9 < b$.

Then, $Q_9(t) \to \infty$. Thus, asymptotically the dynamics of (19) are governed by

$$(u_t)_9 + u_a = -(\mu_9(a_9) + (f_{90})_9)u_9 - (\alpha_9)_9(a_9)N_9$$

$$= -(\mu_9(a_9) + f_{90})u_9 - \alpha_9(a_9)\int_0^\infty u_9(t,a_9)da_9$$

$$u_9(t,0) = \int_0^\infty \beta_9\,(a_9)u_9(t,a_9)da_9$$

Using the method developed by Cushing [2, 3], we suppose

$$u_9(t,a_9) = \tilde{\mathrm{u}}_9(a_9)^{e^{\gamma(t-a9)}}$$

And define $w = \int_0^\infty \tilde{\mathrm{u}}(a)e^{-\gamma a_9}\,da_9$. Then,

A solution of the first differential equation for $\tilde{\mathrm{u}}(a_9)$ gives,

$$\tilde{\mathrm{u}}_9(a_9) = e^{-\int_0^a \mu_9(\tau_9)d\tau - (f_{90})_9 a}\left(\tilde{\mathrm{u}}_9(0) - w\int_0^a e^{\int_0^\tau \mu_9(s_9)ds + (f_{90})}\tau_9\,(\alpha_9)_9(\tau_9)e^{\gamma_9\tau}d\tau\right). \qquad (21)$$

Substituting the expression (21) into the expression for above yields

$$\left(1-\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\right)\tilde{u}_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a(\alpha_9)_9(\tau_9)e^{-\gamma_9\tau_9}e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds}d\tau_9\,da_9$$

$$=0 \qquad (22)$$

Substituting (21) into the definition of of $w$ yields

$$\left(\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}da\right)(\tilde{u}_9(0))_9$$

$$+\left(-1-\int_0^{\infty}e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(\tau_9)d\tau}\int_0^a e^{(f_{90})_9\tau_9+\int_0^\tau \mu_9(s_9)ds+y\tau}\alpha_9(\tau)d\tau\,da\right)w=0.$$

(23)

Define $Y(\gamma,a)=e^{-\gamma_9 a_9-(f_{90})_9 a_9-\int_0^a \mu_9(t_9)d\tau}$. Then

$$\left(1-\int_0^{\infty}\beta_9\,(a_9)Y_9(\gamma_9,a_9)da_9\right)u_9(0)$$

$$+w\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)\int_0^a(\alpha_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9=0,$$

(24)

$$\left(\int_0^{\infty}Y_9(\gamma_9,a_9)da_9\right)\tilde{u}_9(0)$$

$$+\left(-1-\int_0^{\infty} Y_9(\gamma_9,a_9)\int_0^{a}(a_9)_9(\tau)_9(Y^{-9})_9(\gamma_9,\tau_9)d\tau_9\,da_9\right)w=0.$$

Equation (20) has a solution of the form of $u_9(t,a_9)=u_9(a_9)e^{\gamma(t-a)}$ if and only if the system of equation (24) is solvable for $\tilde{u}_9(0)\not\equiv 0$ and $\gamma, w$ in the complwx plane.

Let $I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)Y^{-9}\,(\gamma_9,a_9)d\tau_9$. We obtain the characteristic equation

$$f_{90}^{l}{}_9\left(\int_0^{\infty} Y_9(\gamma_9,a_9)da_9\right)\left(\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$+\left(1-\int_0^{\infty}\beta_9(a_9)Y_9(\gamma_9,a_9)da\right)\left(1+(f_{90}^{l})_9\int_0^{\infty}Y_9(\gamma_9,a_9)I_9(\gamma_9,a_9)da_9\right)$$

$$=0. \qquad (25)$$

Since

$$(\rho_0)_9(a_9)=(\rho_0)_9(0)e^{-\int_0^a\mu_9(\tau_9)d\tau_9-(f_{90})_9a_9}$$

$$I_9(\gamma_9,a_9)=\int_0^a(\rho_0)_9(\tau_9)e^{\gamma_9\tau_9+(f_{90})_9\tau+\int_0^{\tau}\mu_9(s_9)ds_9}d\tau$$

$$=\frac{(\rho_0)_9(0)}{\gamma_9}(e^{\gamma_9a_9}-1)$$

Let

$$E_9(a_9)=e^{-(f_{90})_9a_9-\int_0^a\mu_9(\tau_9)d\tau_9}$$

Then $[Y_9]_9[\gamma_9]_9, a_9) = E_9(a_9)e^{-\gamma_9 a_9}$. by a simple calculation,

$$\int_0^\infty \beta_9(a_9)Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma}\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}\ da\right);$$

$$\int_0^\infty Y_9(\gamma_9, a_9)I(\gamma_9, a_9)da = \frac{[\rho_0]_9(0)}{\gamma_9}\left(\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}\ da_9\right).$$

Substituting in to (25), we get

$$\frac{[\rho_0]_9(0)}{\gamma_9}[f_{90}^9]\int_0^\infty E_9(a_9)e^{-\gamma_9 a_9}da_9\ (\int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9)$$

$$+\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right).$$

$$\left(1 + \frac{(\rho_0)_9(0)}{\gamma_9}(f_{90}^{'})_9\left(\int_0^\infty E_9(\,a_9)da_9 - \int_0^\infty E_9(\,a_9)e^{-\gamma_9 a_9}da_9\right)\right) = 0.$$

Hence the characteristic equation becomes

$$\left(1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9\right)(1 + \frac{(f_{90}^{'})_9}{\gamma_9}(\rho_0)_9(0)\int_0^\infty E_9(\,a_9)da_9) = 0.$$

We need ũ(a) $\not\equiv 0$, namely $w \neq 0$. This necessarily leads, from (94), to

$$1 - \int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)da_9 = 1 - \int_0^\infty \beta_9(a_9)E_9(a_9)e^{-\gamma_9 a_9}da_9 \neq 0,$$

Since $\int_0^\infty \beta_9(a_9)Y_9(\gamma_9 a_9)\ \int_0^a \alpha_9\,(\tau_9)Y^{-9}(\gamma_9, \tau_9)d\tau_9 da_9 > 0$. Therefore, the characteristic solution is

$$\gamma_9 = -(f_{90}^{'})_9(\rho_0)_9 \int_0^{\infty} E_9(a_9)da_9.$$

Since $f_9'(P_0)_9 > 0$, we obtain the following result

Theorem 2. Assume that the hypotheses (H1) and (H2) hold. Then the equilibrium $(\rho_0(a_9), 0)$ is asymptotically stable if $g(P_0) < b$. It is unstable if $g(P_0) > b$.

## 3. COEXISTENCE

In the previous section, we studied the dynamic behavior of the trivial solution $(\rho_0(a_9), 0)$ which is related to the extinction of predators (debit summation types: that which acts as a resource)and to

The possible persistence of the prey population. The purpose of this section is to investigate existence and stability of nontrivial equilibrium, which is related to coexistence of both credit summation and debit summation.

Denote the nontrivial equilibrium by $(p^*(a), Q^*)$. It must satisfy the equations:

$$[p_a^*]_9 + (\mu_9(a_9) + [f_9]_9(P_9^*) + [f_9]_9(Q_9^*))P_9^* = 0, \quad (26)$$

$$p_9^*(0) = h_9(Q_9^*) \int_0^{\infty} \beta_9(a_9)p_9^*(a_J)p_9^*(a_9)da_9 \quad (27)$$

$$g_9(P_9^*) - b_9 = 0, \quad (28)$$

Where $P_9^* = \int_0^{\infty} p_9^*(a_9)\, da_9$

Equation (28) yields $P^* = g^{-9}(b)$,

From (26),

$$p_9^*(a) = p_9^*(0)e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}$$

Substituting into (27), we need to solve

$$\left(1 - h_9(Q_9^*)\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(Q_9^*)a_9}da_9\right)p_9^*(0) = 0$$

For $Q_9^*$

Fix $p_9^*(0)$ and $P_9^*$ Let

$$H_9(z_9) = 1 - h_9(z_9)\int_0^\infty \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9}da_{,9}$$

And assume that

$$[k_3]_9\int_0^\infty \beta_9(a_9)e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(g_9^{-9}(b_9))a_9 - [k_9]_9a_9}\, da_9 < 1. \qquad (H3)$$

Then

$$\lim_{z\to\infty}[H_9(z_9)] > 0.$$

Clearly,

$$\frac{dH_9}{dz_9} = h_9(z)\left( [f_9^{'}]_9(z_9) \right) \int_0^{\infty} a_9\beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - f_9(P_9^*)a_9 - f_9(z_9)a_9} da_9$$

$-h^{'}(z_9) \int_0^{\infty} \beta_9(a_9)\, e^{\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9 - [f_9]_9(z_9)a_9} \quad da_9$

y the assumptions on $\left[f_9^9\right]_9 and\ [h^9]^9 dH_9/dz > 0$. Hence, if

$$H_9(0) = 1 - \int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9(P_9^*)a_9} da_9 < 0, \qquad (29)$$

There exists a unique positive solution $Q^*$

Now, we claim that $H_9(0) < 0$ follows if $g_9^{-}(b_9) < [f_9^{-9}]_9(r_9)$, where $r_9$ is the solution of (12).

Indeed, since

$$R_9(s_9) = \int_0^{\infty} \beta_9(a_9)\, e^{-\int_0^a \mu_9(\tau_9)d\tau_9 e^{-y\frac{a}{9}}} da_9$$

Is a decreasing function of $s_9$ when $P_9^* = [g^{-9}]_9(b_9) < [f_9^{-9}]_9(r_9)$, that is $[f_9]_9(P_9^*) < r, R\left(f_9(P^*)\right) > 1$ which

implies $H(0) < 0$.

Summarizing the above statements, we have shown the existence of the nontrivial equilibrium.

THEOREM 3. Assume that the hypothesis (H3) holds and $R(f_9(g^{-9}(b))) > 1, i.e.,$

$$\int_0^{\infty} \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9)d\tau_9 - [f_9]_9\left(g_9^{-9}(b)\right)a_9} > 1. \qquad \text{(H4)}$$

Then the system (4)-(6) has a unique positive equilibrium $(P_9^*(a_9), Q_9^*)$.

Analyzing the stability of the nontrivial equilibrium $(P_9^*(a_9), Q_9^*)$ is more complicated. We again consider perturbations $m_9(t_9 a_9) = p_9(t_9, a_9) - p_9^*(a_9)$ and $W_9(t) = Q_9(t) - Q^*$.Then

$$M_9(t) \equiv \int_0^\infty m_9(t, a_9) da_9 = P_9(t) - P_9^*,$$

$$[m_9]_9 + [m_a]_9 = [p_9]_9 + [p_a]_9 - [p^*]_9(a_9) \approx -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9$$

$$-[f_{9*}^9]_9 p_9^*(a_9)M - [f_{9*}^9]_9 p_9^*(\mathrm{a})W_9,$$

$$m_9(t,0)p_9(t,0) - p_9^*(0) \approx h_* \int_0^\infty \beta_9(a_9) m_9(t, a_9)\, da + \frac{[h_*^9]_9}{[h_*]_9}\, p_9^*(0) W_9$$

$$\frac{dW_9}{dt} = \frac{dQ_9}{dt} \approx [g_*^9]_9 Q_9^* M_{,9}$$

Where $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}]_9$ denoting $[f_9(P^*)]_9$, $[f_{9*}']_9$ denoting $[f'_9(P^*)]_9 [f_{9*}']_9$ denoting $[f_9]_9$ $(Q_9^*$, $h_{*}^{9}$ denoting $h_*'$ denoting $h_9'$ $(Q_9^*)$, and $[g_*']_9$

Denoting $[g']_9(P_9^*)$.

Now, we study the asymptotic behavior of the solutions $m(t,a)$ and $W(t)$ of the system of equations:

$$[m_9]_9 + [m_a]_9 = -(\mu_9(a_9) + [f_{9*}]_9 + [f_{9*}]_9)m_9 - [f'_{9*}]_9 [p_9^*](a_9)M_9 - [f_{9*}']_9 p_9^*(a_9)W_9$$

$$m_9(t,0) = [h_*]_9 \int_0^\infty \beta_9(a_9) m_9(t, a_9) da_9 + \frac{[h_*^9]_9}{[h^*]_9} [p^*]_9(0)\, W_{,9} \qquad (30)$$

$$\frac{dW_9}{dt} = [g_*^9]_9 [Q_9^*] M_9$$

Integrating $[m_9]_9(t, a_9)$ with respect to a from 0 to ∞, we have

$$\frac{dM_9(t)}{dt} = \int_0^\infty [mt]_9(t, a_9)da_9$$

$$= -\int_0^\infty [m_a]_9\,(t, a_9)da_9 - \int_0^\infty \mu(a_9)m(t, a_9)da_9$$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[f^{'}{}_{9*}\right]_9 \int_0^\infty p_9^*(a_9)da_9\right)M_9 - [f_{9*}^9]_9 \int_0^\infty p_9^*(a_9)da_9 W_9$$

$= m_9(t, 0) - \int_0^\infty \mu_9(a_9)\, m_9(t, a_9)da_9$

$$-\left([f_{9*}]_9 + [f_{9*}]_9 + \left[\dot{f}_{9*}\right]_9 P_9^*\right)M_9 - \left[\dot{f}_{9*}\right]_9 P_9^* W_{.9}$$

From (30), we arrive at the equations

$$\frac{dW_9(t)}{dt} = \int_0^\infty ([h_*]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9$$

$$-\left([f_{9*}] + [f_{9*}]_9 + \left[\acute{f}_{9*}\right]_9 P_9^*\right)M_9 + \left(\frac{[h_*^{'}]_9}{[h_*]^9}\, P_9^*(0) - f_{9*}^{'} P_9^*\right)W_9,$$

(31)

$$\frac{dW_9(t)}{dt}\left[g^{'}{}_*\right]_9 Q_9^*\, M_{.9}$$

By the mean value theorem,

$$\int_0^\infty ([h_{"*}]_9\beta_9(a_9) - \mu_9(a_9))m_9(t, a_9)da_9 = ([h_*]_9\beta_9(\theta_9) - \mu_9(\theta_9))M_{,9}$$

Where $0< \theta_9 = \theta_9(t) < \infty$.

And if $a_9^9 [a_9^9] 2<4 [a_9] 2<4 a_9 e$,

$\mathrm{Re}\,\lambda_9 [\lambda_9]_9 = \mathrm{Re}\lambda_9 [\lambda_9]_9 = -\frac{9}{9} a_9 [a2\ ]_9 < 0.$

Let $\Delta = \max_{9<u<\infty} |h_* [h_*]_9 \beta_9(a_9) - \mu_9(a_9)|$. Then $|n_9(t)|$ ⍰$\Delta \forall t$ ⍰0. Hence we obtain the following result.

**THEOREM 4**

Assume that the hypotheses(H3)and (H4)hold Then the nontrivial equilibrium($\rho^*(a), \boldsymbol{Q}^*$) is asymptotically stable if either $a_9^9 <4 a_9 e$ and $[a_9] > 1\Delta$,or $[a_9^9]$ ⍰$[a_9]e$ and $(a_9 - \sqrt{(a_9^9)_9 - 4(a_9)_9 e) > 1\Delta}$

These conditions sufficient The equilibrium might be unstable under some other circumstances in which cases Hopf-bifurcation might occur We intend to pursue this problem in the future.

Assume $\mu(a) = \mu_0 + \left(\frac{a}{a+9}\right)$ and $\beta_9(a_9) = ae^{-ma}$ Then $\Delta = [\mu_0] + 2$ Simple calculation leads to

$$\int_0^\infty \beta_9(a_9) e^{-\int_0^a \mu_9(\tau_9) + d\tau_9 da} = \frac{1}{(m + \mu_0 + 1)^9} + \frac{1}{(m + \mu_0 + 1)^3}$$

Hence in order to satisfy the hypothesis (H2), we assume $m_9 + [\mu_0]$ ⍰0.5. Assume$[f_9](x_9) = K_9 x_9$ Then the hypothesis (H9) is satisfied We take the conversion rate function $g_9(x_9) = [0x_9]/(2 + x_9)$; the functional responses $[[f_9](x) = [0]\, x_9/(1 + x_9)$and $h_9(x_9) = (1+\omega_9 x_9)/(1 + x_9)$(See Holling [7] and Rashevsky [23])

Equation (24),now has the form of

$$\frac{m_9 + [\mu_0] + [\mathrm{K}][P_0])}{[m_9]_{\,9} + (\mu_0)_{\,9} + [\mathrm{K}][P_0])\,{}^{3}_{9}\ [\mathrm{K}]_{\,9}[P_0]\,{}^{3}_{9}} = 1$$

The equilibrium $[P_0]$ is the solution of this equation

If $\delta P_0/(1+P_0) < b_9$i e $[P_0] < b/(\delta_9 - b_9)$then the equilibrium$(P_0)$ $[P_0]_{\,9}(\mathrm{a}), 0)$ is stable This implies that the

cedit confederation goes to extinction but that the cedit confederation persists $b_9$

$[f_{9*}] = K_9\ \mathrm{P}^{*}\ [f_{9*}] = \frac{\sigma \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*} \quad [h_*] = \frac{9+\omega_9 \mathrm{Q}_9^*}{9+\mathrm{Q}_9^*}$

$[f_{9*}] = K_9\ [\mathrm{f}\,{}^{9}_{9*}] = \frac{\sigma_9}{(9+\mathrm{Q}_9^*)^9}\,[\,\mathrm{g}\,{}^{9}_{*}] = \frac{\sigma_9}{(9+\mathrm{P}_9^*)^9} \quad [\mathrm{h}\,{}^{9}_{*}] = \frac{\omega_9 - 9}{(9+\mathrm{Q}_9^*)^9}$

In order to have (H3) and (H4) hold,$m, \mu_0, \mathrm{K}, \mathrm{b}, \delta,$ and $\omega$ need to satisfy

$$\frac{\omega_9}{(m + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma)^{\,9}} + \frac{1\omega_9}{([m_9]_{\,9} + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 + b_9} + \sigma_9)^{\,3}} < 1$$

And

$$\frac{1}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,9}} + \frac{1\omega_9}{(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9})^{\,3}} > 1$$

When the inequalities (33)and (34)hold there is a unique equilibrium $\mathrm{Q}^{*}$ which can be solved from the following equation:

$(m_9 + [\mu_0]_{\,9} + 1 + \frac{K_9 b_9}{\delta_9 - b_9} + (m_9 + [\mu_0]_{\,9} + \frac{K_9 b_9}{\delta_9 - b_9} + \sigma_9)\ \mathrm{Q}_9^{*})^{\,3}$

$=m_9 + [\mu_0]_9 + 3 + \frac{K_9 b_9}{\delta_9 - b_9}(1+ Q^*)(1 + \omega_9 Q_9^*$

Now

$$\left[a\ {}^{1}_{1}\right] - 4\left[a_{\ 9}\right]e = \left(\frac{1K_9 b_9}{\delta_9 - b_9}\right)^9 + \frac{4K_9 b_9 \sigma_9 Q_9^*}{(\delta_9 - b_9)(9 + Q_9^*)} + \frac{\sigma_9^9 Q_9^{*9}}{(1 + Q_9^*)^9}$$

$$\frac{4\, b_9 \sigma_9 (\delta_{\ 9} - b)Q^*}{\delta_{\ 9}(1 + Q_9^*)^9} - \frac{4\,(\delta_{\ 9} - b_9)^9 (1 - \omega_9) Q_9^*}{\delta_{\ 9}(1 + \omega_9 Q_9^*)(1 + Q_9^*)} P * (0)$$

And

$$\Delta = \lim_{0<a<\infty} |[h_*]_9\, ae^{\ mu}_{\ 9}\ \ [\mu_0]_9 - \frac{a_9}{a_9 + 9}$$

Hence if $[a\ {}^{9}_{9} > 4(a_9)_{\ 9}e$ and

$$[a_9] = \frac{1K_9 b_9}{\delta_9 - b_9} + \frac{\sigma_9 Q_9^*}{1 + Q_9^*} > 1\Delta$$

The nontrivial equilibrium is stable.

IF we define

$$[a_9]_9 = [f_{9*}]_9 + [f_{9*}]_9 + [f_{9*}]\,P^*_9$$

$$[a_9]_9 = \left[f^{9}_{9*}\right] P^*_9 - \frac{h^9_*}{h_*} P^*(0)$$

$$e=[g^9_*]_9\, Q^*_9$$

$$\eta_9(t) = \left[h_*\right]_9 \beta_9 \left(\theta_9(t)\right) - \mu_9(\theta(t)),$$

The system(32)can be written as the following system:

$$[M(t)]\left[a_9\right]M_9 - [a_9]_9 W_9 - [n(t)]M_9$$

$$w_9(t) = eM_9$$

Write

$$A=\begin{pmatrix} -[a_9]_9 - [a_9]_9 \\ e \qquad\quad 0 \end{pmatrix}$$

$$B=(t)=\begin{pmatrix} [\eta(t)]_9 & 0 \\ 0 & 0 \end{pmatrix}$$

And

$$X(t)=\begin{pmatrix} M_9 \\ W_9 \end{pmatrix}$$

The solution of (32)is

$$X_9(\mathrm{t}) = \mathrm{e}^{A,t}\,[\mathrm{X}]_9(0) + \int_0 \mathrm{e}^{At}$$

**Note: There are lots of zeroes corresponding and concomitant to the second law of black holes. Infact as many as that of extant and existential black holes exist. At the outset, it is to be stated that there are hadrons in every system. Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is SO (32) and its axiomatic predications, predicational anteriorities, character constitution shall be extant and existential in very many systems, and the characterstics are taken in to consideration in the classification scheme. Many systems have such fundamental instabilities like that of quantum gravity and characterstics of those systems form the citadel and fulcrum, bulwark and manor, mainstay and reinforcement, alcazar and chateau theory has a fundamental instability on which the classification schémas are valid. There are lots of systems which follow the axioms of string theory. It is the characterstics of this system which are taken in to consideration in the classification scheme. "All institutionalization involves common moral as well as other values. Collectivity obligations are, therefore, an aspect of every institutionalized role. But in certain contexts of orientation-choice, these obligations may be latent ...." (Parsons, 1951, p. 99). There are various systems that have the same bastion, support system, stylobate and sentinel as that of the Deleuzean terms and predications and phenomenological methodologies systemized. Look at this beautiful passage whose stability analysis would be eye opener. To "deconstruct" is not the same as to destroy. Deconstruction attempts to undo logical contradictions, to overturn rigid conceptual oppositions while releasing new concepts and meanings that could not be included in the old system. At the heart of Western metaphysics, for example, Derrida finds the opposition between "speech" and "writing." This binary logic functions in an illicit way to establish speech as the means of giving "presence" to the world, while writing is deemed derivative and inferior In Derrida's sense of "grammatology," however, all**

**production of meaning is writing and subject to the infinite play of signification. By taking away the transcendental signified and advancing the concept of "differance" (language organized around difference and deferred, or mediated, understandings), Derrida, like Nietzsche, wants to leave us without transcendental illusions, metaphysical unities, and foundations that constrain thought and creativity. THE POSTMODERN TURN IN PHILOSOPHY: THEORETICAL PROVOCATIONS AND NORMATIVE DEFICITS by Steven Best and Douglas Kelner http://www.gseis.ucla.edu/faculty/kellner/kellner.html Each and every system has electrons, neutrons and protons and for that matter quarks. There shall be strong nuclear force and weak nuclear force. There are many systems that satisfy the criterion specified by the equation, principle or statement in question. Characterstics of the investigating systems form the bastion for the classification scheme and doxa thereof. Systemic differentiation is conducted. Despite gravity being constant, there exists gravity between two objects, and this could be taken as a system. Depending upon some parametric representationalities, functionalities, advantageousness, appropriateness, benefit, facilitation, fittingness, helpfulness, instrumentality, merit, practicality, serviceability, suitability, and usefulness, utility, these systems could be classified in to various categories. In respect of an equation, there shall be many systems that satisfy the given equation. Equations themselves could be by the utilization of the model solved term by term as has been exemplified and illustrated many time in the previous papers. There is lot of systems that could be brought in to the orbit of and gamut of the theory in question which the investigatable systems satisfy the axiomatic predications and postulation alcovishness of the systems in question. Towards the end of classificational consummation, consolidation, corporation and concatenation we take the characterstics of the systems, the predicational interiorities, ontological consonance and primordial exactitude, accolytish representations, functional topology, apocryphal aneurism and atrophied asseveration, event at contracted points, and other parameters as the bastion and stylobate of the stratification purposes. Such totalistic entities**

**would have easy paradigm of relational content, differentiated system of expressly oriented actions with primary focus and locus of (e) homologues receptiveness and differentially instrumental activity, variable universalism and particularism, imperative compatibilities and structural variabilities, interactional dynamical orientation, institutionalization and internalisation of pattern variables common attitudinal orientation of constituionalisation of internalized dispositions, and a qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction or interactional dynamics, internal differentiation, structural morphology, formal characterization, concept formulation, phenomenological methodologies, constituent structure, transformational minimal condition, paradigmatic feasibilities, programmatic plausibilities, comparative variability, normative aspect of expectational prediction, projection and prognostication as consideration of the investigatory systems. Any scale can be used that is convenient to the classification scheme. It is important to note that the scheme of classification and the stratification doxa must not be adversely antagonistic, inexorably irreconcilable, antithetically antipodal, diametrically opposed, repugnantly retrogressive, inimically inverse, violatively unsimilar, diversely dissimilar, antipathetically anti thetical, conflictingly combative, obstructively pugnacious, inimically obstructive, repellently restrictive, disputatiously gainsaying or conformingly pugnacious to the axiomatic predications and postulation alcovishness of the theory in question or the equation representative or constitutive thereof. Sole intention, main objective and primary aim is twofold. One is towards the end of circumvention of the extra equation and the concomitant and corresponding variable therein. Second is avoidance of clustered congest, swarmed huddle, mustered pack, sardine squash, and swamp throng in the scheme of classification. Only thing that is sought out is the consonance in the entire diaspora and body fabric of the systems under study. This statement is true and holds unmistakably true for all the papers and I sincerely entreat readers to remember the statement and read the paper against this background. When we write A+B we mean by that B is being**

**added to A or vice versa. It is like adding milk to water and water to milk. There may or may not be a time gap. As said earlier there may be many systems that satisfy the conditions of the equations and those systems that are investigatory or investigatable are taken in to consideration based on their characterstics in the classification doxa. When there are more than two entities, we can take logarithm and find the value of that factor to be taken with anti log to obtain prediction and projected values of the model. In case of A-B, we are removing B from A and that means B is eating up A. These factors are taken in to consideration in the application of the model to equations. Log (ab) and Log (a+b) is well defined. Model stands out as universal testament and template for application to each sentence and equation what with the quantification process done and the correlations well defined. In the eventuality of non existence of any connection at some phase, the model would have the accentuation and attrition coefficients and detritions coefficients as zero rendering the equations of concatenation simpler. Projection formula which incorporates in its diaspora the initial values provide autheticative determination, unimpeachable validation, incontrovertible establishment , apodictic evidence, reliable roll out, unfailing cinquecento quattrocento trecento, incontrovertible indication of the final finale, notwithstanding appellation, appellative, brand, cognomen, compellation, designation, flag, handle, identification, label, moniker, nomen, slot, style, surname, tab, tag, term, title designation, appellation, appellative, class, classification, denomination, description, epithet, of the classification scepter, scenario, scimitar, schottische. There is pure and impure consciousness in every one. Gratification producing and deprivation producing one's can be easily classified from individual general ledgers. Similar analogy holds for collective general ledger and cosmic general ledger or nature's general ledger. It is also to be noted that while dealing with equations towards the end of consummation of the measure, it is necessary that the two variables are to be classified in to three sections and each one would have the adventitious and decidedly stated relationship whereby the fundamental equations are drawn up and the analysis made. All the**

**parametric representationalities, conditionalities, orientationalities remain unequivocal as stated in the variables stated in to consideration section and are different from module to module**

## SECTION FIVE

## Efficiently Controllable Graphs

**INTRODUCTION—VARIABLES USED**

**Efficiently Controllable Graphs Can Gokler, Seth Lloyd, Peter Shor, Kevin Thompson (Submitted on 18 Mar 2015 (v1), last revised 10 Mar 2016 (this version, v3))**

**(1)** Authors investigate graphs that can be disconnected into small components by (e) removing a vanishingly small fraction of their vertices.
**(2)** Authors show that when a quantum network is described by (e) such a graph, the network is efficiently controllable, in the sense that universal quantum computation can be performed using (e) a control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems.
**(3)** Authors show that networks corresponding to (e&eb) finite-dimensional lattices are efficently controllable, and explore generalizations to (e) percolation clusters and random graphs.
**(4)** They show that the classical computational complexity of estimating the ground state of Hamiltonians described by controllable graphs is (=) polynomial in the number of subsystems/qubits. Subjects: Quantum Physics (quant-ph) Cite as: arXiv:1503.05270 [quant-ph] (or arXiv:1503.05270v3 [quant-ph] for this version)

**NOTATION**

**Module One**

Authors investigate graphs that can be disconnected into small components by (e) removing a vanishingly small fraction of their vertices

$G_{13}$ : Category one of removing a vanishingly small fraction of their vertices

$G_{14}$ : Category two of SAS

$G_{15}$ : Category three of SAS

$T_{13}$ : Category one of graphs that can be disconnected into small components

$T_{14}$ : Category two of SAS

$T_{15}$ : Category three of SAS

**Module Two**

Authors show that when a quantum network is described by such a graph, the network is efficiently

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| controllable, in the sense that universal quantum computation can be performed using (e) a control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems | |
| $G_{16}$ : Category one of quantum network is described by such a graph<br><br>$G_{17}$ : Category two of SAS<br><br>$G_{18}$ : Category three of SAS | |
| $T_{16}$ : Category one ofnetwork is efficiently controllable, in the sense that universal quantum computation can be performed using (e) a control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems<br><br>$T_{17}$ : Category two of SAS<br><br>$T_{18}$ : Category three of SAS | |
| **Module three**<br><br>Authors show that when a quantum network is described by such a graph, the network is efficiently controllable, in the sense that universal quantum computation can be performed using (e) a control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems | |
| $G_{20}$ : Category one of control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems<br><br>$G_{21}$ : Category two of SAS<br><br>$G_{22}$ : Category three of SAS | |
| $T_{20}$ : Category one of quantum network is described by such a graph, the network is efficiently controllable, in the sense that universal quantum computation can be performed<br><br>$T_{21}$ : Category two of SAS<br><br>$T_{22}$ : Category three of SAS | |
| **Module four**<br><br>Authors show that when a quantum network is described by such a graph, the network is efficiently controllable, in the sense that universal quantum computation can be performed using a control sequence polynomial in the size of the network while (e) controlling a vanishingly small fraction of subsystems | |
| $G_{24}$ : Category one of controlling a vanishingly small fraction of subsystems<br><br>$G_{25}$ : Category two of SAS<br><br>$G_{26}$ : Category three of SAS | |
| $T_{24}$ : Category one ofquantum network is described by such a graph, the network is efficiently controllable, in the sense that universal quantum computation can be performed using a control sequence polynomial in the size of the network<br><br>$T_{25}$ : Category two of SAS | |

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| $T_{26}$ : Category three of SAS | |
| **Module five**<br><br>Authors show that networks corresponding to finite-dimensional lattices are efficently controllable, and explore generalizations to (e) percolation clusters and random graphs | |
| $G_{28}$ : Category one of percolation clusters and random graphs<br><br>$G_{29}$ : Category two of SAS<br><br>$G_{30}$ : Category three of SAS | |
| $T_{28}$ : Category one of networks corresponding to finite-dimensional lattices are efficently controllable, and explore generalizations<br><br>$T_{29}$ : Category two of SAS<br><br>$T_{30}$ : Category three of SAS | |
| **Module six**<br><br>They show that the classical computational complexity of estimating the ground state of Hamiltonians described by controllable graphs is (=) polynomial in the number of subsystems/qubits.<br><br>Subjects: Quantum Physics (quant-ph) Cite as: arXiv:1503.05270 [quant-ph] (or arXiv:1503.05270v3 [quant-ph] for this version) | |
| $G_{32}$ : Category one of classical computational complexity of estimating the ground state of Hamiltonians described by controllable graphs<br><br>$G_{33}$ : Category two of SAS<br><br>$G_{34}$ : Category three of SAS | |
| $T_{32}$ : Category one of polynomial in the number of subsystems/qubits<br><br>$T_{33}$ : Category two of SAS<br><br>$T_{34}$ : Category three of SAS | |
| **Module seven**<br><br>**Local controllability of quantum networks Daniel Burgarth, Sougato Bose, Christoph Bruder, Vittorio Giovannetti (28 Jun 2009 (this version, v2))**<br><br>(1) Authors give a sufficient criterion that guarantees that a many-body quantum system can be controlled by (e&eb) properly manipulating the (local) Hamiltonian of one of its subsystems.<br>(2) The method can be applied to a wide range of systems: it does not depend on the details of the couplings but only on their associated topology. As a special case, they prove that Heisenberg and Affleck-Kennedy-Lieb-Tasaki chains can be controlled by (e&eb) operating on one of the spins at | |

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| their ends.<br>(3) In principle, arbitrary quantum algorithms can be performed on such chains by acting on (e&eb) a single qubit. Subjects: Quantum Physics (quant-ph) Journal reference: Phys. Rev. A 79, 060305(R) (2009) DOI:10.1103/PhysRevA.79.060305 Cite as: arXiv: 0805.3975 [quant-ph] (or arXiv:0805.3975v2 [quant-ph] for this version)<br><br>Authors give a sufficient criterion that guarantees that a many-body quantum system can be controlled by (e&eb) properly manipulating the (local) Hamiltonian of one of its subsystems | |
| $G_{36}$ : Category one of **sufficient criterion that guarantees that a many-body quantum system;** properly manipulating the (local) Hamiltonian of one of its subsystems<br><br>$G_{37}$ : Category two of SAS<br><br>$G_{38}$ : Category three of SAS | |
| $T_{36}$ : Category one of properly manipulating the (local) Hamiltonian of one of its subsystems **;sufficient criterion that guarantees that a many-body quantum system**<br><br>$T_{37}$ : Category two of SAS<br><br>$T_{38}$ : Category three of SAS | |
| **Module eight**<br><br>The method can be applied to a wide range of systems: it does not depend on the details of the couplings but only on their associated topology.<br><br>As a special case, they prove that **Heisenberg and Affleck-Kennedy-Lieb-Tasaki chains** can be controlled by (e&eb) operating on one of the spins at their ends. | |
| $G_{40}$ : Category one of **Heisenberg and Affleck-Kennedy-Lieb-Tasaki chains;** operating on one of the spins at their ends<br><br>$G_{41}$ : Category two of SAS<br><br>$G_{42}$ : Category three of SAS | |
| $T_{40}$ : Category one of operating on one of the spins at their ends **;Heisenberg and Affleck-Kennedy-Lieb-Tasaki chains**<br><br>$T_{41}$ : Category two of SAS<br><br>$T_{42}$ : Category three of SAS | |
| **Module Nine**<br><br>**AKLT model from Wikipedia, the free encyclopedia**<br><br>The AKLT model is an extension of the one-dimensional quantum Heisenberg spin model. The proposal and exact solution of this model by Affleck, Lieb, Kennedy and Tasaki [1] provided crucial insight into the | |

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| physics of the spin-1 Heisenberg chain.[2] It has also served as a useful example for such concepts as valence bond solid mmetry Protected Topological Order[3][4][5] And Matrix Product State Wavefunctions.<br><br>**Definition**<br><br>Affleck et al. were interested in constructing a one-dimensional state with a valence bond between every pair of sites. Because this leads to two spin 1/2s for every site, the result must be the wavefunction of a spin 1 system.<br><br>For every adjacent pair of the spin 1s, two of the four constituent spin 1/2s are stuck in a total spin zero state. Therefore each pair of spin 1s is forbidden from being in a combined spin 2 state. By writing this condition as a sum of projectors, AKLT arrived at the following Hamiltonian<br><br>{\displaystyle {\hat {H}}=\sum _{j}{\vec {S}}_{j}\cdot {\vec {S}}_{j+1}+{\frac {1}{3}}({\vec {S}}_{j}\cdot {\vec {S}}_{j+1})^{2}} \hat H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2<br><br>This Hamiltonian is similar to the spin 1, one-dimensional quantum Heisenberg spin model but has an additional "biquadratic" spin interaction term. | |
| $G_{44}$ : Category one of LHS of the above equation<br><br>$G_{45}$ : Category two of SAS<br><br>$G_{46}$ : Category three of SAS | |
| $T_{44}$ : Category one of RHS opf the above equation<br><br>$T_{45}$ : Category two of SAS<br><br>$T_{46}$ : Category three of SAS | |

**Note: Cross references are given above**